MATH 210C. WEYL GROUP COMPUTATIONS

1. INTRODUCTION

In §2 of the handout on Weyl groups and character lattices, for $n \geq 2$ it is shown that for $G = \mathrm{U}(n) \subset \mathrm{GL}_n(\mathbf{C})$ and $T = (S^1)^n$ the diagonal maximal torus (denoted $\Delta(n)$ in the course text), we have $N_G(T) = T \rtimes S_n$ using the symmetric group S_n in its guise as $n \times n$ permutation matrices. (The course text denotes this symmetric group as $S(n)$.)

The case of $\mathrm{SU}(n)$ and its diagonal maximal torus $T' = T \cap \mathrm{SU}(n)$ (denoted as $S\Delta(n)$ in the course text) was also worked out there, and its Weyl group is also S_n. This case is more subtle than in the case of $\mathrm{U}(n)$ since we showed that the Weyl group of $\mathrm{SU}(n)$ does *not* lift isomorphically to a subgroup of the corresponding torus normalizer inside $\mathrm{SU}(n)$.

Remark 1.1. Consider the inclusion $T' \hookrightarrow T$ between respective diagonal maximal tori of $\mathrm{SU}(n)$ and $\mathrm{U}(n)$. Since $T = T' \cdot Z$ for the *central* diagonally embedded circle $Z = S^1$ in $\mathrm{U}(n)$, we have $N_{\mathrm{SU}(n)}(T') \subset N_{\mathrm{U}(n)}(T)$ and thus get an injection $W(\mathrm{SU}(n), T') \hookrightarrow W(\mathrm{U}(n), T)$ that is an equality for size reasons. During our later study of root systems we will explain this equality of Weyl groups for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ in a broader setting.

For each of the additional classical compact groups $\mathrm{SO}(n)$ $(n \geq 3)$ and $\mathrm{Sp}(n)$ $(n \geq 1)$, we found an explicit self-centralizing and hence maximal torus in HW3 Exercise 4; the maximal torus found in this way for $\mathrm{SO}(2m)$ is also a maximal torus in $\mathrm{SO}(2m+1)$. The aim of this handout is to work out the Weyl group in these additional cases.

The course text explains this material, in 3.3–3.8 in Chapter IV. Our presentation is different in some minor aspects, but the underlying technique is the same: just as the method of determination of the Weyl groups for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ in the earlier handout rested on a consideration of eigenspace decompositions relative to the action of the maximal torus on a "standard" $\mathbf{C}$-linear representation of the ambient compact connected Lie group, we shall do the same for the special orthogonal and symplectic cases with appropriate "standard" representations over $\mathbf{C}$.

2. ODD SPECIAL ORTHOGONAL GROUPS

Let's begin with $G = \mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ with $m \geq 1$. In this case, a maximal torus $T = (S^1)^m$ was found in HW3 Exercise 4: it consists of a string of 2×2 rotation matrices $r_{2\pi\theta_1}, \ldots, r_{2\pi\theta_n}$ laid out along the diagonal of a $(2m+1) \times (2m+1)$ matrix, with the lower-right entry equal to 1 and $\theta_j \in \mathbf{R}/\mathbf{Z}$. In other words, a typical $t \in T$ can be written as

$$t = \begin{pmatrix} r_{2\pi\theta_1} & 0 & \ldots & 0 & 0 \\ 0 & r_{2\pi\theta_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \ldots & r_{2\pi\theta_m} & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

(This torus is denoted as $T(m)$ in the course text.)

View $\mathbf{C}^n$ as the complexification of the standard representation of $\mathrm{SO}(n)$, so the decomposition of the rotation matrix r_θ into its diagonal form over $\mathbf{C}$ implies that the action of T on $\mathbf{C}^{2m+1}$ has as its eigencharacters

$$\{\chi_1, \chi_{-1}, \ldots, \chi_m, \chi_{-m}, 1\}$$

where $\chi_{\pm j}(t) = e^{\pm 2\pi i \theta_j}$. More specifically, any $t \in T$ acts on each plane $P_j = \mathbf{R}e_{2j-1} \oplus \mathbf{R}e_{2j}$ via a rotation r_{θ_j}, so t acts on $(P_j)_{\mathbf{C}}$ with eigenvalues $\chi_{\pm j}(t)$ (with multiplicity).

These T-eigencharacters are pairwise distinct with 1-dimensional eigenspaces in $\mathbf{C}^{2m+1}$, so any $n \in N_G(T)$ must have action on $\mathbf{C}^{2m+1}$ that *permutes* these eigenlines in accordance with its permutation effect on the eigencharacters in $\mathrm{X}(T)$. In particular, n preserves the eigenspace $(\mathbf{C}^{2m+1})^T$ for the trivial chracters, and this eigenspace is the basis line $\mathbf{C}e_{2m+1}$.

Since the action of G on $\mathbf{C}^{2m+1}$ is defined over $\mathbf{R}$, if n acting on T (hence on $\mathrm{X}(T)$) carries χ_k to $\chi_{k'}$ then by compatibility with the componentwise complex conjugation on $\mathbf{C}^{2m+1}$ we see that n acting on T (hence on $\mathrm{X}(T)$) carries the complex conjugate $\overline{\chi}_k = \chi_{-k}$ to $\overline{\chi}_{k'} = \chi_{-k'}$. Keeping track of the χ_k-eigenline via the index $k \in \{\pm 1, \ldots, \pm m\}$, the effect of $W(G,T)$ on the set of eigenlines defines a homomorphism f from $W(G,T)$ into the group $\Sigma(m)$ of permutations σ of $\{\pm 1, \cdots \pm m\}$ that permute the numbers $\pm j$ in pairs; equivalently, $\sigma(-k) = -\sigma(k)$ for all k. (The course text denotes $\Sigma(m)$ as $G(m)$.)

The permutation within each of the m pairs of indices $\{j, -j\}$ constitutes a $\mathbf{Z}/2\mathbf{Z}$, and the permutation induced by σ on the set of m such pairs is an element of S_m, so we see that $\Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ with the standard semi-direct product structure.

Proposition 2.1. *The map*

$$f : W(G,T) \to \Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$$

is an isomorphism.

For injectivity, note that any $g \in N_G(T) \subset \mathrm{GL}_{2m+1}(\mathbf{C})$ representing a class in the kernel has effect on $\mathbf{C}^{2m+1}$ preserving every (1-dimensional) eigenspace of T and so must be diagonal over $\mathbf{C}$ (not just diagonalizable) with entries in S^1 by compactness. Membership in $G = \mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ forces the diagonal entries of g to be ± 1. Such g with $\det(g) = 1$ visibly belongs to $\mathrm{SO}(2m+1) = G$ and hence lies in $Z_G(T) = T$, so the injectivity of $W(G,T) \to \Sigma(m)$ is establshed.

To prove surjectivity, first note that a permutation among the m planes P_j is obtained from a $2m \times 2m$ matrix that is an $m \times m$ "permutation matrix" in copies of the 2×2 identity matrix. This $2m \times 2m$ matrix has determinant 1 since each transposition $(ij) \in S_m$ acts by swapping the planes P_i and P_j via a direct sum of *two* copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, by expanding this to a determinant-1 action on $\mathbf{R}^{2m+1}$ via action by the trivial action on $\mathbf{R}e_{2m+1}$ gives an element of $G = \mathrm{SO}(2m+1)$ that lies in $N_G(T)$ and represents any desired element of $S_m \subset \Sigma(m)$. Likewise, since the eigenlines for $\chi_{\pm j}$ in $(P_j)_{\mathbf{C}}$ are the lines $\mathbf{C}(e_{2j-1} + ie_{2j})$ and $\mathbf{C}(e_{2j-1} - ie_{2j}) = \mathbf{C}(e_{2j} + ie_{2j-1})$ that are swapped upon swapping e_{2j-1} and e_{2j} *without* a sign intervention, we get an element of $N_G(T)$ representing any $(\epsilon_1, \ldots, \epsilon_m) \in (\mathbf{Z}/2\mathbf{Z})^m \subset \Sigma(m)$ by using the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{\epsilon_j} \in \mathrm{O}(2)$ on the plane P_j for each j and using the action by $(-1)^{\sum \epsilon_j}$ on $\mathbf{R}e_{2m+1}$ to ensure an overall sign of 1. This completes our determination of the Weyl group of $\mathrm{SO}(n)$ for odd n.

3. Even special orthogonal groups

Now suppose $G = \mathrm{SO}(2m)$. We have a similar description of a maximal torus $T = (S^1)^m$: it is an array of m rotation matrices $r_{\theta_j} \in \mathrm{SO}(2)$ (without any singleton entry of 1 in the lower-right position). The exact same reasoning as in the case $n = 2m + 1$ defines a homomorphism

$$f : W(G,T) \to \Sigma(m)$$

that is injective due to the exact same argument as in the odd special orthogonal case.

The proof of surjectivity in the case $n = 2m+1$ does not quite carry over (and in fact f will not be surjective, as is clear when $m = 1$ since $\mathrm{SO}(2)$ is commutative), since we no longer have the option to act by a sign on $\mathbf{R}e_{2m+1}$ in order to arrange for an overall determinant to be equal to 1 (rather than -1).

Inside $\Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ we have the index-2 subgroup $A(m)$ that is the kernel of the homomorphism $\delta_m : \Sigma(m) \to \{\pm 1\}$ defined by

$$((\epsilon_1, \dots, \epsilon_m), \sigma) \mapsto (-1)^{\sum \epsilon_j}.$$

(The course text denotes this group as $SG(m)$.) Explicitly, $A(m) = H_m \rtimes S_m$ where $H_m \subset (\mathbf{Z}/2\mathbf{Z})^m$ is the hyperplane defined by $\sum \epsilon_j = 0$.

Note that T is a maximal torus in $\mathrm{SO}(2m+1)$, and $N_{\mathrm{SO}(2m)}(T) \subset N_{\mathrm{SO}(2m+1)}(T)$ via the natural inclusion $\mathrm{GL}_{2m}(\mathbf{R}) \hookrightarrow \mathrm{GL}_{2m+1}(\mathbf{R})$ using the decomposition $\mathbf{R}^{2m+1} = H \oplus \mathbf{R}e_{2m+1}$ for the hyperplane H spanned by $e_1, \dots, e_{2m}$. Hence, we get an injection

$$W(\mathrm{SO}(2m), T) \hookrightarrow W(\mathrm{SO}(2m+1), T).$$

Projection to the lower-right matrix entry defines a character $N_{\mathrm{SO}(2m+1)}(T) \twoheadrightarrow \{\pm 1\}$ that encodes the sign of the action of this normalizer on the line $\mathbf{R}e_{2m+1}$ of T-invariants. This character kills T and visibly has as its kernel exactly $N_{\mathrm{SO}(2m)}(T)$.

Upon passing to the quotient by T, we have built a character

$$\Sigma(m) = W(\mathrm{SO}(2m+1), T) \twoheadrightarrow \{\pm 1\}$$

whose kernel is $W(\mathrm{SO}(2m), T)$. This character on $\Sigma(m)$ is checked to coincide with δ_m by using the explicit representatives in $N_{\mathrm{SO}(2m+1)}(T)$ built in our treatment of the odd special orthogonal case. Thus, we have proved:

Proposition 3.1. *The injection $f : W(\mathrm{SO}(2m), T) \to \Sigma(m)$ is an isomorphism onto $\ker \delta_m = A(m)$.*

4. Symplectic groups

Finally, we treat the case $G = \mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}_n(\mathbf{H})$. Recall that G consists of precisely the matrices

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{U}(2n),$$

and (from Exercise 4 in HW3) a maximal torus $T = (S^1)^n$ of G is given by the set of elements

$$\mathrm{diag}(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n)$$

for $z_j \in S^1$. (This torus is denoted as T^n in the course text.) Note that the "standard" representation of G on $\mathbf{C}^{2n}$ has $T = (S^1)^n$ acting with $2n$ distinct eigencharacters: the component projections $\chi_j : T \to S^1$ and their reciprocals $1/\chi_j = \overline{\chi}_j$. Denoting $1/\chi_j$ as χ_{-j}, the action of $N_G(T)$ on T via conjugation induces a permutation of this set of eigencharacters $\chi_{\pm 1}, \dots, \chi_{\pm n}$.

Keeping track of these eigencharacters via their indices, we get a homomorphism from $W(G,T) = N_G(T)/T$ into the permutation group of $\{\pm 1, \dots, \pm n\}$. Recall that this permutation group contains a distinguished subgroup $\Sigma(n)$ consisting of the permutations σ satisfying $\sigma(-k) = -\sigma(k)$ for all k. We claim that $W(G,T)$ lands inside $\Sigma(n)$. This says exactly that if the action on $\mathrm{X}(T)$ by $w \in W(G,T)$ carries χ_k to $\chi_{k'}$ (with $-n \leq k, k' \leq n$) then it carries χ_{-k} to $\chi_{-k'}$. But by definition we have $\chi_{-k} = 1/\chi_k$, so this is clear.

Proposition 4.1. *The map $W(G,T) \to \Sigma(n)$ is an isomorphism.*

This equality with the same Weyl group as for $\mathrm{SO}(2n+1)$ is not a coincidence, but its conceptual explanation rests on a duality construction in the theory of root systems that we shall see later.

Proof. Suppose $w \in W(G,T)$ is in the kernel. Then for a representative $g \in G$ of w, the g-action on $\mathbf{C}^{2n}$ preserves the χ_k-eigenline for all $-n \leq k \leq n$, so g is diagonal in $\mathrm{GL}_{2n}(\mathbf{C})$. Thus, $g \in Z_G(T) = T$, so $w = 1$. Using the inclusion $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$$

that carries the diagonal maximal torus T_n of $\mathrm{U}(n)$ isomorphically onto our chosen maximal torus T of $\mathrm{Sp}(n)$, we get an injection

$$S_n = W(\mathrm{U}(n), T_n) \hookrightarrow W(G,T)$$

that coincides (check!) with the natural inclusion of S_n into $\Sigma(n) = (\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$.

It remains to show that each of the standard direct factors $\mathbf{Z}/2\mathbf{Z}$ of $(\mathbf{Z}/2\mathbf{Z})^n$ lies in the image of $W(G,T)$ inside $\Sigma(n)$. This is a problem inside each

$$\mathrm{SU}(2) = \mathrm{Sp}(1) \subset \mathrm{GL}(\mathbf{C}e_j \oplus \mathbf{C}e_{j+n})$$

for $1 \leq j \leq n$, using its 1-dimensional diagonal maximal torus that is one of the standard direct factors S^1 of $T = (S^1)^n$. But we already know $W(\mathrm{SU}(2), S^1) = \mathbf{Z}/2\mathbf{Z}$, with non-trivial class represented by the unit quaternion $j \in \mathrm{SU}(2) \subset \mathbf{H}^\times$ whose conjugation action normalizes the unit circle $S^1 \subset \mathbf{C}^\times$ via inversion, so we are done. ■