Math 145. Homework 8

1. If X and Y are C^∞ manifolds, show that a continuous map $f : X \to Y$ between the underlying topological spaces is a C^∞ map if for all open $U \subset Y$ and $\varphi \in C^\infty(U)$ the pullback function $\varphi \circ f : f^{-1}(U) \to \mathbf{R}$ lies in $C^\infty(f^{-1}(U))$.

2. (gluing spaces) Let $\{X_i\}$ be a set of topological spaces. Suppose we are given *open* sets $X_{ij} \subseteq X_i$ for all i, j with $X_{ii} = X_i$ and homeomorphisms $f_{ij} : X_{ij} \simeq X_{ji}$ for all i, j such that f_{ii} is the identity and

$$f_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}, \;\; f_{jk}|_{X_{jk} \cap X_{ji}} \circ f_{ij}|_{X_{ij} \cap X_{ik}} = f_{ik}|_{X_{ik} \cap X_{ij}}$$

for all i, j, k. We want to glue the X_i's along the f_{ij}'s (converting X_{ij}, X_{ji} into $X_i \cap X_j$). Draw pictures to illustrate your arguments below.

(i) Define $X = (\coprod X_i)/\sim$, where $x_i \sim f_{ij}(x_i)$ for $x_i \in X_{ij}$. Show that the natural maps $f_i : X_i \to X$ are injective, with $f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ f_{ij}$, $f_i(X_{ij}) = f_j(X_{ji}) = f_i(X_i) \cap f_j(X_j)$. Thus, we may view X_i as a subset of X and as such $X_{ij} = X_{ji}$ is identified with the overlap $X_i \cap X_j$ inside of X (the identification being f_{ij}).

(ii) We define $U \subseteq X$ to be *open* if and only if $f_i^{-1}(U)$ is open in X_i for all i. Show that this makes X a topological space with respect to which $f_i(X_i)$ is open and f_i is a homemorphism of X_i onto $f_i(X_i)$. Thus, we may view X_i as an open subset of X. We call $(X, \{f_i\})$ the *gluing of the X_i's along the f_{ij}'s.* Prove the universal property that for any topological space Y and continuous maps $g_i : X_i \to Y$ that "agree on overlaps" in the sense that $g_i|_{X_{ij}} = g_j|_{X_{ji}} \circ f_{ij}$ for all i, j, there is a unique continuous map $g : X \to Y$ such that $g \circ f_i = g_i$ for all i.

3. (gluing sheaves) Let X be a topological space with an open covering $\{X_i\}$ and let $\mathscr{O}_i$ be a sheaf of F-valued functions on X_i for a field F. We define $X_{i_1 \dots i_n} = X_{i_1} \cap \cdots \cap X_{i_n}$. Suppose that for all i, j we have $\mathscr{O}_i(U) = \mathscr{O}_j(U)$ for all open $U \subset X_{ij}$. For every open $U \subseteq X$, define

$$\mathscr{O}(U) = \{(s_i) \in \prod \mathscr{O}_i(U \cap X_i) \,|\, s_i|_{U \cap X_{ij}} = s_j|_{U \cap X_{ij}} \text{ for all } i, j\}.$$

(i) Prove that $\mathscr{O}$ is of local nature. We call it the *gluing* of the $\mathscr{O}_i$'s.

(ii) For any ringed space $(X, \mathscr{O})$ and open subset $V \subset X$, we define the sheaf of functions $\mathscr{O}|_V$ on V by the rule $U \mapsto \mathscr{O}(U)$ for open $U \subset V$. For a field F, formulate what it means to glue a set of ringed spaces $\{(X_i, \mathscr{O}_i)\}$ over F along isomorphisms akin to Exercise 2 between open subspaces $X_{ij} \subset X_i$ (equipped with $\mathscr{O}_i|_{X_{ij}}$). and prove the corresponding universal property within the framework of ringed spaces over F. In particular, show that if $(X, \mathscr{O}_X)$ is a ringed space over F and $\{U_i\}$ is an open covering of X, then to give a map from $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is "the same" as to give maps $f_i : (U_i, \mathscr{O}_X|_{U_i}) \to (Y, \mathscr{O}_Y)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j.

4. For any finitely generated reduced k-algebra A, let $\mathrm{MaxSpec}(A) = (\mathrm{Max}(A), \mathscr{O}_A)$. We define *affine n-space* over k to be $\mathbf{A}^n_k = \mathrm{MaxSpec}(k[t_1, \dots, t_n])$ (k^n equipped with the "sheaf of regular functions" on varying Zariski-open subsets of k^n).

(i) Show that $\mathscr{O}_{\mathbf{A}^1_k}(\mathbf{A}^1_k - \{0\}) = k[x, 1/x]$ and that if $n \geq 2$ then the natural restriction map

$$k[t_1, \dots, t_n] = \mathscr{O}_{\mathbf{A}^n_k}(\mathbf{A}^n_k) \to \mathscr{O}_{\mathbf{A}^n_k}(\mathbf{A}^n_k - \{0\})$$

is an *isomorphism.* (Hint: for $X = \mathbf{A}^n_k$, $X - \{0\}$ is covered by the non-vanishing loci $X_{t_1}, \dots, X_{t_n}$.)

(ii) Using (ii), show that if $n \geq 2$ then $\mathbf{A}^n_k - \{0\}$ as a ringed space over k is *not* isomorphic to $\mathrm{MaxSpec}(A)$ for any finitely generated reduced k-algebra A. (Hint: if it were, show via (i) that pullback along the natural inclusion $j : \mathbf{A}^n_k - \{0\} \to \mathbf{A}^n_k$ would have to induce an isomorphism $k[t_1, \dots, t_n] \simeq A$. Deduce a contradiction since j is *not* an isomorphism.) This is the precise sense in which "$\mathbf{A}^n_k - \{0\}$ is not affine" when $n \geq 2$.