Extensions and $H^2(G, A)$

To say that a group E is an extension of group G by group A means there is an exact sequence as in the top row of the diagram below. Two extensions are equivalent if there is an isomorphism ϕ so that the diagram commutes.

$$\begin{array}{ccccccccc} 1 & \to & A & \to & E & \to & G & \to & 1 \\ & & \| & & \downarrow \phi & & \| & & \\ 1 & \to & A & \to & E' & \to & G & \to & 1 \end{array}$$

Assume A is abelian. An extension produces a G-module structure on A, by ${}^x a = \hat{x} a \hat{x}^{-1}$, where $\hat{x} \in E$ is any lifting of $x \in G$. The conjugation is well-defined because A is abelian and the only other $\hat{x}$'s would be $b\hat{x}$ with $b \in A$. Equivalent extensions induce the same actions, since a map ϕ defining an equivalence is assumed to satisfy $\phi|_A = Id$, and it is assumed ϕ induces the identity on G.

(When A is not abelian, all you get out of this is a homomorphism $G \to Out(A)$, the outer automorphism group of A, rather than a G-action $G \to Aut(A)$).

Even with the action or outer action of G on A fixed, classifying extensions up to equivalence is rather far from classifying all such E up to isomorphism of groups. An isomorphism could discombobulate the A's and may not give rise to any map at all between exact sequences. Nonetheless, classifying extensions is considered important.

THEOREM: Equivalence classes of extensions of G by an abelian group A, inducing a fixed G-module structure on A, are in bijective correspondence with elements of the cohomology group $H^2(G, A)$.

We will liesurely prove this theorem in the next couple of pages.

Choose a set $\{f(x) \in E \mid x \in G\}$ of inverse images of all elements of G. All elements of E can be uniquely written $af(x)$, for some $a \in A, x \in G$. One could allow $f(1) \neq 1 \in A$, but the formulas below then get a bit ugly. Therefore, by a set of representatives we will always mean such an f with $f(1) = 1$. Note ${}^x b = f(x)bf(x)^{-1}$ doesn't depend on the choice of f. Note f is a group homomorphism splitting of the extension sequence if and only if $f(x)f(y)f(xy)^{-1} = 1$ for all $x, y \in G$.

Let's compute general products in E,

$$(af(x))(bf(y)) = a({}^x b)(f(x)f(y)f(xy)^{-1})f(xy).$$

The function $u_f(x, y) = f(x)f(y)f(xy)^{-1} \in A$ looks important. The product formula shows that the group structure on E is determined by the G-module structure on A and the function u_f, since if we identify E setwise with $A \times G$ via $af(x) \leftrightarrow (a, x)$, we can write

$$(a, x)(b, y) \leftrightarrow (af(x))(bf(y)) = a({}^x b)u_f(x, y)f(xy) \leftrightarrow (a({}^x b)u_f(x, y), xy).$$

Moreover, if $\phi : E \to E'$ is an isomorphism of extensions, then $\{\phi f(x)\}$ is a set of representatives for G in E'. Since $\phi|_A = Id$, it follows that $u_f = u_{\phi f}$.

Recall that a function $u : G \times G \to A$ determines a cocycle in $Z^2(G, A)$, for the free standard $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$, if one has

$$ud_3(x, y, z) = xu(y, z) - u(xy, z) + u(x, yz) - u(x, y) = 0 \in A.$$

Except we are writing A as an abelian multiplicative group here. It is much more convenient to write the cocycle condition as

$${}^x u(y, z)u(x, yz) = u(x, y)u(xy, z).$$

Exercise 1. Given a group G, a G-module structure $v : G \to Aut(A)$ on an abelian group A, and a function $u : G \times G \to A$, define a product on the *set* $A \times G$ by $(a, x)(b, y) = (a({}^x b)u(x, y), xy)$. Show that u is a cocycle in $Z^2(G, A)$ if and only if this product is associative. If $u(x, y) = 1$ for all $x, y \in G$, then we just have the semidirect product $A \times_v G$.

Exercise 2. If u is any cocycle, check that ${}^x u(1, 1) = u(x, 1)$ and $u(1, y) = u(1, 1)$, for all $x, y \in G$. Show that if u is a cocycle with $u(1, 1) = 1$, then the product of Exercise 1 has $(1, 1)$ as a 2-sided identity. Find a formula for inverses $(a, x)^{-1}$ in this product structure. Thus the product structure becomes a group, $E_u = A \times_{v,u} G$, with an obvious projection homomorphism $E_u \to G$. The kernel is isomorphic to group A by the inclusion $i(a) = (a, 1) \in A \times_{v,u} G$.

Exercise 3. Assuming the cocycle u satisfies $u(1, 1) = 1$, show that the following relations hold in E_u:

$$(1, x)(1, y)(1, xy)^{-1} = (u(x, y), 1) \text{ and } (1, x)(a, 1)(1, x)^{-1} = ({}^x a, 1).$$

[Hint: Do *not* use the formula for inverses. Since you know E_u is a group, you can multiply the first equation by $(1, xy)$ and the second equation by $(1, x)$, and *then* verify.]

At this point, if we *have* an extension $1 \to A \to E \to G \to 1$ and a set of representatives $\{f(x)\} \subset E$, then since E is associative we have a cocycle $u_f(x, y) = f(x)f(y)f(xy)^{-1} \in A$. Clearly $u_f(1, 1) = 1$, since $f(1) = 1$. Because of the explicit product formulas, we see that the extensions E and $E_{u_f} = A \times_{v,u_f} G$ are equivalent, via the isomorphism $\phi(af(x)) = (a, x)$. Note $\phi(a) = (a, 1) = i(a) \in E_u$. The set of representatives $\{f(x)\} \subset E$ maps to the set of representatives $\{(1, x)\} \subset E_{u_f}$.

We have worked with (certain) 2-cocycles, now we bring in (certain) 2-coboundaries. Beginning with an extension E and a set of representatives $\{f(x)\}$, what are other possible sets of representatives? Obviously just sets $\{b(x)f(x)\}$, where $b : G \to A$ is any function with $b(1) = 1$. In the standard resolution, b extends to a G-map $F_1 \to A$, with coboundary $\delta(b) : F_2 \to A$ determined by

$$\delta(b)(x, y) = bd(x, y) = xb(y) - b(xy) + b(x),$$

in additive notation. Multiplicatively this can be written

$$\delta(b)(x, y) = {}^x b(y)b(x)b(xy)^{-1}.$$

Note that $b(1) = 1$ implies $\delta(b)(1, 1) = b(1) = 1$.

Exercise 4. Show that $u_{bf}(x,y) = \delta(b)u_f(x,y)$. [Hint: $f(x)b(y) = {}^{x}b(y)f(x)$.]

(Note that an immediate consequence of Exercise 4 is that two sets of representatives f and bf determine the same cocyle, $u_f = u_{bf}$, if and only if $b \in Z^1(G,A)$, the 1-cocycles, which is exactly the condition that $b(x) = (b(x), x)$ defines a group homomorphism section of the semidirect product $A \times_v G \to G$. But this is peripheral to our present discussion.)

Exercise 4 is easy, but has serious consequences. Beginning with an extension E, choose a set of representatives f and construct the cocycle u_f. Exercise 4 shows that the cohomology class $[u_f] \in H^2(G,A)$ is independent of the choice of set of representatives. So we can call this class $[u(E)] \in H^2(G,A)$. We also observed above that if $\phi : E \to E'$ is an isomorphism of extensions then $u_f = u_{\phi f}$, hence $[u(E)] = [u(E')]$. So we have a well-defined map from equivalence classes of extensions (inducing a given G-module structure on A) to $H^2(G,A)$.

We want to prove this correspondence is bijective. But here we must pay a little price for our decision to only consider cocycles u with $u(1,1) = 1$ and coboundaries δb with $b(1) = 1$. Here, the arguments $(1,1)$ and (1) refer to basis elements in F_2 and F_1 of the standard resolution, and $1 \in A$ is the identity.

We have $H^2(G,A) = \frac{Z^2(G,A)}{B^2(G,A)}$, cocycles mod coboundaries. We now go back to *additive* notation in a general abelian G-module A. Let $Z^2_0(G,A) \subset Z^2(G,A)$ denote those cocycles with $u(1,1) = 0$ and let $B^2_0(G,A) \subset B^2(G,A)$ denote coboundaries δb, where $b(1) = 0$. We will prove the following claim after finishing the proof of the THEOREM.

CLAIM: $\frac{Z^2_0(G,A)}{B^2_0(G,A)} \to \frac{Z^2(G,A)}{B^2(G,A)}$ is an isomorphism.

In Exercises 1 and 2, beginning with any cocycle u with $u(1,1) = 1$, we constructed an extension $E_u = A \times_{v,u} G$. In Exercise 3 it was verified that E_u induces the given G-module structure v on A, and also that $u_f = u$ for the natural factor set $\{f(x)\} = \{(1,x)\} \subset E_u$. Thus, by the claim, the map from equivalence classes of extensions to $H^2(G,A)$ is surjective. But we also conclude the map is injective. If extension E yields cocycle u with one choice of representatives f, then E and E_u are equivalent. But replacing f by bf, where $b(1) = 1$, doesn't change E, and now we see by Exercise 4 that E is equivalent to $E_{\delta(b)u}$. In other words, again by the claim, the association $u \mapsto E_u$ determines a well-defined map from $H^2(G,A)$ to equivalence classes of extensions, inverse to the association $E \mapsto [u(E)]$. This proves the THEOREM.

PROOF OF CLAIM: Given any cocycle $v \in Z^2$ and a 1-cochain c, note

$$(v + \delta c)(1,1) = v(1,1) + cd(1,1) = v(1,1) + c(1),$$

since $d(1,1) = (1)$ in the standard resolution. But $c(1)$ is arbitrary, so we can change any cocycle by a coboundary and get $v + \delta c \in Z^2_0$. This proves the map in the claim is surjective. But also, if $u \in Z^2_0$ and if $u = \delta b$ then $0 = u(1,1) = \delta b(1,1) = bd(1,1) = b(1)$, which proves $b \in B^2_0$, hence the map in the claim is injective.