# Math 210C. The adjoint representation

Let $G$ be a Lie group. One of the most basic tools in the investigation of the structure of $G$ is the conjugation action of $G$ on itself: for $g \in G$ we define $c_g : G \to G$ to be the $C^\infty$ automorphism $x \mapsto gxg^{-1}$. (This is not interesting when $G$ is commutative, but we will see later that *connected* commutative Lie groups have a rather simple form in general.)

The *adjoint representation* of $G$ on its tangent space $\mathfrak{g} = \mathrm{T}_e(G)$ at the identity is the homomorphism

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$$

defined by $\mathrm{Ad}_G(g) = \mathrm{d}c_g(e)$. This is a homomorphism due to the Chain Rule: since $c_{g'} \circ c_g = c_{g'g}$ and $c_g(e) = e$, we have

$$\mathrm{Ad}_G(g'g) = \mathrm{d}c_{g'}(e) \circ \mathrm{d}c_g(e) = \mathrm{Ad}_G(g') \circ \mathrm{Ad}_G(g).$$

In this handout we prove the smoothness of $\mathrm{Ad}_G$ (which the course text seems to have overlooked), compute the derivative

$$\mathrm{d}(\mathrm{Ad}_G)(e) : \mathfrak{g} \to \mathrm{T}_1(\mathrm{GL}(\mathfrak{g})) = \mathrm{End}(\mathfrak{g})$$

at the identity, and use $\mathrm{Ad}_G$ to establish a very useful formula relating the Lie bracket rather directly to the group law on $G$ near $e$.

## 1. Smoothness and examples

To get a feeling for the adjoint representation, let's consider the case $G = \mathrm{GL}_n(\mathbf{F})$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. For any $X \in \mathfrak{g} = \mathrm{Mat}_n(\mathbf{F})$, a parametric curve in $G$ through the identity with velocity vector $X$ at $t = 0$ is $\alpha_X(t) := \exp(tX)$. Thus, the differential $\mathrm{Ad}_G(g) = \mathrm{d}c_g(e)$ sends $X = \alpha'_X(0)$ to the velocity at $t = 0$ of the parametric curve

$$c_g \circ \alpha_X : t \mapsto g \exp(tX) g^{-1} = 1 + gtXg^{-1} + \sum_{j \geq 2} \frac{t^j}{j!} gX^j g^{-1},$$

so clearly this has velocity $gXg^{-1}$ at $t = 0$. In other words, $\mathrm{Ad}_G(g)$ is $g$-conjugation on $\mathrm{Mat}_n(\mathbf{F})$. This is visibly smooth in $g$.

We can use a similar parametric curve method to compute $\mathrm{d}(\mathrm{Ad}_G)(e)$ for $G = \mathrm{GL}_n(\mathbf{F})$, as follows. Choose $X \in \mathfrak{g}$, so $\alpha_X(t) := \exp(tX)$ is a parametric curve in $G$ with $\alpha'_X(0) = X$. Hence, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)$ is the velocity at $t = 0$ of the parametric curve $\mathrm{Ad}_G(\exp(tX)) \in \mathrm{GL}(\mathfrak{g})$. In other words, it is the derivative at $t = 0$ of the parametric curve $c_{\exp(tX)} \in \mathrm{GL}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$. For $Y \in \mathfrak{g}$,

$$\exp(tX) \circ Y \circ \exp(-tX) = (1 + tX + t^2(\cdot)) \circ Y \circ (1 - tX + t^2(\cdot)) = (Y + tXY + t^2(\cdot)) \circ (1 - tX + t^2(\cdot)),$$

and this is equal to $Y + t(XY - YX) + t^2(\cdots)$, so its $\mathrm{End}(\mathfrak{g})$-valued velocity vector at $t = 0$ is the usual commutator $XY - YX$ that we know to be the Lie bracket on $\mathfrak{g}$.

Next we take up the proof of smoothness in general. First, we localize the problem near the identity using the elementary:

**Lemma 1.1.** *Let $G$ and $H$ be Lie groups. A homomorphism of groups $f : G \to H$ is continuous if it is continuous at the identity, and it is $C^\infty$ if it is $C^\infty$ near the identity.*

*Proof.* The left-translation $\ell_g : G \to G$ is a homeomorphism carrying $e$ to $g$, and likewise $\ell_{f(g)} : H \to H$ is a homeomorphism. Since

$$f \circ \ell_g = \ell_{f(g)} \circ f \tag{1.1}$$

(as $f$ is a homomorphism), continuity of $f$ at $g$ is equivalent to continuity of $f$ at $e$. This settles the continuity aspect. In a similar manner, since left translations are diffeomorphisms and $\ell_g$ carries an open neighborhood of $e$ onto one around $g$ (and similarly for $\ell_{f(g)}$ on $H$), if $f$ is $C^\infty$ on an open $U$ around $e$ then $f$ is also $C^\infty$ on the open $\ell_g(U)$ around $g$ due to (1.1). Since the $C^\infty$-property is local on $G$, it holds for $f$ if it does so on an open set around every point. ■

Finally, we prove smoothness of $\mathrm{Ad}_G$. Since the conjugation-action map $c : G \times G \to G$ defined by $(g, g') \mapsto g g' g^{-1}$ is $C^\infty$ and $c(e, e) = e$, we can choose a open coordinate domains $U \subset U' \subset G$ around $e$ so that $c(U \times U) \subset U'$. Let $\{x_1, \ldots, x_n\}$ be a coordinate system on $U'$ with $x_i(e) = 0$, and define $f_i = x_i \circ c : U \times U \to \mathbf{R}$ as a function on $U \times U \subset \mathbf{R}^{2n}$. Let $\{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ denote the resulting product coordinate system on $U \times U$.

Each $f_i$ is smooth and $c_g : U \to U'$ has $i$th component function $f_i(g, z_1, \ldots, z_n)$ with $g \in U$. Thus, the matrix $\mathrm{Ad}_G(g) = \mathrm{d}(c_g)(e) \in \mathrm{Mat}_n(\mathbf{R})$ has $ij$-entry equal to $(\partial f_i / \partial z_j)(0)$. Hence, smoothness of $\mathrm{Ad}_G$ on $U$ reduces to the evident smoothness of each $\partial f_i / \partial z_j$ in the first $n$ coordinates $y_1, \ldots, y_n$ on $U \times U$ (after specializing the second factor $U$ to $e$). By the preceding Lemma, this smoothness on $U$ propagates to smoothness for $\mathrm{Ad}_G$ on the entirety of $G$ since $\mathrm{Ad}_G$ is a homomorphism.

## 2. Key formula for the Lie bracket

For our Lie group $G$, choose $X, Y \in \mathfrak{g}$. In class we mentioned the fact (to be proved next time) that there is a unique Lie group homomorphism $\alpha_X : \mathbf{R} \to G$ satisfying $\alpha_X'(0) = X$. The automorphism $\mathrm{Ad}_G(\alpha_X(t))$ of $\mathfrak{g}$ therefore makes sense and as a point in the open subset $\mathrm{GL}(\mathfrak{g})$ of $\mathrm{End}(\mathfrak{g})$ it depends smoothly on $t$ (since $\mathrm{Ad}_G$ is smooth). Evaluation on $Y$ for this matrix-valued path defines a smooth path

$$t \mapsto \mathrm{Ad}_G(\alpha_X(t))(Y)$$

valued in $\mathfrak{g}$. We claim that the velocity of this latter path at $t = 0$ is $[X, Y]$. In other words:

**Theorem 2.1.** *For any $X, Y \in \mathfrak{g}$,*

$$[X, Y] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathrm{Ad}_G(\alpha_X(t))(Y)).$$

Observe that the left side uses the construction of *global* left-invariant differential operators whereas the right side is defined in a much more localized manner near $e$. The "usual" proof of this theorem uses the notion of Lie derivative, but the approach we use avoids that.

*Proof.* Since $Y$ is the velocity at $s = 0$ of the parametric curve $\alpha_Y(s)$, for any $g \in G$ the vector $\mathrm{Ad}_G(g)(Y) = \mathrm{d}(c_g)(e)(Y) \in \mathfrak{g}$ is the velocity at $s = 0$ of the parametric curve $c_g(\alpha_Y(s))$. Thus, for any $t$, $\mathrm{Ad}_G(\alpha_X(t))(Y)$ is the velocity at $s = 0$ of

$$a(t, s) := c_{\alpha_X(t)}(\alpha_Y(s)) = \alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$

Note that $a(t,0)=e$ for all $t$, so for each $t$ the velocity to $s\mapsto a(t,s)\in G$ lies in $\mathrm{T}_e(G)=\mathfrak{g}$; this velocity is nothing other than $\mathrm{Ad}_G(\alpha_X(t))(Y)$, but we shall suggestively denote it as $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}a(t,s)$. Since this is a parametric curve valued in $\mathfrak{g}$, we can recast our problem as proving the identity

$$[X,Y]\stackrel{?}{=}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}a(t,s)$$

where $a(t,s)=\alpha_X(t)\alpha_Y(s)\alpha_X(-t)$. We shall compute each side as a point-derivation at $e$ on a smooth function $\varphi$ on $G$ and get the same result.

For the right side, an exercise to appear in HW3 (Exercise 9 in I.2) shows that its value on $\varphi$ is the ordinary 2nd-order multivariable calculus derivative

$$\frac{\partial^2}{\partial_t\partial_s}|_{(0,0)}\varphi(a(t,s))$$

of the smooth function $\varphi\circ a:\mathbf{R}^2\to\mathbf{R}$. By a clever application of the Chain Rule, it is shown in the course text (on page 19, up to swapping the roles of the letters $s$ and $t$) that this 2nd-order partial derivative is equal to the difference

$$\frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_X(t)\alpha_Y(s))-\frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_Y(s)\alpha_X(t)).$$

Letting $\widetilde{X}$ and $\widetilde{Y}$ respectively denote the left-invariant vector fields extending $X$ and $Y$ at $e$, we want this difference of 2nd-order partial derivatives to equal the value $[X,Y](\varphi)=[\widetilde{X},\widetilde{Y}](\varphi)(e)$ at $e$ of $\widetilde{X}(\widetilde{Y}(\varphi))-\widetilde{Y}(\widetilde{X}(\varphi))$, so it suffices to prove in general that

$$\frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_X(t)\alpha_Y(s))=X(\widetilde{Y}\varphi)$$

(and then apply this with the roles of $X$ and $Y$ swapped).

In our study next time of the construction of 1-parameter subgroups we will see that for any $g\in G$, $(\widetilde{Y}\varphi)(g)=(\partial_s|_{s=0})(\varphi(g\alpha_Y(s)))$. Thus, setting $g=\alpha_X(t)$, for any $t$ the $s$-partial at $s=0$ of $\varphi(\alpha_X(t)\alpha_Y(s))$ is equal to $(\widetilde{Y}\varphi)(\alpha_X(t))$. By the same reasoning now applied to $X$ instead of $Y$, passing to the $t$-derivative at $t=0$ yields $(\widetilde{X}(\widetilde{Y}\varphi))(e)=X(\widetilde{Y}(\varphi))$. ■

## 3. Differential of adjoint

Finally, we connect the Lie bracket to the adjoint representation of $G$:

**Theorem 3.1.** *Let $G$ be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then $\mathrm{d}(\mathrm{Ad}_G)(e)\in\mathrm{End}(\mathfrak{g})$ is equal to $\mathrm{ad}_{\mathfrak{g}}$. In other words, for $X\in\mathfrak{g}$, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)=[X,\cdot]$.*

*Proof.* Choose $X\in\mathfrak{g}$, so $\alpha_X(t)$ is a parametric curve in $G$ with velocity $X$ at $t=0$. Consequently, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)$ is the velocity vector at $t=0$ to the parametric curve $\mathrm{Ad}_G(\alpha_X(t))$ valued in the open subset $\mathrm{GL}(\mathfrak{g})$ of $\mathrm{End}(\mathfrak{g})$.

Rather generally, if $L:(-\epsilon,\epsilon)\to\mathrm{End}(V)$ is a parametric curve whose value at $t=0$ is the identity then for any $v\in V$ the velocity to $t\mapsto L(t)(v)$ at $t=0$ is $L'(0)(v)$. Indeed, the second-order Taylor expression $L(t)=1+tA+t^2B(t)$ for a smooth parametric curve $B(t)$ valued in $\mathrm{End}(V)$ implies that $L(t)(v)=v+tA(v)+t^2B(t)(v)$, so this latter curve valued in $V$ has velocity $A(v)$. But clearly $A=L'(0)$, so our general velocity identity is proved.

Setting $L = \mathrm{Ad}_G \circ \alpha_X$ and $v = Y$, we conclude that $\mathrm{Ad}_G(\alpha_X(t))(Y)$ has velocity at $t = 0$ equal to the evaluation at $Y$ of the velocity at $t = 0$ of the parametric curve $\mathrm{Ad}_G \circ \alpha_X$ valued in $\mathrm{End}(\mathfrak{g})$. But by the Chain Rule this latter velocity is equal to

$$\mathrm{d}(\mathrm{Ad}_G)(\alpha_X(0)) \circ \alpha'_X(0) = \mathrm{d}(\mathrm{Ad}_G(e))(X),$$

so $\mathrm{d}(\mathrm{Ad}_G(e))(X)$ carries $Y$ to the velocity at $t = 0$ that equals $[X, Y]$ in Theorem 2.1. ■