MATH 210C. CENTERS OF SIMPLY CONNECTED SEMISIMPLE GROUPS

As we have discussed in class, if G is a connected compact Lie group that is semisimple and simply connected and T is a maximal torus then for a basis B of $(X(T)_{\mathbf{Q}}, \Phi(G,T))$ and the corresponding basis $B^\vee$ of the dual root system $(X_*(T)_{\mathbf{Q}}, \Phi(G,T)^\vee)$, we have a canonical description of T as a direct product of circles:

$$\prod_{b \in B} S^1 \simeq T$$

defined by $(x_b) \mapsto \prod_{b \in B} b^\vee(x_b)$. The center Z_G lies inside T, and is dual to the cokernel P/Q of the weight lattice $P = (\mathbf{Z}\Phi^\vee)'$ modulo the root lattice $Q = \mathbf{Z}\Phi$. Moreover, the *simply connected* G is determined up to isomorphism by its root system (this is part of the Isomorphism Theorem who proof requires work with Lie groups and Lie algebras over $\mathbf{C}$, so beyond the scope of this course).

Hence, it is natural to seek an explicit description of the resulting inclusion

$$\mathrm{Hom}(P/Q, S^1) = Z_G \hookrightarrow T = \mathrm{Hom}(P, S^1) = \prod_{b \in B} b^\vee(S^1)$$

in terms of the "coroot parameterization" of T (the final equality using that $B^\vee$ is a $\mathbf{Z}$-basis of $\mathbf{Z}\Phi^\vee$). From the tables at the end of volume II (Chapters 4-6) of Bourbaki's *Lie groups and Lie algebras*, one can read off this information (essentially the inclusion of Q into P in terms of a Cartan matrix relative to B and $B^\vee$).

Below we tabulate all of this information for all of the irreducible (and reduced) root systems, using the labeling of vertices in the diagram by the elements of B; the notation next to a vertex b indicates what is evaluated inside $b^\vee : S^1 \to T$. (The same works for split connected semisimple linear algebraic groups over any field, once one learns that theory and its relation to root systems.) We are very grateful to Jason Starr for his assistance in rendering a hand-written table in modern typography.

Here are some general comments to help with reading the tables:

(1) For the group $\mathrm{SU}(n+1)$ the description below indicates that the center is the group μ_{n+1} of $(n+1)$th roots of unity, and that if we label the vertices in the Dynkin diagram from left to right as $b_1, \dots, b_n$ then an inclusion $\mu_{n+1} \hookrightarrow \prod_{j=1}^n b_j^\vee(S^1) = T$ is given by

$$t \mapsto b_1^\vee(t) b_2^\vee(t^2) \cdots b_n^\vee(t^n).$$

Let's see that this really does encode the usual way of describing the center of $\mathrm{SU}(n+1)$ as diagonal scalar matrices with a common $(n+1)$th root of unity along the diagonal.

Let D be the diagonal torus of $\mathrm{U}(n+1)$ (with the usual bases $\{e_j\}_{1 \le j \le n+1}$ and $\{e_j^*\}_{1 \le j \le n+1}$ of its respective character and cocharacter lattices, corresponding to its $n+1$ entries in order from upper left to lower right). Let $D' = D \cap \mathrm{SU}(n+1)$ be the corresponding diagonal maximal torus of $\mathrm{SU}(n+1)$. A basis of the associated root system $\Phi(\mathrm{SU}(n+1), D') = \Phi(\mathrm{U}(n+1), D)$ is given by characters $b_j = e_j - e_{j+1}$

$(1 \leq j \leq n)$ which send a diagonal $d' \in D' \subset D$ to the ratio $d'_j/d'_{j+1} \in S^1$, and the corresponding collection of coroots consists of the cocharacters

$$b_j^\vee = e_j^* - e_{j+1}^* : S^1 \to D' \subset D \ (1 \leq j \leq n)$$

sending z to the diagonal matrix whose jth entry is z and $(j+1)$th entry is $1/z$.
Hence, for $t \in \mu_{n+1}$ we have

$$\begin{aligned}\prod_{j=1}^{n} b_j^\vee(t^j) = \prod_{j=1}^{n}(e_j^*(t^j)/e_{j+1}^*(t^j)) &= e_1^*(t)\left(\prod_{j=2}^{n} e_j^*(t^j)/e_j^*(t^{j-1})\right)(1/e_{n+1}^*(t^n)) \\ &= e_1^*(t)\left(\prod_{j=2}^{n} e_j^*(t)\right)e_{n+1}^*(1/t^n) \\ &= \prod_{j=1}^{n+1} e_j^*(t),\end{aligned}$$

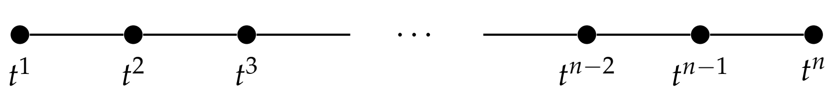
where the final equality uses that $1/t^n = t$ since $t^{n+1} = 1$. But this final product is exactly the scalar diagonal matrix with t as the common entry along the diagonal, so we have indeed recovered the usual identification of the center of $\mathrm{SU}(n+1)$ with μ_{n+1}.

(2) Here are some comments for the compact special orthogonal groups, or rather their spin double covers. For the root system B_n with $n \geq 2$ (i.e., the group $\mathrm{Spin}(2n+1)$) the table says that the center is identified with μ_2 sitting inside the direct factor of $T = \prod_{b \in B} b^\vee(S^1)$ corresponding to the coroot associated to the unique short root in the basis.

Turning to type D_n (i.e., the group $\mathrm{Spin}(2n)$), the center for type D_n depends on the parity of n (this corresponds to the fact that the group structure of P/Q for type D_n depends on the parity of n, being $\mathbf{Z}/(4)$ for odd n and $(\mathbf{Z}/(2)) \times (\mathbf{Z}/(2))$ for even n). In the case of D_n for odd n, we write "$\zeta^2 = -1$" to indicate that the formula as written is for ζ a primitive 4th root of unity (though it really works as written for all 4th roots on unity: the coroot contributions of ζ^2 are trivial when $\zeta \in \mu_2 \subset \mu_4$).

(3) At the end, we provide a short list of exceptional isomorphisms arising from the coincidence of equality of very low-rank root systems in the various infinite families. (It is reasonable to consider D_2 to be $\mathrm{A}_1 \times \mathrm{A}_1$ if you stare at the diagram for type D, and likewise it is reasonable to regard B_1 as being A_1 since $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$.)

SU$(n+1)$ A_n



$t^{n+1} = 1$

Spin$(2n+1)$ $\downarrow$ **SO**$(2n+1)$

B_n $(n \geq 2)$

$B_1 = A_1$

$\zeta^2 = 1$

Sp(n)

C_n $(n \geq 2)$

$C_1 = A_1$

$C_2 = B_2$

$\zeta^2 = 1$

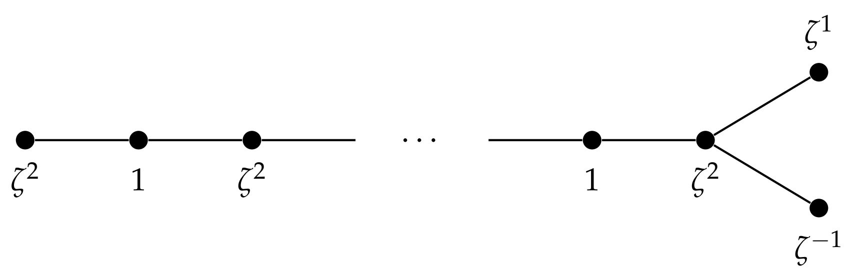
Spin$(2n)$ $\downarrow$ **SO**$(2n)$

not adjoint

D_n $(n \geq 4)$

(n odd)

$D_3 = A_3$



$\zeta^4 = 1$

"$\zeta^2 = -1$"

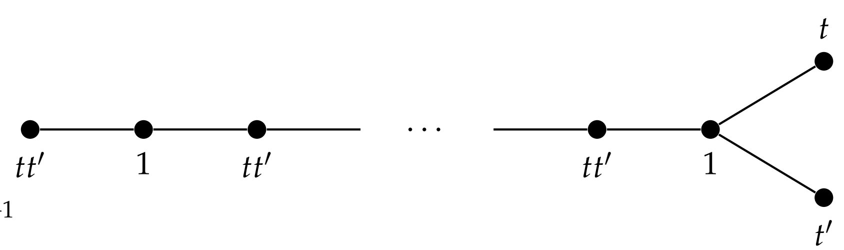
Spin$(2n)$ $\downarrow$ **SO**$(2n)$

not adjoint

D_n $(n \geq 4)$

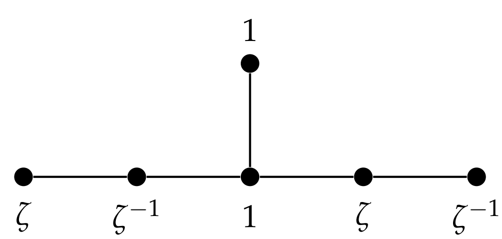
(n even)

$D_2 = A_1 \times A_1$



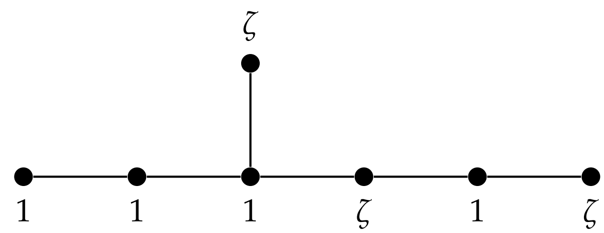
$(t, t') \in \mu_2 \times \mu_2$

E_6



$\zeta^3 = 1$

E_7



$\zeta^2 = 1$

$$E_8,\ F_4,\ G_2 : \text{trivial center}$$

$$\mathbf{SO}(6) \simeq \mathbf{SU}(4)/\mu_2 \quad (\, D_3 = A_3 \,)$$

$$\mathbf{SO}(5) \simeq \mathbf{Sp}(2)/\text{center} \quad (\, B_2 = C_2 \,)$$

$$\mathbf{SO}(4) \simeq (\mathbf{SU}(2) \times \mathbf{SU}(2))/\Delta(\mu_2) \quad (\, D_2 = A_1 \times A_1 \,)$$

$$\mathbf{SO}(3) \simeq \mathbf{SU}(2)/\mu_2 \quad (\, B_1 = A_1 \,)$$

$$\mathbf{SO}(2) \simeq S^1$$