Math 210B. Artin–Rees and completions

1. Definitions and an example

Let A be a ring, I an ideal, and M an A-module. In class we defined the *I-adic completion* of M to be

$$\widehat{M} = \varprojlim M/I^n M.$$

We will soon show that $\widehat{M}$ is naturally identified with the A-module of equivalence classes of Cauchy sequences $\{x_n\}$ in M relative to the I-adic topology, where $\{x_n\} \sim \{y_n\}$ when $x_n - y_n \to 0$. The importance of this alternative description of $\widehat{M}$ is that it expresses the completion in terms which involve only the underlying topology on M and not the specific ideal I (as many ideals J can define the same topology). To establish the description of the completion in terms of Cauchy sequences, we first define an A-linear map from the A-module of equivalence classes of Cauchy sequences to $\widehat{M}$. Then we prove this map is an isomorphism.

Consider a Cauchy sequence $x = \{x_n\}$ in M relative to the I-adic topology. For any $r > 0$, the elements x_n for n sufficiently large (depending on r) all differ from each other by elements of $I^r M$. That is, the sequence $\{x_n \bmod I^r M\}$ becomes constant for large n (depending on r); this property for all r is just a reformulation of the Cauchy condition. Define $m_r \in M/I^r M$ to be this terminal value. It is clear that $m_{r+1} \bmod I^r M = m_r$. Hence, the sequence (m_r) is an element of $\varprojlim M/I^r M = \widehat{M}$ depending only on the initial Cauchy sequence x. We denote this element of $\widehat{M}$ as $\widehat{x}$. It is easy to see that $\widehat{x}$ only depends on the equivalence class of the Cauchy sequence x with which we began, and that moreover it depends A-linearly on x. It remains to show:

Proposition 1.1. *The A-linear map $x \mapsto \widehat{x}$ from the A-module of equivalence classes of Cauchy sequence into the A-module $\widehat{M}$ is an isomorphism.*

Proof. For surjectivity, pick an element $(m_r) \in \varprojlim M/I^r M$. Choose a representative $x_r \in M$ of m_r for all r. The sequence $x = \{x_r\}$ in M is visibly I-adically Cauchy (since $x_n \bmod I^r M = m_r$ for all $n \geq r$), and by the definition of $\widehat{x}$ we see that $\widehat{x} = (m_r)$. For injectivity, by additivity of $x \mapsto \widehat{x}$ we just have to check that if $\widehat{x} = 0$ then $x \sim 0$. The vanishing of $\widehat{x}$ says that for each r and sufficiently large n (depending on r) we have $x_n \in I^r M$, so $x_n \to 0$ in M relative to the I-adic topology. This says exactly that $x \sim 0$, as desired. ∎

In class we saw explicitly that for a field k, $A = k[x_1, \dots, x_n]_{(x_1,\dots,x)n)}$ equipped with its max-adic topology has completion identified with the local ring $k[\![x_1, \dots, x_n]\!]$ (with its own max-adic topology). Here are more computations of completions.

Example 1.2. For any ring R, the ring $R[x]$ equipped with the x-adic topology has completion identified with the $R[x]$-algebra $R[\![x]\!]$ of formal power series over R in the variable x (as one might naturally guess). Indeed, an element of this completion is simply a sequence of polynomials $f_n(x) \in R[x]$ with $\deg(f_n) < n$ such that $f_{n+1} \bmod x^n = f_n$, which is to say $f_n = r_0 + r_1 x + \cdots + r_{n-1} x^{n-1}$ for $r_j \in R$ that have nothing to do with n. Thus, $f = \sum r_j x^j \in R[\![x]\!]$ is a well-defined element, and is uniquely determined by the condition $f \bmod x^n = f_n$ under the evident isomorphism $R[x]/(x^n) \simeq R[\![x]\!]/(x^n)$ (both sides finite free over R with basis $\{1, x, \ldots, x^{n-1}\}$). The map $(f_n) \mapsto f$ defines an $R[x]$-algebra map $R[x]^\wedge \to R[\![x]\!]$, and by construction it is clearly injective and surjective.

As a special case, for a ring k (not necessarily a field) consider $A = k[x, y]$ with the x-adic topology. Writing A as $k[y][x]$, taking $R = k[y]$ above yields the description $\widehat{A} = k[y][\![x]\!]$. We want to describe this in another way, by using the natural identification $k[\![x, y]\!] = (k[\![x]\!])[\![y]\!]$, so we first review how this latter identification is defined in a way that requires no messy computations to justify compatibility with ring structures. (The method carries over to any finite set of variables.) Since $k[\![x, y]\!]$ is (x, y)-adically separated and complete, it is also y-adically separated and complete (why?). Thus, the evident injection of $k[x, y]$-algebras

$$f_0 : k[\![x]\!][y] \to k[\![x, y]\!]$$

(defined by the evident k-algebra inclusion $k[\![x]\!] \hookrightarrow k[\![x, y]\!]$ and the $k[y]$-algebra structure on the target) factors uniquely through a $k[x, y]$-algebra map from the y-adic completion of the source (just like the universal property for completions of metric spaces!):

$$f : k[\![x]\!][\![y]\!] \to k[\![x, y]\!].$$

To prove f is an isomorphism, we inspect what f is: we claim it carries $\sum_n (\sum_m a_{nm} x^m) y^n$ to $\sum_{n,m} a_{nm} x^n y^m$ ($a_{nm} \in k$). The abstract f is y-adically continuous, and the proposed formula is y-adically continuous (why?), so to prove the correctness of the proposed formula it suffices to compare on a y-adically dense subset. The subset $k[\![x]\!][y]$ is y-adically dense in $k[\![x]\!][\![y]\!]$, so we are reduced to computing f on $k[\![x]\!][y]$, which is to say (by definition of f in terms of f_0) that we have to show that f_0 is given by the proposed formula with y^n appearing for only finitely many n. But the proposed formula is visibly k-linear and compatible with multiplication by y, so it is $k[y]$-linear. Hence, we are reduced to checking that it correctly computes f_0 on $k[\![x]\!]$, which in turn is obvious from the definition of f_0.

Now using the composite $k[x, y]$-algebra isomorphism

$$k[\![y]\!][\![x]\!] = k[\![x, y]\!] = k[\![x]\!][\![y]\!]$$

(which amounts to a big rearrangement of infinite series), we wish to describe the image of $\widehat{A} = k[y][\![x]\!]$ in $k[\![x]\!][\![y]\!]$. We claim that it goes over to the set $k[\![x]\!]\{y\}$ of "restricted" formal power series $\sum h_n(x) y^n$: formal power series in y with coefficients $h_n \in k[\![x]\!]$ that x-adically tend to 0 as $n \to \infty$. (For example, $\sum_n x^n y^n \in k[\![x]\!]\{y\}$ but $\sum y^n \notin k[\![x]\!]\{y\}$ inside $k[\![x, y]\!]$.) To see this, we first observe that the elements $\sum_{n,m} a_{nm} x^n y^m \in k[\![x, y]\!]$ which lie in $k[y][\![x]\!]$ are precisely the ones that, when written in the form $\sum_n (\sum_m a_{nm} y^m) x^n$ have each $\sum_m a_{nm} y^m \in k[y]$ (inside $k[\![y]\!]$). That is, for each n we have $a_{nm} = 0$ for sufficiently large m with largeness perhaps depending on n. Now we consider the other rearrangement $\sum_m (\sum_n a_{nm} x^n) y^m = \sum_m h_m(x) y^m$. Each fixed power x^n does not appear in $h_m(x)$ for m sufficiently large (depending on n). Equivalently, by taking m large enough for the values $n \leq N$, this says that if m is sufficiently large (depending on N) then h_m only involves monomials x^n with $n > N$. In other words, the condition is exactly that $h_m \to 0$ in $k[\![x]\!]$ relative to the x-adic topology.

2. The Artin–Rees Lemma

In complete generality, if M is a module over a ring A then for an ideal I of A the I-adic completion $\widehat{M}$ has a natural topology: for $x = (x_n) \in \varprojlim M/I^nM \subset \prod_{n\geq 0} M/I^nM$, a base of open neighborhoods of x is given by those sequences $x' = (x'_n)$ such that $x'_n = x_n$ for all $n \leq N$ (with growing N). This is the subspace topology from the product topologies of the discrete spaces M/I^nM, so it is Hausdorff. More explicitly, if $x \neq x'$ then $x_{n_0} \neq x'_{n_0}$ for some n_0, so the open neighborhoods of x and x' given by equality with the first n_0 components are disjoint.

Exactly as for completions of metric spaces, essentially by construction $\widehat{M}$ is *complete* for this topology: all Cauchy sequences in $\widehat{M}$ converge. Indeed, by definition of the topology we see that a sequence in $\widehat{M}$ is Cauchy if and only if the induced sequences of nth components in each M/I^nM are eventually constant, with this resulting set of terminal constants $x_n \in M/I^nM$ providing an element $x = (x_n) \in \widehat{M}$ that is a limit of the given Cauchy sequence. However, what is *not* at all obvious (and in fact can fail in some cases) is whether or not this topology on $\widehat{M}$ coincides with the I-adic topology. More specifically, by definition there are natural surjective A-linear "forgetful" maps $\widehat{M} \to M/I^nM$ for all n, inducing surjective maps

$$\widehat{M}/I^n\widehat{M} \to M/I^nM,$$

and it is not clear if these maps are isomorphisms. In particular, although $\widehat{M}$ is rigged to be complete for its "inverse limit" topology, it's not clear if the completion is *complete* ... for the I-adic topology on $\widehat{M}$!

The key to good properties of completion in the noetherian setting is the following fundamental result.

Theorem 2.1 (Artin–Rees Lemma)**.** *Let A be a noetherian ring, I an ideal, and M a finitely generated A-module. For any A-submodule $M' \subset M$, there exists $n_0 > 0$ such that for all $n \geq n_0$,*

$$M' \bigcap I^nM = I^{n-n_0}(M' \bigcap I^{n_0}M).$$

Proof. The problem is to find $c > 0$ so that $M' \cap I^nM \subset I^{n-c}(M' \cap I^cM)$ for all $n > c$, as the reverse inclusion is clear. It is equivalent to say that $M' \cap I^{n+1}M = I(M' \cap I^nM)$ for all $n \geq c$. That is, we claim that the decreasing chain of submodules $M'_n = M' \cap I^nM$ which satisfies $IM'_n \subset M'_{n+1}$ for all n necessarily satisfies $IM'_n = M'_{n+1}$ for all large n. The property of such equality for all large n is called I*-stable* in §5 of Chapter X of Lang's "Algebra". We refer the reader to that section of Lang's book (see in particular Theorem 5.4) for an elegant self-contained proof of the Artin–Rees lemma by using the Hilbert basis theorem and modules over the auxiliary ring $A \oplus I \oplus I^2 \oplus I^3 \oplus \ldots$ (using the ring structure in which the direct summands I^n and I^m have their product under multiplication assigned to live in the direct summand I^{n+m}). ■

The Artin–Rees lemma clearly implies that the I-adic topology on M induces the I-adic topology on M' as the subspace topology, though in general $M' \cap I^nM$ is strictly larger than I^nM'. Loosely speaking, the Artin–Rees Lemma expresses a precise sense in which the discrepancy between $M' \cap I^n M$ and $I^n M'$ is "uniformly bounded" as n grows. In the next section we will use this fact to deduce many nice properties of completion for finitely generated modules over noetherian rings.

Here is a very important consequence of the Artin–Rees Lemma that is analogous to the determination of an analytic function by its power series expansion:

Corollary 2.2 (Krull intersection theorem: local case)**.** *If $(A, \mathfrak{m})$ is a local noetherian ring and M is a finitely generated A-module equipped with the $\mathfrak{m}$-adic topology then $M \to \widehat{M}$ is injective.*

Proof. The kernel $M' = \ker(M \to \widehat{M})$ is visibly $\bigcap_{n \geq 0} \mathfrak{m}^n M$. This is a mysterious A-submodule of M, but at least it is *finitely generated* since A is noetherian. But the Artin–Rees Lemma gives some $n_0 > 0$ such that

$$\mathfrak{m}^{n-n_0}(M' \cap \mathfrak{m}^{n_0} M) = M' \cap \mathfrak{m}^n M = M'$$

for all $n \geq n_0$, and the left side is $\mathfrak{m}^{n-n_0} M'$. Taking $n = n_0 + 1$, we get $M' = \mathfrak{m} M'$, so $M' = 0$ by Nakayama's Lemma! ■

The Krull intersection theorem beyond the local case is a bit more involved. In general, for noetherian A with an ideal I and finitely generated A-module M, often $1 + I$ contains some $a \in A$ that is a non-unit in the non-local case. Such an a is a unit in $\widehat{A}$ (why?), so if M is a nonzero $A/(a)$-module then $\widehat{M}$ is a module over $\widehat{A}/(a) = 0$, forcing $\widehat{M} = 0$. The same argument shows that in general the kernel of $M \to \widehat{M}$ contains the $(1 + I)$-torsion in M. The general Krull intersection theorem says that this is always the exact kernel of the map to the completion (for noetherian A and finitely generated M); in the local case this recovers the above injectivity result.

3. Properties of noetherian completion

Let A be a noetherian ring, and I an ideal of A. Consider an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of finitely generated A-modules. Since the submodules $\{M' \cap I^n M\}$ of M' define the I-adic topology (by the Artin–Rees lemma), the natural map

$$\widehat{M'} = \varprojlim M'/I^n M' \to \varprojlim M'/(M' \cap I^n M)$$

is an isomorphism. (This is most easily visualized by thinking in terms of equivalence classes of Cauchy sequences). But the target is visibly the kernel of the natural map $\widehat{M} \to \widehat{M''}$, and in HW11 it is shown that this latter map is surjective. Hence, we get an induced short exact sequence

$$0 \to \widehat{M'} \to \widehat{M} \to \widehat{M''} \to 0,$$

so I-adic completion is an *exact* functor on finitely generated A-modules. Moreover, HW11 shows that the image of $\widehat{M'}$ in $\widehat{M}$ is the *closure* of the image of M' in $\widehat{M}$ for the "inverse limit" topology on $\widehat{M}$ (which we will see below is the same as its I-adic topology). Thus, $\widehat{M'}$ is a *closed* submodule of $\widehat{M}$ relative to the "inverse limit" topology on $\widehat{M}$.

The key to relating this inverse limit topology to the I-adic topology on $\widehat{M}$, and more generally unlocking the mysteries of the completion process, is the fact that $\widehat{M}$ can be directly constructed from M and the completion $\widehat{A}$. To make this precise, first note that by definition $\widehat{M}$ is naturally an $\widehat{A}$-module and the natural map $M \to \widehat{M}$ is linear over the natural map $A \to \widehat{A}$. This underlies:

Proposition 3.1. *The natural $\widehat{A}$-linear map $\phi_M : \widehat{A} \otimes_A M \to \widehat{M}$ is an isomorphism.*

If one tried to "cheat" by taking the left side as the *definition* of the right side (for noetherian A and A-finite M) then one would immediately get stuck on the fact that it would not be clear if this has good exactness properties in M (whereas we have seen above, partially by HW11, that completion – correctly defined! – is an exact functor for finitely generated modules over a noetherian ring).

Proof. Both sides are *right exact* in M, and it is clear that ϕ_M is compatible with direct sums in M and is an isomorphism for $M = A$. This permits the following rather formal argument (which shows in general that a natural transformation between right exact additive functors on finitely generated A-modules is a natural isomorphism when it is so on $M = A$.)

The compatibility with direct sums and the isomorphism property for $M = A$ implies that $\phi_{A^{\oplus n}}$ is an isomorphism for any $n \geq 0$. Now choose a right exact sequence

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0,$$

as we may do since A is *noetherian* (ensuring that any finite generating set $m_1, \dots, m_n$ of M has finitely generated A-module of "relations" in $A^{\oplus n}$). Thus, we get a commutative diagram of right exact sequences

$$\begin{array}{ccccccc} \widehat{A} \otimes_A A^{\oplus m} & \longrightarrow & \widehat{A} \otimes_A A^{\oplus n} & \longrightarrow & \widehat{A} \otimes_A M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \phi_{A^{\oplus m}}} & & \downarrow{\scriptstyle \phi_{A^{\oplus n}}} & & \downarrow{\scriptstyle \phi_M} & & \\ (A^{\oplus m})^\wedge & \longrightarrow & (A^{\oplus n})^\wedge & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

in which the left and middle vertical arrows are isomorphisms. The induced map ϕ_M between the cokernels is therefore an isomorphism too. ■

Corollary 3.2. *For any noetherian ring A and ideal I of A, the completion $\widehat{A}$ is a flat A-algebra and $\widehat{J} = \widehat{A} \otimes_A J \simeq J\widehat{A}$ via multiplication for any ideal J of A. In particular, for ideals $J, J' \subset A$ we have $\widehat{JJ'} = \widehat{J}\widehat{J'}$ inside $\widehat{A}$.*

Proof. The isomorphism $\widehat{A} \otimes_A M \simeq \widehat{M}$ for finitely generated A-modules implies that the functor $\widehat{A} \otimes_A (\cdot)$ is *exact* on finitely generated A-modules. In general, if an A-module N makes $N \otimes_A (\cdot)$ an exact functor on finitely generated A-modules then N is A-flat. To see this, for any finitely generated A-module M, pick a generating set and consider the resulting exact sequence

$$0 \to M' \to A^{\oplus n} \to M \to 0.$$

Note that M' is finitely generated since A is noetherian. Tensoring against N over A then yields an exact sequence

$$\mathrm{Tor}^1_A(N, A^{\oplus n}) \to \mathrm{Tor}^1_A(N, M) \to N \otimes_A M' \to N \otimes_A A^{\oplus n} \to N \otimes_A M \to 0,$$

but the last 3 terms define a short exact sequence by hypothesis. Thus, the map between Tor-terms is surjective. The first Tor-term vanishes, so we conclude that $\mathrm{Tor}^1_A(N, M) = 0$ for all finitely generated A-modules M. By general nonsense with universal δ-functors, $\mathrm{Tor}^j_A(N, \cdot)$ is naturally compatible with direct limits for any j (since it is true for $j = 0$), so by expressing any A-module M as the direct limit of its finitely generated A-submodules we conclude that $\mathrm{Tor}^1_A(N, M) = 0$ for *every* A-module M. Thus, N is A-flat, as desired.

Now we address the assertions concerning the behavior of I-adic completion on ideals J in A. By flatness of $\widehat{A}$ over A, the inclusion $J \hookrightarrow A$ induces an injection $\widehat{A} \otimes_A J \to \widehat{A}$. But this map is visibly "multiplication" on elementary tensors (using $J \to A \to \widehat{A}$), so the image of this injective map is the ideal $J\widehat{A} \subset \widehat{A}$. This establishes the first assertion concerning completion. The second assertion holds because $(J\widehat{A})(J'\widehat{A}) = JJ'\widehat{A}$ as ideals of $\widehat{A}$. ■

The importance of the flatness in the preceding corollary cannot be overestimated, especially when A is local and I is the maximal ideal. This flatness property pervades many applications of completion in algebraic geometry, commutative algebra, and number theory, as it underlies techniques of Grothendieck which permit one to transfer properties between A and $\widehat{A}$. (An especially striking example is when A is the local ring at a point x on a "smooth" affine variety Z over $\mathbf{C}$ and A' is the local ring at the "same" point on the associated complex-analytic manifold Z^{an}. By deep results of Oka, A' is noetherian, so the equality $\widehat{A} = \widehat{A}'$ implies that any results which transfer properties between a local *noetherian* ring and its completion permit transferring properties between the algebraic local ring A and the analytic local ring A', thereby linking local algebraic and analytic geometry over $\mathbf{C}$.)

Corollary 3.3. *For any finitely generated A-module M, $\widehat{M}$ is a finitely generated $\widehat{A}$-module and the natural map*

$$q_n : \widehat{M}/I^n\widehat{M} \to M/I^nM$$

is an isomorphism. In particular, the inverse limit topology on $\widehat{M}$ coincides with its adic topology relative to the ideal $\widehat{I} = I\widehat{A}$, so $\widehat{M}$ is $\widehat{I}$-adically separated and complete and $\widehat{A}$ is separated and complete for the $I\widehat{A}$-adic topology.

Proof. The map $j : M \to \widehat{M}$ induces a map $j_n : M/I^nM \to \widehat{M}/I^n\widehat{M}$ whose composition with q_n is the identity map of M/I^nM (check!). Thus, to show that q_n is an isomorphism it suffices to prove that j_n is an isomorphism. By the exactness of completion on finitely generated modules, we have naturally

$$\widehat{M}/I^n\widehat{M} \simeq (M/I^nM)^\wedge,$$

and the composition of this with j_n is the natural map

$$M/I^nM \to (M/I^nM)^\wedge.$$

This is just the map from M/I^nM to its I-adic completion, and so is an isomorphism since the I-adic topology on M/I^nM is *discrete* (hence separated and complete). ■

Example 3.4. Let $(A, \mathfrak{m})$ be a local noetherian ring. We saw in class that the $\mathfrak{m}$-adic completion $\widehat{A}$ is local, and $\widehat{A}/\widehat{\mathfrak{m}} = (A/\mathfrak{m})^\wedge = A/\mathfrak{m}$ is a field. That is, the ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$ must be the unique maximal ideal of the local ring $\widehat{A}$, so $\widehat{A}$ is max-adically separated and complete. We will soon see that $\widehat{A}$ is also *noetherian*, so it is a "complete local noetherian ring" (meaning complete for its own max-adic topology). There are examples of non-noetherian local $(A, \mathfrak{m})$ such that the local $\mathfrak{m}$-adic completion $\widehat{A}$ has maximal ideal strictly larger than $\mathfrak{m}\widehat{A}$ and which is *not* separated and complete for the topology defined by its own maximal ideal. In general one has to be very careful when working with completions beyond the noetherian setting.

Finally, we address the important problem of preservation of the noetherian condition under the completion process. As a special case, if R is noetherian then so is $R[x_1, \dots, x_n]$ (Hilbert basis theorem), but how about its $(x_1, \dots, x_n)$-adic completion $R[\![x_1, \dots, x_n]\!]$?

Theorem 3.5 (Formal Hilbert basis theorem)**.** *For any noetherian ring R, $R[\![x_1, \dots, x_n]\!]$ is noetherian.*

Proof. In Example 1.2 we saw that there is a natural ring isomorphism

$$R[\![x_1, \dots, x_n]\!] \simeq (R[\![x_1, \dots, x_{n-1}]\!])[\![x_n]\!].$$

(This was explained in the case $n = 2$, but the method was sufficiently general that it works the same way for any $n > 1$, as the reader may check.) Hence, by induction (as in the proof of the usual Hilbert basis theorem) it suffices to treat the case $n = 1$. That is, we seek to show that $R[\![x]\!]$ is noetherian when R is noetherian. The proof of this case is modeled on the case of $R[x]$, except using induction on least-degree terms instead of top-degree terms. See Theorem 9.4 in Chapter IV of Lang's "Algebra" for the details. ∎

There is a nifty isomorphism that allows us to bootstrap the formal Hilbert basis theorem into a proof of noetherianity for completions of noetherian rings in general:

Proposition 3.6. *If $\{a_1, \dots, a_n\}$ is a generating set of A then*

$$\widehat{A} \simeq A[\![X_1, \dots, X_n]\!]/(X_1 - a_1, \dots, X_n - a_n)$$

as A-algebras. In particular, $\widehat{A}$ is noetherian.

Moreover, every finitely generated $\widehat{A}$-module is separated and complete for the $I\widehat{A}$-adic topology, and every submodule of a finitely generated $\widehat{A}$-module is closed for this topology.

Proof. Let $B = A[x_1, \dots, x_n]$ with the $(x_1, \dots, x_n)$-adic topology, and define the A-algebra map $q : B \to A$ by $x_i \mapsto a_i$. Let $J = (x_1, \dots, x_n) \subset B$. Viewing A as a B-module in this way, the I-adic topology on A coincides with its J-adic topology as a finitely generated B-module, so we can compute $\widehat{A}$ with the B-module viewpoint using the J-adic topology. Note that B is noetherian, by the usual Hilbert basis theorem.

Consider the right exact sequence

$$B^{\oplus n} \to B \xrightarrow{q} A \to 0$$

of finitely generated B-modules, where the left map is $(b_i) \mapsto \sum (X_i - a_i) b_i$. Passing to J-adic completions then yields a right-exact sequence

$$\widehat{B}^{\oplus} \to \widehat{B} \to \widehat{A} \to 0$$

where the first map is $(b'_i) \mapsto \sum (X_i - a_i)b'_i$ and the second map is the natural map $\widehat{q}$ induced between completions. In particular, extension by continuity from the dense image of B in $\widehat{B} = A[\![x_1, \dots, x_n]\!]$ shows that $\widehat{q}$ is a ring homomorphism, and its kernel is the ideal generated by the elements $X_i - a_i$. This establishes the asserted description of $\widehat{A}$.

To prove the assertions concerning finitely generated $\widehat{A}$-modules, now we can replace (A, I) with $(\widehat{A}, I\widehat{A})$ without losing the noetherian condition (!), and this bring us to the case that A is itself I-adically separated and complete (i.e., the natural map $A \to \widehat{A}$ is an isomorphism). Hence, for any finitely generated A-module M we have naturally $\widehat{A} \otimes_A M \simeq \widehat{M}$ yet $A \to \widehat{A}$ is an isomorphism, so we conclude that the natural map $M \to \widehat{M}$ is an isomorphism. That is, M is I-adically separated and complete.

Finally, continuing to assume $A = \widehat{A}$, if $j : M' \hookrightarrow M$ is an inclusion between finitely generated A-modules, to show that M' is closed in M for the I-adic topology we recall that in general the image of $\widehat{M'}$ in $\widehat{M}$ under $\widehat{j}$ is the closure of the image of M' in $\widehat{M}$ for the inverse limit topology on $\widehat{M}$ (by HW11). The natural identifications $M' = \widehat{M'}$ and $M = \widehat{M}$ identify $\widehat{j}$ with j, and we have also seen that the inverse limit topology on $\widehat{M}$ is its adic topology for the ideal $I\widehat{A} = I$, so M' is its own closure in M for the I-adic topology. ∎

Here is an interesting "formal" variant of Nakayama's Lemma.

Corollary 3.7. *Let A be a noetherian ring that is separated and complete for the topology defined by an ideal I. An A-linear map $f : M' \to M$ between finitely generated A-modules is surjective if $f \bmod I$ is surjective.*

In particular, a set of elements $m_1, \dots, m_n$ of M is a spanning set if and only if it is an spanning set for M/IM, and if $M/IM = 0$ then $M = 0$.

Proof. For the cokernel $M'' = \operatorname{coker} f$ we have $M''/IM'' = 0$, and we want to show that $M'' = 0$. More generally, for any finitely generated A-module M we claim that if $M = IM$ then $M = 0$. Note that $M = I^n M$ for all $n > 0$, by induction. Thus, $M/I^n M = 0$ for all $n > 0$, so $\widehat{M} = 0$. But $M \to \widehat{M}$ is an isomorphism due to the completeness hypothesis on A, so $M = 0$. ∎

4. Discrete valuation rings

Let R be a discrete valuation ring, and $t \in R$ a uniformizer (i.e., generator of the unique maximal ideal $\mathfrak{m}$ of R). Whereas the completion of a local noetherian domain can generally fail to be a domain (as we have seen at singularities on plane curves), for discrete valuation rings this problem does not occur:

Proposition 4.1. *The completion $\widehat{R}$ is a discrete valuation ring with t as a uniformizer.*

Proof. The element t is not a zero-divisor in $\widehat{R}$. Indeed, suppose $x = (x_n) \in \varprojlim R/(t^n)$ and $tx = 0$, so $tx_n = 0$ in $R/(t^n)$ for all $n > 0$. Then $x_n \in t^{n-1}R/t^nR$ since a representative $x'_n \in R$ of x_n becomes divisible by t^n after multiplication by t and hence must be divisible by t^{n-1} because R is a discrete valuation with unifomizer t. But with $x_n \in (t^{n-1})/(t^n)$ for all $n > 0$, the compatibility of the x_n's under the reduction maps yields $x_n = x_{n+1} \bmod t^n = 0$ for all $n > 0$, so $x = 0$ as desired.

We shall next prove that every nonzero element of $\widehat{R}$ has the form ut^n with $u \in \widehat{R}^\times$ and $n \geq 0$. Once this is proved, it follows that $\widehat{R}$ is a domain (as t is not a zero divisor in $\widehat{R}$, so a product of elements of the form ut^n and $u't^{n'}$ with $u, u' \in \widehat{R}^\times$ and $n, n' \geq 0$ is nonzero). Thus, it would even be a PID in which every element is a unit multiple of a power of the element t that is a non-unit (since t is killed under the natural map $\widehat{R} \to R/(t)$), and so (t) must be a maximal ideal and in fact the only one. Being a local PID, it must then be a discrete valuation ring with uniformizer t as claimed.

To verify the asserted description of nonzero elements ξ of $\widehat{R}$, writing $(\xi_n) \in \varprojlim R/(t^n)$ for ξ, since $\xi \neq 0$ some $\xi_{n_0} \neq 0$ we have $\xi_{n_0} = t^e u_{n_0} \bmod t^{n_0}$ for some $e < n_0$ and $u_{n_0} \in R^\times$. Thus, for alll $n > n_0$ we have $\xi_n \bmod t^{n_0} = \xi_{n_0} = t^e u \bmod t^{n_0}$ is nonzero. A representative $\xi'_n \in R$ for $\xi_n \in R/(t^n)$ is a power of t times a unit, and modulo t^{n_0} this becomes $t^e u$, so ξ_n cannot be divisible by any power of t higher than the eth power (or else that would be detected modulo t^{n_0} with $n_0 > e$. Writing $\xi_n = t^{e_n} u_n$ with $0 \leq e_n \leq e$ and $u_n \in R^\times$, we have $t^{e_n} u_n \equiv t^e u_{n_0} \bmod t^{n_0}$ with $n_0 > e \geq e_n$. Dividing throughout by t^e then gives $t^{e_n - e} \equiv u_{n_0}/u_n \bmod t^{n_0 - e}$ with $n_0 - e > 0$, $e_n - e \geq 0$, and $u_{n_0}/u_n \in R^\times$. But a unit is nonzero modulo the maximal ideal tR, so this forces $e_n - e = 0$; i.e., $\xi_n = t^e u_n \bmod t^n$ for some $u_n \in R^\times$ and a common integer $e \geq 0$ for all $n \geq n_0$.

The congruence $\xi'_{n+1} \equiv \xi'_n \bmod t^n$ for all $n \geq n_0$ says $t^e u_{n+1} \equiv t^e u_n \bmod t^n$ for all $n \geq n_0$, so $u_{n+1} \equiv u_n \bmod t^{n-e}$ for all $n \geq n_0 > e$. Thus, if we define $v_m := u_{m+n_0} \bmod t^m$ for $m > 0$ then $v_{m+1} \equiv v_m \bmod t^m$ for all $m > 0$. This says that $v := (v_m) \in \varprojlim R/(t^m) = \widehat{R}$ and this element is a unit (as each v_m is a unit), with $\xi_m = \xi_{m+n_0} \bmod t^m = t^e u_{n_0+m} \bmod t^m = t^e v_m$ in $R/(t^m)$ for all $m > 0$. In other words, $\xi = t^e v$ for $v \in \widehat{R}^\times$, as desired. ■