

1. MOTIVATION

Let $\{\Gamma_i\}$ be an inverse system of finite groups with surjective transition maps, and define $\Gamma = \varprojlim \Gamma_i$ equipped with its “inverse limit” topology (i.e., the closed subspace topology inside the compact Hausdorff product space $\prod \Gamma_i$ in which the finite factors Γ_i are discrete). The natural maps $\Gamma \rightarrow \Gamma_i$ are all surjective (why?), and by definition of the topology we see that the kernel $U_i = \ker(\Gamma \rightarrow \Gamma_i)$ is an *open* normal subgroup with these U_i a base of open neighborhoods of 1.

Such Γ are called *profinite*, the most important examples of interest being $\mathbf{Z}_p = \varprojlim \mathbf{Z}/(p^n)$ and especially Galois groups $\text{Gal}(K/k)$ with the Krull topology, where K/k is an arbitrary Galois extension (perhaps of infinite degree). In this latter case, the finite groups Γ_i can be taken to be the Galois groups $\text{Gal}(K_i/k)$ for the directed system $\{K_i\}$ of k -finite Galois subextensions of K/k (with the Galois groups made into an inverse system via “restriction”). We are most interested in the case of absolute Galois groups $\Gamma = \text{Gal}(k_s/k)$ for a field k , but it clarifies matters below to contemplate a general profinite Γ (equipped with a choice of inverse system presentation via some $\{\Gamma_i\}$).

In class we introduced the notion of a *discrete* Γ -module, namely a Γ -module M such that each $m \in M$ has open stabilizer in Γ . (If Γ is finite then it has the discrete topology and *every* Γ -module is discrete. The concept is therefore only of interest when Γ is infinite.) In earlier homeworks we have seen the interest in this concept for the purposes of Galois descent when $\Gamma = \text{Gal}(K/k)$ is a Galois group and M is a K -vector space equipped with a Γ -action that is semilinear over the natural Γ -action on K . We have also seen several other interesting examples of discrete Γ -modules in this Galois group setting.

Remark 1.1. The “discrete module” terminology is justified by the fact that this condition on a Γ -module is equivalent to the property that the action map $\Gamma \times M \rightarrow M$ is continuous when M is equipped with the discrete topology (and Γ is given its inverse limit topology). We will never use this fact, but the interested reader may enjoy checking it as an instructive exercise.

Remark 1.2. The surjectivity of the transition maps $\Gamma_j \rightarrow \Gamma_i$ for $j \geq i$ can be dropped without affecting the definition of a profinite group. Indeed, given a general inverse system $\{\Gamma_i\}$ whose transition maps are not assumed to be surjective, inside each Γ_i consider the inverse system $\Gamma_{j,i} \subset \Gamma_i$ of images of maps $\Gamma_j \rightarrow \Gamma_i$ for $j \geq i$. This is an inverse system of subsets of a finite set, ordered by reverse inclusion, so by finiteness of the ambient set Γ_i we see that for some $j(i) \geq i$, $\Gamma_{j,i} = \Gamma_{j(i),i}$ for all $j \geq j(i)$; i.e., $\Gamma_{j,i}$ stabilizes for $j \gg i$.

Letting $\Gamma'_i := \Gamma_{j(i),i} \subset \Gamma_i$, for $j \gg i$ we have that $\Gamma_j \rightarrow \Gamma_i$ has image Γ'_i . But for each $j \geq i$ if $j' \gg j$ (large enough to ensure $j' \gg i!$) then $\Gamma_{j'} \rightarrow \Gamma_j$ has image Γ'_j and $\Gamma_{j'} \rightarrow \Gamma_i$ has image Γ'_i , so $\Gamma'_j \rightarrow \Gamma'_i$ is *surjective* whenever $j \geq i$ (not only for $j \gg i!$). Consider the map $\varprojlim \Gamma'_i \rightarrow \varprojlim \Gamma_i$ between compact Hausdorff spaces. This is visibly continuous and bijective, hence a homeomorphism. Thus, by replacing $\{\Gamma_i\}$ with $\{\Gamma'_i\}$ we get the same topological group as at the outset, but with the gain on the inverse system of finite groups that its transition maps are surjective.

In this handout we explain a variant of group cohomology that is adapted to the setting of profinite groups Γ and discrete Γ -modules. Loosely speaking, this cohomology theory is a “limit” of ordinary group cohomologies for the finite quotients Γ_i of Γ , but building a theory base on such an approach is a bit messy and also looks ad hoc (though historically it is what was originally done, in the days before Grothendieck). We will take another approach, based on the principle of derived functors, and will *prove* as a theorem that our initial abstract definition does recover the “limit of finite group cohomologies” viewpoint.

The merit of basing everything on derived functors is twofold: conceptual elegance (i.e., nothing will look ad hoc), and the immediate availability of the usual long exact sequence formalism. Nonetheless, it is important to make contact with reality by proving that our abstract definitions can be computed using suitable ordinary group cohomologies at finite level. This underlies the proof of Hilbert 90 for infinite-degree extensions as given in class, and in general it emphasizes the dichotomy which arises in many cohomology theories: abstract definitions are best-suited to getting a theory off the ground with good functorial properties, but we need concrete descriptions in order to actually compute anything (e.g., proving vanishing theorems for some higher cohomologies).

2. DISCRETIZATION

Our first order of business is to show that the abelian category $\text{Mod}_{\text{disc}}(\Gamma)$ of discrete Γ -modules has enough injectives, so it is amenable to the theory of (right) derived functors. This will be shown by using a trick to bootstrap from the case of modules over ordinary groups (without topology).

Definition 2.1. Let M be an abstract Γ -module (i.e., no discreteness condition). The *discretization* M^{disc} of M is the subset of elements $m \in M$ such that the stabilizer $\text{Stab}_{\Gamma}(m)$ is open in Γ (equivalently, one of the open normal subgroups $U_i = \ker(\Gamma \rightarrow \Gamma_i)$ acts trivially on m).

Pick $m, m' \in M^{\text{disc}}$, so their respective stabilizers $H = \text{Stab}_{\Gamma}(m)$ and $H' = \text{Stab}_{\Gamma}(m')$ are open. Hence, $m \pm m'$ is fixed by the open subgroup $H \cap H'$, so its stabilizer subgroup in Γ is open. That is, $m \pm m' \in M^{\text{disc}}$. Likewise, if $\gamma \in \Gamma$ then $\gamma \cdot m$ has Γ -stabilizer $\gamma \text{Stab}_{\Gamma}(m) \gamma^{-1}$ which is open. We conclude that M^{disc} is a Γ -submodule of M , and it is a discrete Γ -module by its very definition. Our interest in the discretization is due to:

Lemma 2.2. *For any discrete Γ -module N , $\text{Hom}_{\Gamma}(N, M) = \text{Hom}_{\Gamma}(N, M^{\text{disc}})$. That is, every Γ -equivariant map $f : N \rightarrow M$ lands inside M^{disc} .*

Proof. Pick an f , so for $n \in N$ we seek to prove that $f(n) \in M^{\text{disc}}$. For $\gamma \in \Gamma$ we have $\gamma \cdot f(n) = f(\gamma \cdot n)$, and by the discreteness of N we have $\gamma \cdot n = n$ for γ in an open subgroup $H \subset \Gamma$. Thus, $f(n) \in M^H \subset M^{\text{disc}}$. ■

The usefulness of discretization is that it provides enough injectives in $\text{Mod}_{\text{disc}}(\Gamma)$. Indeed, for a discrete Γ -module M we can forget the topology and just view M as a $\mathbf{Z}[\Gamma]$ -module, so by general nonsense there is a Γ -linear injection $M \hookrightarrow J$ into an injective $\mathbf{Z}[\Gamma]$ -module J . But M is *discrete*, so this injection factors through J^{disc} . To show that $\text{Mod}_{\text{disc}}(\Gamma)$ has enough injectives it is therefore enough to prove:

Proposition 2.3. *If J is an injective $\mathbf{Z}[\Gamma]$ -module then J^{disc} is injective in $\text{Mod}_{\text{disc}}(\Gamma)$. That is, the functor $\text{Hom}_{\Gamma}(\cdot, J^{\text{disc}})$ on the category $\text{Mod}_{\text{disc}}(\Gamma)$ is exact.*

Proof. By the preceding lemma, if M is a discrete Γ -module then naturally in M we have

$$\text{Hom}_{\Gamma}(M, J^{\text{disc}}) = \text{Hom}_{\Gamma}(M, J).$$

In other words, the functor of interest is the composition of the exact forgetful functor $\text{Mod}_{\text{disc}}(\Gamma) \rightarrow \text{Mod}(\mathbf{Z}[\Gamma])$ and the functor $\text{Hom}_{\mathbf{Z}[\Gamma]}(\cdot, J)$ on $\text{Mod}(\mathbf{Z}[\Gamma])$ that is exact due to the assumed injectivity property of J . ■

It now makes sense to apply the general theory of derived functors:

Definition 2.4. The δ -functor $H^{\bullet}(\Gamma, \cdot) : \text{Mod}_{\text{disc}}(\Gamma) \rightarrow \text{Ab}$ is the right derived functor of $M \rightsquigarrow M^{\Gamma}$.

Strictly speaking, there could be notational confusion, since we are using the same notation as for ordinary group cohomology. But in practice one *never* works with the ordinary group cohomology of an infinite profinite group, since it is not interesting. Hence, the context will always make the intended meaning clear. Moreover, there really is difference:

Example 2.5. Let M be a Γ -module with trivial Γ -action, so it is certainly discrete. If we view Γ “discretely” and consider ordinary group cohomology, say denoted $H_{\text{disc}}^{\bullet}(\Gamma, M)$ to avoid confusion, then $H_{\text{disc}}^1(\Gamma, M) = \text{Hom}(\Gamma, M)$ is the abelian group of abstract group homomorphisms from Γ to M . However, in Example 3.5 we will see that when using profinite group cohomology as defined above, $H^1(\Gamma, M) = \text{Hom}_{\text{cont}}(\Gamma, M)$ is the abelian group of *continuous* group homomorphisms $\Gamma \rightarrow M$ when M is viewed discretely and Γ is equipped with its inverse limit topology. This continuity condition is exactly the property that the kernel is open, and so contains some U_i , which is to say that these are the homomorphisms which factor through one of the quotient maps $\Gamma \twoheadrightarrow \Gamma_i$.

To illustrate the effect of imposing the continuity condition on homomorphisms, consider the case $M = \mathbf{F}_p$ and $\Gamma = \prod_{i=1}^{\infty} \mathbf{F}_p$ with the product topology (and trivial action on M). A nontrivial continuous homomorphism $\Gamma \rightarrow M$ corresponds to an *open* subgroup of Γ of index p , of which there are just countably many. In contrast, a nontrivial group homomorphism $\Gamma \rightarrow M$ ignoring topologies is merely a codimension-1 \mathbf{F}_2 -subspace of Γ , of which there are uncountably many.

The importance of remembering the topology on Γ for the purposes of cohomological applications is on par with its importance in the formulation of infinite Galois theory.

3. RELATION WITH COHOMOLOGY OF FINITE GROUPS

The definition of $H^{\bullet}(\Gamma, M)$ is extremely abstract, and so to compute it we seek a more hands-on description. The approach we will use, inspired by the bar resolution, is based on the concept of *continuous cochains*; these are functions $f : \Gamma^j \rightarrow M$ continuous relative to the discrete topology on M and the natural compact Hausdorff topology on each Γ^j .

[The method of continuous cochains is sometimes used to *define* of profinite group cohomology. However, this can seem ad hoc, much like using the bar resolution method as the *definition* of ordinary group cohomology, though the bar resolution is actually what was done

first historically, before the advent of the unifying idea of derived functors that subsumed all other cohomology theories in topology and algebra.]

Lemma 3.1. *A map of sets $f : \Gamma^j \rightarrow M$ into a discrete Γ -module is a continuous cochain if and only if it factors as*

$$\Gamma^j \rightarrow (\Gamma_i)^j \rightarrow M^{U_i} \rightarrow M$$

for some Γ_i .

Note that M^{U_i} is a module for the group $\Gamma/U_i = \Gamma_i$.

Proof. The sufficiency is clear, so for necessity we assume f is continuous. The image of f is compact in the discrete topological space M , so it is finite. By the discreteness of the Γ -module M , this finite image lies in $M^{U_{i_0}}$ for sufficiently small U_{i_0} . For each of the finitely many m in the image of f and each $x \in f^{-1}(m)$, some open neighborhood $x(U_{i_x})^j$ is contained in $f^{-1}(m)$. As we vary x , the open sets $x(U_{i_x})^j$ cover the compact space $f^{-1}(m)$, so finitely many of them also cover this fiber. That is, for fixed m in the image of f , the non-empty fiber $f^{-1}(m)$ is a union of finitely many sets of the form $x_r(U_{i_r})^j$ ($1 \leq r \leq n$) for some indices i_1, \dots, i_n . Thus, any $U_i \subset \cap U_{i_r}$ has the property that right multiplication by $(U_i)^j$ on Γ^j preserves $f^{-1}(m)$.

For each $m \in f(\Gamma^j)$, pick an index $i(m)$ so that right multiplication by $(U_{i(m)})^j$ preserves $f^{-1}(m)$. Taking U_i small enough so that it lies in $U_{i(m)}$ for all such finitely many m , $f : \Gamma^j \rightarrow M$ is invariant under right multiplication by $(U_i)^j$. That is, f factors through the quotient map $\Gamma^j \rightarrow (\Gamma/U_i)^j = (\Gamma_i)^j$. By shrinking U_i further if necessary, we can arrange that the finite set $f(\Gamma^j)$ also lies inside M^{U_i} . ■

For a discrete Γ -module M , let $C^j(\Gamma, M)$ denote the abelian group of *continuous j -cochains*. For each i we may view M^{U_i} as a module over the finite group $\Gamma/U_i = \Gamma_i$ (as well as a discrete Γ -module, since U_i is normal in Γ), and there is a natural “inflation” map

$$\alpha_{j,i} : C^j(\Gamma_i, M^{U_i}) \rightarrow C^j(\Gamma, M^{U_i}) \rightarrow C^j(\Gamma, M)$$

sending any function $f : (\Gamma_i)^j \rightarrow M^{U_i}$ to its composition with the inclusion $M^{U_i} \hookrightarrow M$ and the projection $\Gamma^j \twoheadrightarrow (\Gamma_i)^j$.

When $U_{i'} \subset U_i$ (so $\Gamma_{i'}$ dominates Γ_i as quotients of Γ), there is likewise a natural “inflation” map

$$C^j(\Gamma_i, M^{U_i}) \rightarrow C^j(\Gamma_{i'}, M^{U_i}) \rightarrow C^j(\Gamma_{i'}, M^{U_{i'}})$$

and it is straightforward to check that the maps $\alpha_{j,i}$ respect this change in i so as to define a natural map

$$\alpha_j : \varinjlim C^j(\Gamma_i, M^{U_i}) \rightarrow C^j(\Gamma, M)$$

for every $j \geq 0$. The content of Lemma 3.1 is exactly that α_j is an isomorphism for each $j \geq 0$. The explicit formulas which compute the differentials in the cochain complex for ordinary group cohomology are taken as the *definition* of the differentials

$$d_{j,M} : C^j(\Gamma, M) \rightarrow C^{j+1}(\Gamma, M)$$

for continuous cochains. (It is easy to check – do it – that $d_{j,M}$ preserves continuity of cochains; this uses that M is discrete as a Γ -module.) This really is a complex: $d_{j+1,M} \circ d_{j,M} =$

0. To see this, we simply observe that the isomorphisms α_j respect the differentials at finite level in the sense that

$$d_{j,M} \circ \alpha_j = \alpha_{j+1} \circ \varinjlim d_{j,M^{U_i}},$$

so the vanishing of d^2 on continuous cochains for Γ valued in M is a consequence of the same for ordinary cochains for the finite groups Γ_i (valued in the modules M^{U_i}).

It now makes sense to consider the homologies of the continuous cochain complex $C^\bullet(\Gamma, M)$. Since homology commutes with direct limits, we get natural isomorphisms

$$(1) \quad \varinjlim H^j(\Gamma_i, M^{U_i}) \simeq H^j(\varinjlim C^\bullet(\Gamma_i, M^{U_i})) \simeq H^j(C^\bullet(\Gamma, M))$$

for each $j \geq 0$. Note that the left side is defined concretely in terms of “inflation” of cochain functions relative to the surjections $(\Gamma_{i'})^j \twoheadrightarrow (\Gamma_i)^j$ and inclusions $M^{U_i} \hookrightarrow M^{U_{i'}}$, and this *does* compute the cohomological inflation δ -functor (as proved in Remark 2.2 in the handout on annihilation of cohomology) so there is no confusion as to the meaning of the limit process on the left side of (1).

Example 3.2. For $j = 0$, we have $H^0(C^\bullet(\Gamma, M)) = M^\Gamma$ because the differential $d_{0,M} : C^0(\Gamma, M) \rightarrow C^1(\Gamma, M)$ is the map $m \mapsto (\gamma \mapsto \gamma m - m)$.

The key point is that the continuous cochain complex really does compute profinite group cohomology. More specifically, we first get a δ -functor structure:

Lemma 3.3. *If $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$ is a short exact sequence of discrete Γ -modules then the induced map of complexes of continuous cochains*

$$0 \rightarrow C^\bullet(\Gamma, M') \rightarrow C^\bullet(\Gamma, M) \rightarrow C^\bullet(\Gamma, M'') \rightarrow 0$$

is short exact in each degree.

Proof. The exactness at the left and middle terms is elementary, and the problem is surjectivity at the third term: $C^j(\Gamma, M) \rightarrow C^j(\Gamma, M'')$ is surjective for each j . Now comes the subtle problem: by Lemma 3.1, this map is passage to the limit over i on the maps $C^j(\Gamma_i, M^{U_i}) \rightarrow C^j(\Gamma_i, M''^{U_i})$, but for any specific i the map $M^{U_i} \rightarrow M''^{U_i}$ may not be surjective!

To circumvent this problem, we pick a specific continuous cochain $f'' \in C^j(\Gamma, M'')$ that we wish to lift to $C^j(\Gamma, M)$. The image of f'' lies in a *finite* subset of M'' . Each of these finitely many elements of M'' lift to elements of M , and if we choose such lifts then they constitute a finite subset of M and hence lie in M^{U_i} for some common U_i . That is, the image of f'' lies in the image of M^{U_i} in M''^{U_i} . In other words, if we form the short exact sequence of Γ_i -modules

$$0 \rightarrow M^{U_i} \rightarrow M^{U_i} \rightarrow Q_i \rightarrow 0$$

where $Q_i \subset M''^{U_i}$ is the image of M^{U_i} then $f'' \in C^j(\Gamma_i, Q_i)$. This lifts to some $f \in C^j(\Gamma_i, M^{U_i})$ by the theory of cochains without topologies, so the image of f under

$$C^j(\Gamma_i, M^{U_i}) \hookrightarrow C^j(\Gamma, M)$$

is a lift of $f'' \in C^j(\Gamma, M'')$ as desired. ■

Using the snake lemma, we conclude that the sequence of functors

$$M \rightsquigarrow H^n(C^\bullet(\Gamma, M))$$

on $\text{Mod}_{\text{disc}}(\Gamma)$ has a natural structure of δ -functor with 0th term $H^0(C^\bullet(\Gamma, M)) = M^\Gamma$. For example, the connecting map

$$\delta : M^{\prime\prime\Gamma} \rightarrow H^1(\Gamma, M)$$

carries $m'' \in M^{\prime\prime\Gamma}$ to the cohomology class of the *continuous* 1-cocycle $\gamma \mapsto \gamma m - m$ where $m \in M$ is an arbitrary lift of m'' ; this is the same formula as in ordinary group cohomology (the point being that this 1-cocycle does automatically satisfy the continuity condition to lie in $C^1(\Gamma, M')$). The link between our δ -functor construction with continuous cochains and the derived functor of Γ -invariants rests on:

Proposition 3.4. *For an injective discrete Γ -module J , the complex $C^\bullet(\Gamma, J)$ is exact in positive degrees. In particular, the δ -functor $M \rightsquigarrow (H^n(C^\bullet(\Gamma, M)))_{n \geq 0}$ is erasable.*

Proof. The j th homology of this complex has been seen to be identified with the direct limit of the finite group cohomologies $H^j(\Gamma_i, J^{U_i})$, so it suffices to show that the latter vanish for all $j > 0$. More specifically, it suffices to show that for any open normal subgroup $H \subset \Gamma$, J^H is an injective Γ/H -module. That is, we want the functor $\text{Hom}_{\Gamma/H}(\cdot, J^H)$ on the category of $\mathbf{Z}[\Gamma/H]$ -modules to be exact. But for any Γ/H -module M , any Γ -equivariant map $M \rightarrow J$ must land inside J^H since the source has trivial H -action when viewed as a Γ -module. Hence,

$$\text{Hom}_{\Gamma/H}(M, J^H) = \text{Hom}_\Gamma(M, J),$$

and the right side is the composition of the exact functor $\text{Hom}_\Gamma(\cdot, J)$ on $\text{Mod}_{\text{disc}}(\Gamma)$ and the exact forgetful functor from the category of Γ/H -modules into the category $\text{Mod}_{\text{disc}}(\Gamma)$ of *discrete* Γ -modules. (This forgetful functor is valued in the category of discrete Γ -modules because H is open in Γ .) Being a composition of exact functors, it is exact. ■

We conclude by the universality of erasable δ -functors that there is a *unique* isomorphism of δ -functors

$$H^n(\Gamma, M) \simeq H^n(C^\bullet(\Gamma, M))$$

on $\text{Mod}_{\text{disc}}(\Gamma)$ extending the identification of the 0th term on both sides with the functor $M \rightsquigarrow M^\Gamma$. That is, the continuous cochain complex computes profinite group cohomology as a δ -functor.

Example 3.5. Consider M with trivial Γ -action, so (as without topologies) $B^1(\Gamma, M) = 0$ and $Z^1(\Gamma, M) = \text{Hom}_{\text{cont}}(\Gamma, M)$. That is, naturally $H^1(\Gamma, M) \simeq \text{Hom}_{\text{cont}}(\Gamma, M)$.

By (1), for each $j \geq 0$ we obtain natural isomorphisms

$$\varinjlim H^j(\Gamma_i, M^{U_i}) \rightarrow H^j(\Gamma, M)$$

for discrete Γ -modules M . This is essentially a proof that profinite group cohomology is “computed” as a limit of finite group cohomologies, except for an important technical point: we built ρ_j by bare hands for each j separately via the continuous cochain complex, whereas in class we introduced such maps by entirely conceptual means via the composition of maps

$$\rho_j : H^j(\Gamma_i, M^{U_i}) \rightarrow H^j(\Gamma, M^{U_i}) \rightarrow H^j(\Gamma, M)$$

(the first being an instance of the inflation map $H^j(\Gamma_i, \cdot) \rightarrow H^j(\Gamma, \cdot)$) defined via universal δ -functoriality on the category of Γ_i -modules).

It behooves us to check that we have really built the *same* map in two different ways (so that our cochain map construction can also be understood in conceptual terms). For the purpose of proving *vanishing* theorems for specific M and j (e.g., Hilbert 90 for Galois extensions of possibly infinite degree) such agreement of maps is irrelevant: there's at most one isomorphism to the zero group! But as a matter of general principles (to avoid ambiguity), in the next section we prove this agreement of maps.

4. COMPATIBILITY OF MAPS

We have constructed two maps

$$(2) \quad \varinjlim H^j(\Gamma_i, M^{U_i}) \rightrightarrows H^j(\Gamma, M)$$

for each $j \geq 0$ and each discrete Γ -module M : one via the identification of both sides with j th homology of the continuous cochain complex, and another via limits of inflation maps. This section proves the equality of both maps.

The method of comparison is to show that the left side of (2), like the right side, admits a natural structure of erasable δ -functor such that *both* sequences of maps in question for varying j respect the δ -functor structures on the source and target (as we vary M). Once this is proved, the two construction must agree since they clearly coincide in degree 0 (namely, the natural isomorphism $\varinjlim M^{U_i} \simeq M$). The left side of (2) vanishes for $j > 0$ when M is an injective discrete Γ -module (since M^{U_i} is then an injective Γ_i -module for all i , as shown in the proof of Proposition 3.4), so erasability must hold once we have a δ -functor structure.

The identification of the derived functor $H^\bullet(\Gamma, M)$ with the homology of the continuous cochain complex was *defined* as an isomorphism of δ -functors, so we are motivated to directly define a δ -functor structure on the left side of (2) which is compatible with the snake lemma δ -functor structure on the homology of the continuous cochain complex. The problem will then be to prove that this δ -functor structure on the left side of (2) makes the other definition of (2), via cohomological inflation maps, a morphism of δ -functors.

Consider a short exact sequence of discrete Γ -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

The main problem in making the left side of (2) a δ -functor as j varies is that the induced sequence of U_i -invariants

$$0 \rightarrow M'^{U_i} \rightarrow M^{U_i} \rightarrow M''^{U_i}$$

is generally only left exact, not surjective on the right. We will circumvent this by exploiting the idea that this failure “vanishes in the limit” over shrinking U_i 's. To be precise, let $Q_i \subset M''^{U_i}$ denote the image of M^{U_i} , so we have compatible short exact sequences

$$0 \rightarrow M'^{U_i} \rightarrow M^{U_i} \rightarrow Q_i \rightarrow 0$$

for all i . Thus, we get a long exact sequence in Γ_i -cohomologies

$$\cdots \rightarrow H^j(\Gamma_i, M'^{U_i}) \rightarrow H^j(\Gamma_i, M^{U_i}) \rightarrow H^j(\Gamma_i, Q_i) \xrightarrow{\delta} \cdots$$

By *definition*, the inflation maps in ordinary group cohomology are δ -functorial, so passage to the direct limit over i makes sense and yields (by exactness of direct limits) a long exact sequence

$$\cdots \rightarrow \varinjlim T_i^j(M'^{U_i}) \rightarrow \varinjlim T_i^j(M^{U_i}) \rightarrow \varinjlim T_i^j(Q_i) \xrightarrow{\delta} \cdots$$

where $T_i^j = H^j(\Gamma_i, \cdot)$ and the inflation map $H^\bullet(\Gamma_i, N) \rightarrow H^\bullet(\Gamma_{i'}, N)$ relative to $\Gamma_{i'} \twoheadrightarrow \Gamma_i$ for Γ_i -modules N (viewed as $\Gamma_{i'}$ -modules in the evident manner) is defined by δ -functorial extension of the evident equality $N^{\Gamma_i} = N^{\Gamma_{i'}}$ in degree 0.

Since $\varinjlim Q_i = M''$ (as any element of M'' lifts into $M = \varinjlim M^{U_i}$ and hence lies in the image Q_i of M^{U_i} for sufficiently small U_i), an elementary argument chasing cocycles and coboundaries (left to the reader) verifies both surjectivity and injectivity of the natural map

$$\varinjlim H^j(\Gamma_i, Q_i) \rightarrow \varinjlim H^j(\Gamma_i, M''^{U_i})$$

for each $j \geq 0$. Thus, we obtain a δ -functor structure on the left side of (2). (This really is a δ -functor; check the functoriality in the given short exact sequence!) A review of our construction of the δ -functor structure on the homologies of the continuous cochain complex shows that it is compatible with the δ -functor structure we have just built on the left side of (2) (in which the direct limit is taken with respect to cohomological inflation) since inflation maps in ordinary group cohomology are computed at the level of cochain complexes by inflation of functions.

The δ -functorial cohomological inflation $H^\bullet(\Gamma_i, \cdot) \rightarrow H^\bullet(\Gamma, \cdot)$ is also computed by inflation, using *continuous* cochains for Γ_i and Γ , and the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'^{U_i} & \longrightarrow & M^{U_i} & \longrightarrow & Q_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

may be viewed as one of Γ -modules (so the induced long exact sequences of profinite Γ -cohomology for the top and bottom yield a commutative long rectangle diagram with their respective connecting maps). Thus, we just need to check that for each $j \geq 0$ the composite map

$$\varinjlim H^j(\Gamma_i, Q_i) \simeq \varinjlim H^j(\Gamma_i, M''^{U_i}) \simeq H^j(\Gamma, M'')$$

coincides with the composite map

$$\varinjlim H^j(\Gamma_i, Q_i) \rightarrow \varinjlim H^j(\Gamma, Q_i) \rightarrow \varinjlim H^j(\Gamma, M'').$$

But the implicit cohomological inflations in both composites can be computed at the level of (continuous) j -cochains via composing functions with $\Gamma^j \twoheadrightarrow (\Gamma_i)^j$, so everything comes down to a commutative diagram for the coefficient modules in these cohomologies: the diagram

$$\begin{array}{ccc} Q_i & \longrightarrow & M''^{U_i} \\ & \searrow & \downarrow \\ & & M'' \end{array}$$

commutes, by the *definition* of the map across the top side.