Math 145. Homework 3

1.25b, 1.33 (for (b), also show that V is an algebraic set in $\mathbf{C}^3$), 1.39, 1.40

1. For a prime ideal $P \subset k[X_1, \dots, X_n]$, $Z := \underline{Z}(P) \subseteq k^n$, and $A = k[X_1, \dots, X_n]/P$, call $K = \mathrm{Frac}(A)$ the *function field* of the irreducible (!) Z and define $k(Z) \stackrel{\text{def}}{=} K$, $k[Z] \stackrel{\text{def}}{=} A$.

(i) For any $b \in A = k[Z]$, let $Z_b = \{z \in Z \mid b(z) \neq 0\}$. Prove Z_b is a non-empty open set in Z for $b \neq 0$. For every $f = a/b = a'/b' \in k(Z)$ with $a, b, a', b' \in k[Z]$, $b, b' \neq 0$, show that the maps $Z_b \cap Z_{b'} \rightrightarrows k$ defined by $z \mapsto a(z)/b(z), a'(z)/b'(z)$ coincide. Compare addition and multiplication in $k(Z)$ with addition and multiplication of such functions on a dense open (be precise about domains of definition).

(ii) Say f is *defined* at $z \in Z$ if $f = a/b$ as above with $z \in Z_b$. If $A = k[X, Y, U, V]/(XY - UV)$ (which we know to be a domain), prove that $Y \neq 0$ in A and that U/Y is defined at $(x, y, u, v) \in \underline{Z}(XY - UV)$ if and only if $v \neq 0$ or $y \neq 0$. If $A = k[X, Y]$ and $f \in K$ is defined on $k^2 - \{(0, 0)\}$, prove $f \in A$.

(iii) The *pole set* of f is defined to be the set of $z \in Z$ at which f is not defined. Prove that this is a proper Zariski-closed set in Z. For any $f \in K$, on the Zariski open complement U of the pole set of f prove that the function $f : U \to k^1$ "defined" by f really is well-defined (use (i)) and is *continuous* with respect to the Zariski topologies. If A is a UFD, describe the pole set of $f \in K$ in terms of a "reduced form" of f, and draw a picture for $A = k[X, Y]$.

2. Let $A = k[X_1, \dots, X_n]/I$, $B = k[Y_1, \dots, Y_m]/J$ with *radical* ideals I, J. Consider a map of k-algebras $\widetilde{\varphi} : k[\underline{Y}] \to k[\underline{X}]$ such that $\widetilde{\varphi}(J) \subseteq I$, so (as shown in class) the corresponding "polynomial map" $\widetilde{\psi} : k^n \to k^m$ takes $\underline{Z}(I)$ into $\underline{Z}(J)$. Prove that this map has dense image (for the Zariski topology) if and only if the k-algebra map $B \to A$ is injective.

3. Read the handout from Lang's *Algebra* on transcendence bases and prove that if $K_0 \to K \to K'$ are maps of fields, with $\mathrm{trdeg}(K/K_0) = n < \infty$, $\mathrm{trdeg}(K'/K) = m < \infty$, then $\mathrm{trdeg}(K'/K_0) = n + m < \infty$ (hint: first consider the case $n = 0$). This is important in dimension theory for algebraic sets.

4. (i) For a surjective continuous map $X \to Y$ with X irreducible, prove Y is irreducible.

(ii) If X is irreducible, prove any two non-empty open subsets have non-empty intersection (so all non-empty opens are dense). Conclude that all non-empty opens are irreducible.

5. Let $A = k[X_1, \dots, X_n]/I$ with k an algebraically closed field. For $f \in k[\underline{X}]$ representing $a \in A$, let $J \subseteq k[X_1, \dots, X_n, T]$ be the ideal generated by I and $fT - 1$, so $k[X_1, \dots, X_n, T]/J \simeq A_a$.

(i) Prove that $(x, t) \mapsto x$, $x \mapsto (x, f(x)^{-1})$ define a homeomorphism between $\underline{Z}(J) \subseteq k^{n+1}$ and the non-empty open set $\underline{Z}(I)_f = \{z \in \underline{Z}(I) \mid f(z) \neq 0\} \subseteq \underline{Z}(I)$. Establish a bijection between the set of irreducible closed sets in $\underline{Z}(I)$ meeting $\underline{Z}(I)_f$ and the set of irreducible closed sets in $\underline{Z}(I)_f = \underline{Z}(J)$ via $Z \mapsto Z \cap \underline{Z}(I)_f$ and $Z' \mapsto \overline{Z}'$ (closure for the Zariski topology in $\underline{Z}(I)$). (Hint: closure commutes with intersecting with an open set).

(ii) For the (possibly non-injective) natural map $\varphi : A \to A_a$, show that $\mathfrak{p} \mapsto \mathfrak{p} \cdot A_a$ and $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$ induce bijections between the set of prime ideals of A not containing a and the set of prime ideals of A_a. (Hint: use mapping properties to identify $(A/\mathfrak{a})_a$ with $A_a/\mathfrak{a} \cdot A_a$ for ideals $\mathfrak{a}$ of A.) Explain why this "is" exactly the bijection in (i).