# Math 145. Homework 6

Do 2.24, 2.26, 2.28–2.32 in the book. As usual, below $k$ denotes an algebraically closed field.

1. Let $R$ be a ring, $I$ an ideal all of whose elements are nilpotent.

(*i*) If $r \in R$ maps into $(R/I)^{\times}$, prove $r \in R^{\times}$. (Do not invoke the existence of maximal ideals in a non-zero ring.) Make this explicit for $R = k[X]/X^3$, $r = -1 + X$, $I = (X)$.

(*ii*) An element $e \in R$ is called *idempotent* if $e^2 = e$. Using $e$ and $1 - e$, show that specifying an idempotent is 'the same' as specifying an ordered decomposition $R \simeq R_1 \times R_2$ for rings $R_1$, $R_2$. Show that for every idempotent $\overline{e} \in R/I$, there is a unique idempotent $e \in R$ with $e \bmod I = \overline{e}$. Find all of the idempotents in $R = k[Y] \times k[Y] \times k[Y] \simeq k[X,Y]/X(X-1)(X-\lambda)$ where $\lambda \in k - \{0,1\}$ and determine the associated decomposition of $R$ as an ordered product in each case. Draw pictures.

(*iii*) If $Z$ is an affine algebraic set, interpret (with proof!) idempotents in $k[Z]$ in terms of connected components of $Z$. Use this to show that $k[Z]$ has only finitely many idempotents.

(*iv*) If $A$ is a finite $k$-algebra, possibly non-reduced, show that there is an isomorphism $A \simeq A_1 \times \cdots \times A_n$ where the $A_i$ are local finite $k$-algebras with nilpotent maximal ideal, corresponding to the finitely many maximal ideals of $A$. Make this explicit for $A = k[X]/(f)$ for a non-constant monic polynomial $f$.

2. Suppose $k$ does not have characteristic 2 and let $f \in k[X]$ be non-constant. Prove that $Y^2 - f(X) = 0$ defines an irreducible *smooth* curve in $k^2$ if and only if $f$ has distinct roots.

3. Let $Z \subset k^n$ be an irreducible closed set. This exercise explores tangent spaces.

(i) Let $\mathfrak{m}_z \subset k[Z]$ be the maximal ideal associated to $z \in Z$, and $M_z$ the maximal ideal of $\mathscr{O}_{Z,z} = k[Z]_{\mathfrak{m}_z}$. Show that the natural map $\mathfrak{m}_z/\mathfrak{m}_z^2 \to M_z/M_z^2$ is an isomorphism (so we can compute tangent spaces either "globally" using $k[Z]$ or "locally" using $\mathscr{O}_{Z,z}$).

(ii) For $f : Z' \to Z$ with $Z'$ irreducible and $z' \in Z'$ with $f(z') = z$, show $f^* : k[Z] \to k[Z']$ uniquely extends to a map $\mathscr{O}_{Z,z} \to \mathscr{O}_{Z',z'}$ which moreover carries $M_z$ into $M_{z'}$. Defining $\mathrm{d}f(z') : \mathrm{T}_{z'}(Z') \to \mathrm{T}_z(Z)$ to be the linear dual of $M_z/M_Z^2 \to M_{z'}/M_{z'}^2$, establish a *Chain Rule* $\mathrm{d}(f \circ f')(z'') = \mathrm{d}f(z') \circ \mathrm{d}f'(z'')$ for any $f' : Z'' \to Z'$ and $z'' \in Z''$ with $f'(z'') = z'$.

(iii) Let $h_1, \ldots, h_\delta \in \mathfrak{m}_z$ represent a basis of $\mathfrak{m}_z/\mathfrak{m}_z^2 = M_z/M_z^2$. We will soon prove in class that such $h_i$ necessarily generate the ideal $M_z = \mathfrak{m}_z\mathscr{O}_{Z,z} \subset \mathscr{O}_{Z,z}$ (but not necessarily $\mathfrak{m}_z \subset k[Z]$!). Deduce via denominator-chasing that the $h_i$'s generate $\mathfrak{m}_z k[Z]_a$ for some $a \in k[Z]$ that is non-vanishing at $z$, and conclude that $\underline{Z}(h_1, \ldots, h_\delta) \cap Z_a = \{z\}$. Using this, prove that $h = (h_1, \ldots, h_\delta) : Z \to k^\delta$ has $z$ as an isolated point in the fiber $h^{-1}(0)$ (thereby completing the proof of the result in class that $\dim Z \leq \dim \mathrm{T}_z(Z)$).

4. Let $Z$ be an affine algebraic set in $k^n$, not necessarily irreducible. Let $A = k[Z]$.

(i) For $f \in A$, explain how the (reduced!) $k$-algebra $A_f$ for $f \in A$ is a $k$-subalgebra of the $k$-algebra of functions $Z_f \to k$, and check that the operation of restriction of a function on $Z$ to a function on $Z_f$ is compatible with the natural map $A \to A_f$.

(ii) If $f, g \in A$, show that $f$ maps to a unit in $A_g$ iff $Z_g \subseteq Z_f$, in which case $A \to A_g$ uniquely factors through $A \to A_f$. Show in such cases that the map $A_f \to A_g$ corresponds *exactly* to the restriction of a $k$-valued function on $Z_f$ to a $k$-valued function on $Z_g$. (This is the "functoriality" of the Rabinowitz trick.)

(iii) Show that a collection of opens $\{Z_{f_j}\}$ covers $Z$ if and only if the $f_j$'s generate 1 in $A$.