Math 145. Closed subspaces, products, and rational maps

The purpose of this handout is to develop a good notion of *product* for abstract algebraic sets, and to work out some examples. We certainly expect that $\mathbf{A}^{n+m}$ should be a "product" of $\mathbf{A}^n$ and $\mathbf{A}^m$ (via projection to the first n and last m coordinates), but already for $n = m = 1$ we see that the Zariski topology on $\mathbf{A}^{n+m}$ is *not* the product topology. In general $X \times Y$ will have underlying set given by the product but its topology will be more subtle.

1. Closed subsets

Let X be an abstract algebraic set, and $Z \subset X$ a closed subset. We wish to equip Z with a natural structure of abstract algebraic set in its own right, recovering the familiar construction in the affine case that assigns to any $Z \subset \mathrm{MaxSpec}(A)$ the structure $\mathrm{MaxSpec}(A/\underline{I}(Z))$. We will also insist on a good mapping property. This is all captured by:

Theorem 1.1. *For any closed subset Z in an abstract algebraic set X there is a unique sheaf $\mathscr{O}$ of k-valued functions on Z so that the ringed space $(Z, \mathscr{O})$ is an abstract algebraic set and for every map of abstract algebraic sets $f : Y \to X$ with $f(Y) \subset Z$ the induced set-theoretic map $f : Y \to Z$ is a map of ringed spaces over k.*

Explicitly, if $U = \mathrm{MaxSpec}(A)$ is an open affine subspace of X then $Z \cap U$ with its induced ringed space structure as an open subspace of U is $\mathrm{MaxSpec}(A/J)$ where J is the radical ideal of elements of A that vanish on $Z \cap U$.

We call such a pair $(Z, \mathscr{O})$ a *closed subspace* of the abstract algebraic set X. We emphasize that its structure sheaf is part of the data, but the theorem says that this is determined by the algebro-geometric structure on X.

Even when X is affine, this theorem has content: in such cases it says that for *any* map of abstract algebraic sets $f : Y \to \mathrm{MaxSpec}(A)$ which factors through $\mathrm{MaxSpec}(A/J)$ set-theoretically, the induced map of sets $Y \to \mathrm{MaxSpec}(A/J)$ is a morphism of ringed spaces over k. This is important in practice (so we do not leave the world of morphisms of ringed spaces over k).

Proof. The uniqueness of the ringed space structure on Z (granting existence!) is formal, as follows. Suppose there are two sheaves of k-algebras $\mathscr{O}'$ and $\mathscr{O}''$ on Z that solve the problem, and write Z' and Z'' to denote the resulting ringed spaces over k with underlying topological space Z. We wish to prove that $\mathscr{O}' = \mathscr{O}''$ as sheaves of k-valued functions on Z. This amounts to checking that the identity map of Z promotes to morphisms of ringed spaces $Z' \to Z''$ and $Z'' \to Z'$ (as the first gives $\mathscr{O}''(V) \subset \mathscr{O}'(V)$ for all open $V \subset Z$ and the second gives the reverse inclusion). But by hypothesis the inclusion $Z \to X$ promotes to maps of ringed spaces $j' : Z' \to X$ and $j'' : Z'' \to X$ over k, each of which lands in Z set-theoretically. Thus, by applying the universal property of Z'' to the morphism j' we get a morphism $Z' \to Z''$ (i.e., map of ringed spaces over k) that is the identity map on underlying sets, and similarly we get the morphism in the reverse direction. This proves the uniqueness in general, so the problem is one of existence of the required sheaf of k-valued functions on Z.

We first handle the case of affine X (for which we have seen that there is something to be done), and then we will bootstrap to the general case by gluing. So for now assume $X = \mathrm{MaxSpec}(A)$, and let $f : Y \to X$ be a morphism landing in the closed set Z corresponding to a radical ideal $J \subset A$ via the Nullstellensatz. We seek to show that the induced map of sets $Y \to \mathrm{MaxSpec}(A/J)$ is a morphism of ringed spaces. This latter map is at least continuous, since $\mathrm{MaxSpec}(A/J) \to \mathrm{MaxSpec}(A)$ is topologically a homeomorphism onto its image. Thus, the problem is now one of compatibility with structure sheaves, and it is *local on Y* due to the local nature of sheaves of functions. (This just expresses "gluing for morphisms" of ringed spaces as at the end of Exercise 3(ii) in HW8.) In other words, if $\{U_i\}$ is an open cover of Y and the *continuous* map $Y \to Z$ becomes a morphism of ringed spaces upon restriction to each $U_i \subset Y$ then it is a morphism of ringed spaces. (Make sure you understand this!)

We may now choose a covering of Y by open affines to thereby reduce to the case (for the given affine X) when Y is also affine! But now we are in a familiar territory: $Y = \mathrm{MaxSpec}(B)$ and $f : Y \to X$ corresponds to a k-algebra map $\varphi : A \to B$. By hypothesis the map f lands in $\underline{Z}(J)$. In the affine setting we know that for any ideal I in A, the preimage of $\underline{Z}(I)$ under f is $\underline{Z}(\varphi(I)B)$. Hence, the ideal $\varphi(J)B$ vanishes everywhere on Y (as $f^{-1}(\underline{Z}(J))$ is the entirety of Y, by hypothesis), which is to say that it is zero since B is reduced. In other words, $\varphi(J) = 0$, so $\varphi : A \to B$ factors as $A \twoheadrightarrow A/J \to B$ for a k-algebra map $\overline{\varphi} : A/J \to B$. This latter map defines a morphism $\overline{f} : Y \to \mathrm{MaxSpec}(A/J)$ whose composition with the inclusion $\mathrm{MaxSpec}(A/J) \to \mathrm{MaxSpec}(A)$ (corresponding to the canonical quotient map $A \twoheadrightarrow A/J$) is exactly the map $\mathrm{MaxSpec}(\varphi) = f$. Hence, $\underline{f}$ is the map of interest on underlying sets, so we have recovered that map of sets as associated to a map of ringed spaces over k, as desired. This settles the case when X is affine.

Before we turn to the case of general X, it will be convenient to record a "localization" observation which will help in the gluing. Suppose that the problem has been solved for a given pair (X, Z). Then for *any* open subset $U \subset X$ (not necessarily affine, even if X is affine) we may view the closed set $Z \cap U$ in U as an *open subset* of the ringed space Z and hence equip it with a structure of abstract algebraic set (via the structure sheaf on Z, not X!). In this way, we claim that the problem is solved for the pair $(U, Z \cap U)$. For any map of abstract algebraic sets $f : Y \to U$ that lands in $Z \cap U$ set-theoretically, we may view f as a map into X that lands in Z set-theoretically. Thus, the induced map of sets $Y \to Z$ is a morphism of ringed spaces over k (by our hypothesis that the problem is solved for (X, Z)), and this lands inside the open subset $Z \cap U$. Since we equip $Z \cap U$ with the structure sheaf arising from the one of Z (evaluated just on open subsets of the open subset $Z \cap U$), it follows that this map $f : Y \to Z \cap U$ is a morphism of ringed spaces over k, as desired.

Now we handle general X. Letting $\{X_i\}$ be a choice of open cover of X by affines, we can apply the preceding case to the closed set $Z_i = Z \cap X_i$ in X_i. This provides a structure sheaf $\mathscr{O}_i$ on Z_i making $(Z_i, \mathscr{O}_i)$ an abstract algebraic set (even affine!) so that the injection $Z_i \to X_i$ has the desired universal mapping property (for maps to X_i that factor set-theoretically through Z_i). To define the global abstract algebraic set structure on Z, we glue the $\mathscr{O}_i$'s as follows (being careful about the fact that the overlaps $X_{ij} = X_i \cap X_j$ may not be affine). Consider the overlap $Z_{ij} = Z_i \cap Z_j$. This is an open subset of Z, so it is open in both Z_i and Z_j. Our "localization" argument above can be applied to $U = Z_{ij}$ for the pairs (X_i, Z_i) *and* (X_j, Z_j), so the two sheaves $\mathscr{O}_i|_{Z_{ij}}$ and $\mathscr{O}_j|_{Z_{ij}}$ on Z_{ij} both solve the problem for (X_{ij}, Z_{ij}). But we have shown the uniqueness already in general, so $\mathscr{O}_i|_{Z_{ij}} = \mathscr{O}_j|_{Z_{ij}}$ as sheaves on Z_{ij}. Hence, we can glue the $\mathscr{O}_i$'s in accordance with Exercise 3 on HW8 to get a sheaf $\mathscr{O}$ of k-valued functions on Z with $\mathscr{O}|_{Z_i} = \mathscr{O}_i$ for all i. This makes $(Z, \mathscr{O})$ into a ringed space over k such that the open subsets Z_i becomes the ringed spaces $(Z_i, \mathscr{O}_i)$ that are assumed to be abstract algebraic sets. Hence, there are opens $U_{ij} \subset Z_i$ covering Z_i such that each ringed space $(U_{ij}, \mathscr{O}_i|_{U_{ij}})$ is affine. But

$$\mathscr{O}_i|_{U_{ij}} = (\mathscr{O}|_{Z_i})|_{U_{ij}} = \mathscr{O}|_{U_{ij}},$$

so the opens U_{ij} that cover $(Z, \mathscr{O})$ are all affine. That is, $(Z, \mathscr{O})$ is an abstract algebraic set.

It remains to prove that $\mathscr{O}$ "works". That is, if $f : Y \to X$ is a map of ringed spaces over k such that $f(Y) \subset Z$ then we claim that the induced set map $f : Y \to Z$ is a map of ringed spaces over k. The map $f : Y \to Z$ is at least continuous, since Z has the subspace topology from X and

$Y \to X$ is continuous, and the opens Z_i that cover Z have preimages $Y_i = f^{-1}(X_i)$ that are an open cover of Y. Our problem is one of compatibility of the structure sheaves of Y and Z, so by the local nature of sheaves it suffices to treat the pairs (Y_i, Z_i). That is, we want to show that each map $Y_i \to Z_i$ is a map of ringed spaces when Z_i is equipped with the sheaf of functions $\mathscr{O}|_{Z_i} = \mathscr{O}_i$. But the composite map of sets

$$f_i : Y_i \to Z_i \to X_i$$

is induced by restriction to open sets for the map of ringed spaces $f : Y \to X$, so each f_i is a map of ringed spaces. Thus, by the mapping property for each $(Z_i, \mathscr{O}_i)$ relative to $Z_i \to X_i$ (due to how each X_i was chosen and $\mathscr{O}_i$ defined) we conclude that $Y_i \to Z_i$ is a map of ringed spaces over k as desired. ■

2. Products

Let V and V' be abstract algebraic sets. A *product* of V and V' is a triple (P, f, f') where P is an abstract algebraic set and $f : P \to V$ and $f' : P \to V'$ are morphisms of abstract algebraic sets with the mapping property of a product as in any category: if $h : Y \to V$ and $h' : Y \to V'$ are morphisms then there is a unique morphism $g : Y \to P$ such that $f \circ g = h$, $f' \circ g = h'$. We will build such a product with underlying set (but not underlying topological space!) $P = V \times V'$. The real content will be that the computations on underlying sets correspond to morphisms of ringed spaces. We will build up in stages: first with affine spaces, then with affine algebraic sets in general, and finally the general case.

Lemma 2.1. *The affine space $\mathbf{A}^{n+m}$ equipped with its projections $f : \mathbf{A}^{n+m} \to \mathbf{A}^n$ and $f' : \mathbf{A}^{n+m} \to \mathbf{A}^m$ to the first n and last m components is a categorical product.*

That is, if $h : Y \to \mathbf{A}^n$ and $h' : Y \to \mathbf{A}^m$ are morphisms of ringed spaces then the map of sets $g : Y \to \mathbf{A}^{n+m}$ defined by $y \mapsto (h(y), h'(y))$ is a morphism of ringed spaces.

Proof. The problem of continuity of g is "local on Y", and once this is settled then the problem of compatibility of structure sheaves is local on Y. That is, if $\{Y_i\}$ is an open cover of Y then it suffices to prove the result for all maps restricted to a fixed (but arbitrary) choice of open Y_i in Y. By taking the Y_i's to be affine (as we may do), we thereby reduce to the case that $Y = \mathrm{MaxSpec}(A)$ for a reduced finitely generated k-algebra A.

The given maps h and h' correspond to respective k-algebra maps

$$h^* : k[X_1, \dots, X_n] \to A, \;\; h'^* : k[X_{n+1}, \dots, X_{n+m}] \to A.$$

These two together define an evident k-algebra map

$$\varphi : k[X_1, \dots, X_{n+m}] \to A$$

via $X_j \mapsto h^*(X_j)$ for $j \le n$ and $X_i \mapsto h'^*(X_i)$ for $i > n$. The morphism $\mathrm{MaxSpec}(\varphi) : Y \to \mathbf{A}^{n+m}$ does the job (i.e., it is as expected on underlying sets) since MaxSpec is a contravariant functor and the projections f and f' are obtained (!) from the respective k-algebra inclusions

$$k[X_1, \dots, X_n], k[X_{n+1}, \dots, X_{n+m}] \to k[X_1, \dots, X_{n+m}].$$

■

Now we can make products for general affine algebraic sets. Let A and B be reduced finitely generated k-algebras. Picking k-algebra isomorphisms

$$k[X_1, \dots, X_n]/I \simeq A, \;\; k[Y_1, \dots, Y_m]/J \simeq B$$

for necessarily radical ideals I and J, consider the k-algebra

$$C = k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(I, J).$$

There are evident k-algebra maps $A \to C$ and $B \to C$, and provided that C is reduced we can then make ringed space maps

$$\mathrm{MaxSpec}(C) \rightrightarrows \mathrm{MaxSpec}(A), \mathrm{MaxSpec}(B)$$

which we claim will do the job, and moreover we claim that the induced map of sets

$$\mathrm{MaxSpec}(C) \to \mathrm{MaxSpec}(A) \times \mathrm{MaxSpec}(B)$$

is bijective (so we naturally identify the underlying set with the product set).

Lemma 2.2. *The ideal (I, J) is radical; i.e., C is reduced.*

The statement of this lemma makes sense algebraically for general fields k, but it is false for certain k that are not algebraically closed.

Proof. We need to prove that if $f(X, Y) \in k[\underline{X}, \underline{Y}]$ vanishes wherever on $\mathbf{A}^{n+m}$ both I and J vanish (i.e., f lies inside of all maximal ideals containing (I, J)) then $f \in (I, J)$. This would imply the radiciality of (I, J) due to the Nullstellensatz formula for the radical of an ideal in terms of "geometric zeros," and so here we will have used in an essential way that k is algebraically closed.

Recall that over any field K, all vector spaces V over K have bases and any basis of a subspace extends to a basis of the entire space (this requires Zorn's Lemma; the argument below could be modified to avoid this, but it would lead to awkwardness in exposition). In particular, subspaces always split off as direct summands. We will use this in a moment.

Using the k-algebra structure on A, it makes sense to consider the A-submodule $J_A \subseteq A[\underline{Y}]$ generated by J. Note that this is an ideal in this polynomial ring over A (why?). There is a natural k-algebra isomorphism

$$k[\underline{X}, \underline{Y}]/(I, J) = A[\underline{Y}]/J_A.$$

Choose a k-basis of $k[\underline{Y}]$ extending a k-basis of J. This makes explicit that J_A is a free A-module and is a direct summand as such of $A[\underline{Y}]$ (check!). It follows that if $\{e_j\}$ is the induced k-basis of $k[\underline{Y}]/J$, then the e_j's form a basis of the A-module $A[\underline{Y}]/J_A$; in particular, this is a *free* A-module. But if M is a free A-module on a basis $\{\epsilon_\alpha\}$, then by looking at coefficients of the basis it is clear that an element $m \in M$ vanishes if and only if it vanishes in the the $A/\mathfrak{m} = k$-vector space $M/\mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$ of A (as this criterion on coefficients just says $\cap \mathfrak{m} = 0$, which is true because A is *reduced*!). Applying this to the free A-module $M = A[\underline{Y}]/J_A$, we see that an element $f \in M$ vanishes if and only if for all maximals $\mathfrak{m}$ of A the image of f vanishes in $M/\mathfrak{m}M$. But thinking about bases shows that as k-modules

$$M/\mathfrak{m}M \simeq (A/\mathfrak{m})[\underline{Y}]/J_{A/\mathfrak{m}} = k[\underline{Y}]/J = B,$$

where we have used that $k \to A \to A/\mathfrak{m} = k$ is the *identity* map in order to know that the result on the right really does involve forming the quotient by the original ideal J (i.e., its k-coefficients map into A and then recover themselves upon passage to any $A/\mathfrak{m}$).

In concrete terms, what this says is that an element $f \in k[\underline{X}, \underline{Y}]/(I, J)$ vanishes if and only if for all $x \in \underline{Z}(I) = \mathrm{Max}(A)$, $f(x, Y) \in k[\underline{Y}]/J = B$ vanishes. But now we use that B is reduced to conclude this this vanishing (for fixed x) is equivalent to the vanishing of $f(x, Y)|_{Y=y} = f(x, y)$ for all $y \in \mathrm{Max}(B)$. Thus, we conclude that if $f \in k[\underline{X}, \underline{Y}]$ vanishes at every geometric zero of (I, J) in the product affine space, then f vanishes in the ring $k[\underline{X}, \underline{Y}]/(I, J)$; i.e., $f \in (I, J)$. It follows that (I, J) is radical. ■

Now we claim that $\mathrm{MaxSpec}(C)$ with its induced maps to $\mathrm{MaxSpec}(A)$ and $\mathrm{MaxSpec}(B)$ is a product, with the product set as its underlying set in the natural way. To verify this, we work in the more familiar language of closed sets of affine spaces, since the problems have all been solved for affine spaces.

For the product set structure on underlying sets, we simply observe that by design

$$\mathrm{MaxSpec}(C) = \underline{Z}(I, J) \subset \mathbf{A}^{n+m} = \mathbf{A}^n \times \mathbf{A}^m,$$

and this closed subset is visibly the product set $\underline{Z}(I) \times \underline{Z}(J)$ which in turn is exactly $\mathrm{MaxSpec}(A) \times \mathrm{MaxSpec}(B)$. The reader can verify that this composite bijection is exactly the desired map of interest (ultimately because we know how the projections $\mathbf{A}^{n+m} \rightrightarrows \mathbf{A}^n, \mathbf{A}^m$ are expressed algebraically in terms of the first n and last m variables).

But why does this construction have the expected universal mapping property? We will deduce the result from the settled case of affine spaces. Consider ringed space maps $Y \rightrightarrows \underline{Z}(I), \underline{Z}(J)$. Viewing these as maps into $\mathbf{A}^n$ and $\mathbf{A}^m$ respectively, from the case of affine spaces we get that the product map of sets

$$Y \to \mathbf{A}^n \times \mathbf{A}^m = \mathbf{A}^{n+m}$$

is a morphism of ringed spaces over k. By construction this lands inside the subset $\underline{Z}(I) \times \underline{Z}(J) = \underline{Z}(I, J) = \mathrm{MaxSpec}(C)$ with C the quotient of $k[\underline{X}, \underline{Y}]$ modulo the *radical* ideal (I, J). Hence, by the universal property for closed subsets (even just in the affine target case, but with possibly non-affine source Y) especially in §1, the resulting set map $Y \to \mathrm{MaxSpec}(C)$ really is a morphism of ringed spaces over k! One readily checks (e.g., by composing with the inclusion into $\mathbf{A}^{n+m}$ and using the settled case of affine spaces) that this morphism of ringed spaces is the desired map on the underlying set $\underline{Z}(I, J)$ when identified in the natural way as the direct product set

$$\underline{Z}(I) \times \underline{Z}(J) = \mathrm{MaxSpec}(A) \times \mathrm{MaxSpec}(B).$$

This settles the case of products of affine algebraic sets in general (with source Y that is *not* necessarily affine).

Before we construct general products $V \times V'$ by gluing, we verify a compatibility property of products (when they exist!) with open subsets of the factors. This has some content, since the actual product structure is generally not the product topology.

Lemma 2.3. *Suppose $V \times V'$ exists with underlying set identified with the product set via the given projections $f : V \times V' \to V$ and $f' : V \times V' \to V'$. For open $U \subset V$ and $U' \subset V'$, the open subset $f^{-1}(U) \cap f'^{-1}(U') \subset V \times V'$ is a categorical product of U and U', and its underlying set is $U \times U'$ inside $V \times V'$.*

The proof of this lemma is extremely formal, as the reader will see.

Proof. It is trivial to check that the underlying set is the product set: this is the set-theoretic computation

$$(U \times V') \cap (V \times U') = U \times U'.$$

In particular, this product set is *open* in $V \times V'$ since it is an intersection of open subsets $f^{-1}(U)$ and $f'^{-1}(U')$.

The problem is now entirely one of compatibility with structure sheaves. That is, if $h : Y \to U$ and $h' : Y \to U'$ are morphisms of ringed spaces over k (with Y an abstract algebraic set) then we want the map to the product set $(h, h') : Y \to U \times U'$ to be a morphism of ringed spaces over k when $U \times U'$ is equipped with the *topology* and structure sheaf it acquires as an *open* subset of $V \times V'$. But in general, we claim that a map g of sets from a ringed space Y over k into an open subset Ω of another ringed space Z over k is a morphism of ringed spaces (i.e., continuous on topological spaces, and compatible with structure sheaves) if and only if the same holds after composition with the inclusion of Ω into Z. This holds because we equip Ω with the subspace topology as an *open* subset of Z and moreover with structure sheaf given by restriction of the one on Z (evaluated only on open subsets of Ω). Thus, to verify the desired properties for the product set map (h, h') it is harmless to first compose with the inclusion into $V \times V'$. This brings us to exactly the known (by hypothesis) mapping property for this latter product relative to the composite maps of ringed spaces over k

$$Y \to U \to V, \;\; Y \to U' \to V'.$$

■

Now, finally, we can build general products. The reader will see that this too is a largely formal game. All of the real work was in the affine case. For abstract algebraic sets V and V', we have open covers (e.g., by affine opens!) $\{V_i\}$ and $\{V'_j\}$ of V and V' such that the products $V_i \times V'_j$ are known to exist as abstract algebraic sets and have underlying set identified with the product set via the natural map. We will use this together with Lemma 2.3 to make $V \times V'$ via gluing. To be precise, we will first choose i_0 and use the products $V_{i_0} \times V'_j$ to build $V_{i_0} \times V'$. Then the exact same argument applied to the products $V_i \times V'$ for varying i will build $V' \times V$. This formalism only works because we do *not* assume the V_i and V'_j are affine; rather, we just assume that they are open subsets for which the relevant products exist (since when we apply this formalism to build $V \times V'$ by gluing the products $V_i \times V'$ we cannot assume that V' is affine, even if all V_i are affine).

To summarize, it suffices (by symmetry in the two factors) to show that if $\{V'_j\}$ is an open cover of V for which the products $P_j = V \times V'_j$ exist with the product set as the underlying set (via the natural map) then $V \times V'$ exists with the product set as the underlying set (via the natural map). Let $q_j : P_j \to V'_j$ be the projection, and for any j, j', consider the subset

$$P_{jj'} = q_j^{-1}(V'_j \cap V'_{j'}).$$

This is open since q_j is continuous, and its underlying set is the product set $V \times (V_j \cap V'_j)$. By Lemma 2.3, the triple

$$(P_{jj'}, P_{jj'} \to V, P_{jj'} \to V'_j \cap V'_j)$$

is a product of V and $V'_j \cap V'_{j'}$. But $P_{j'j}$ is also such a product! Are the resulting topologies and structure sheaves on the product set $V \times (V'_j \cap V'_{j'})$ the same? Indeed they are, as we now explain.

By the universal property of a product applied twice, there are unique maps of ringed spaces

$$P_{jj'} \to P_{j'j}, \;\; P_{j'j} \to P_{jj'}$$

which respect the projections to V and to $V'_j \cap V'_{j'}$. But these projections are *exactly* the mechanism by which we are identifying the underlying set of the categorical product with the product set, so it follows that these "abstract" maps of ringed spaces are literally the *identity map* on the underlying product set. Hence, the topologies coincide and the structure sheaves coincide.

We summarize our situation as follows. First, the product set $P = V \times V'$ is covered by the subsets P_j which are each equipped with a structure of ringed space, and in this way the overlaps $P_j \cap P_{j'}$ are identified as *open* subsets of P_j and $P_{j'}$ with the *same* topology obtained as the subspace topology from each. Hence, we can "glue topologies" to get a unique topology on the product set P which makes each P_j open and gives as the subspace topology exactly the topology imposed upon P_j from its given nature as a ringed space. The same exact formalism applies now to the structure sheaves: if $\mathscr{O}_j$ denotes the structure sheaf on P_j then $\mathscr{O}_j|_{P_j \cap P_{j'}}$ and $\mathscr{O}_{j'}|_{P_j \cap P_{j'}}$ coincide as sheaves of k-algebras on $P_j \cap P_{j'}$. Thus, we can likewise glue these to a sheaf of k-algebras $\mathscr{O}$ on P uniquely determined by the condition $\mathscr{O}|_{P_j} = \mathscr{O}_j$ for all j. Since each open subspace $(P_j, \mathscr{O}_j)$ of $(P, \mathscr{O})$ is an abstract algebraic set, it follows that so is $(P, \mathscr{O})$. It remains to prove:

Proposition 2.4. *The natural set maps $P \rightrightarrows V, V'$ are morphisms of ringed spaces over k, and have the universal property of a product in the category of abstract algebraic sets.*

Proof. To check the continuity it suffices to work on the constituents of an open cover of P, and then likewise for compatibility with the structure sheaves. Thus, we can restrict to each open $P_j \subset P$. The map $P_j \to V$ is fine since $P_j = V_j \times V'$ as a categorical product of abstract algebraic sets, and the map $P_j \to V'$ is fine since it factors as the projection $P_j \to V'_j$ arising from the mapping property as a categorical product followed by the inclusion $V'_j \to V'$ as an open subspace (with the restricted structure sheaf!). Thus, both maps $P \rightrightarrows V, V'$ are morphisms of ringed spaces over k.

It remains to show that if $Y \rightrightarrows V, V'$ are maps of abstract algebraic sets over k then the induced map to the product set $Y \to P = V \times V'$ is a morphism of ringed spaces over k. As usual, it suffices to check this on the constituents of an open cover of Y. Using the given map $f' : Y \to V'$ we get the open preimages $Y_j \subset Y$ of the opens $V'_j \subset V'$ that cover V', so the Y_j's are an open cover of Y. We may work on each Y_j separately, so in other words we can assume that Y lands inside some V'_{j_0}. By design $P_{j_0} = V \times V'_{j_0}$ is the preimage of V'_{j_0} under the projection morphism $P \to V'$, and P_{j_0} has an open subspace topology from P as well as the restricted structure sheaf on this open subset. Hence, our problem is entirely intrinsic to the triple (P_{j_0}, V, V'_{j_0})! But by hypothesis P_{j_0} is a categorical product (as an abstract algebraic set) and it has the underlying product set via the natural map, so we are done. ■

We end this section by discussing how the formation of products interacts with open and closed subspaces, and computing an important class of examples.

Proposition 2.5. *Let X and X' be abstract algebraic sets, and let $Y \subset X$ and $Y' \subset X'$ be subsets. If these subsets are open then $Y \times Y' \to X \times X'$ is an open subspace* (*open subset with subspace topology and restricted structure sheaf*)*, and if they are closed then $Y \times Y' \to X \times X'$ is a closed subspace* (*closed subset equipped with its natural structure of abstract algebraic set via* §1)*.*

Proof. The open case is Lemma 2.3. The closed case will be largely a game of formalism using the open case repeatedly in order to eventually reduce to the case of affine X and X' for which we can see everything by inspection.

Let $\{X_i\}$ and $\{X'_j\}$ be respective open covers of X and X', so $\{X_i \times X'_j\}$ is an open cover of $X \times X'$ (with the subspace topology as well as restricted structure sheaf!). Letting $Y_i = Y \cap X_i$ (*open* in Y, closed in X_i) and $Y'_j = Y \cap X'_j$, clearly the open subset $X_i \times X'_j$ in $X \times X'$ meets $Y \times Y'$ in the subset $Y_i \times Y'_j$ that is *open* in $Y \times Y'$. We equip $Y_i \times Y'_j$ with the open subspace structure from $Y \times Y'$.

Our problem is to show two things: $Y \times Y' \to X \times X'$ is a homeomorphism onto a closed subset of $X \times X'$, and as such its structure sheaf coincides with the one provided by Theorem 1.1. Since closedness of a subset of a topological space is a local property (i.e., it suffices to check it after intersecting with each of the constituents of an open cover), for the closedness of the image of $Y \times Y'$ in $X \times X'$ it suffices to work with each open subset $X_i \times X'_j$ and its preimage $Y_i \times Y'_j$ in $Y \times Y'$. But these preimages are open in $Y \times Y'$, so even for the homeomorphism property of $Y \times Y'$ onto a closed image it is enough to check that the maps $Y_i \times Y'_j \to X_i \times X'_j$ are homeomorphisms onto closed images. Moreover, once that is verified, to check that the structure sheaf on $Y \times Y'$ in its nature as a product abstract algebraic set coincides with the closed subspace structure from $X \times X'$, the problem is of local nature on $Y \times Y'$ (as for checking equality of sheaves of functions in general!). That is, we will only need to compare the restricted sheaves on the open subsets $Y_i \times Y'_j$. Hence, it will suffice to show that this is a closed subspace of $X_i \times X'_j$ when the latter is equipped with its open subspace structure from $X \times X'$ as a ringed space.

To summarize, for everything we need to prove it is enough to treat each of the maps $Y_i \times Y'_j \to X_i \times X'_j$ separately. In this sense, our entire problem is local on X and X'. We can take the X_i and X'_j to all be *affine*, in which case each Y_i and Y'_j is also affine with its closed subspace structure (due to Theorem 1.1). Hence, we have reduced to the affine case. That is, now

$$X = \mathrm{MaxSpec}(A), Y = \mathrm{MaxSpec}(A/J), X' = \mathrm{MaxSpec}(A'), Y' = \mathrm{MaxSpec}(A'/J')$$

for radical ideals $J \subset A$ and $J' \subset A'$. The question is whether the product affine algebraic set $Y \times Y'$ is a closed subspace (as a ringed space) in the affine $X \times X'$ via the natural map. Choosing k-algebra isomorphisms

$$k[t_1, \dots, t_n]/I \simeq A, \;\; k[t_{n+1}, \dots, t_{n+m}]/I' \simeq A',$$

so J and J' respectively correspond to radical ideals $\widetilde{J} \supseteq I$ and $\widetilde{J}' \supseteq I'$. In particular, we have compatible k-algebra isomorphisms

$$k[t_1, \dots, t_n]/\widetilde{J} \simeq A/J, \;\; k[t_{n+1}, \dots, t_{n+m}]/\widetilde{J}' \simeq A'/J'.$$

By construction of affine products, inside of $\mathbf{A}^{n+m}$ we have as closed subspaces

$$X \times X' = \underline{Z}(I, I'), \;\; Y \times Y' = \underline{Z}(J, J').$$

In other words, $X \times X' = \mathrm{MaxSpec}(C)$ and $Y \times Y' = \mathrm{MaxSpec}(C/K)$ where

$$C = k[t_1, \dots, t_{n+m}]/(I, I'), \;\; K = (J, J')/(I, I') \subset C,$$

and the natural set map $Y \times Y' \to X \times X'$ is indeed induced by the natural surjective map $C \twoheadrightarrow C/K$ (check!). Hence, $Y \times Y' \to X \times X'$ is a closed subspace. ■

Example 2.6. Consider $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ defined by $([x_0, x_1], [y_0, y_1]) \mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. This is a well-defined morphism (see HW9) and if we denote the homogenous "coordinates" on $\mathbf{P}^3$ as Z_{ij} then the image lands in the zero locus of the homogeneous polynomial $Z_{11}Z_{00} - Z_{10}Z_{01}$. In fact it is an isomorphism onto this quadric surface (viewed as a closed subspace of $\mathbf{P}^3$). This is a special case of the following general considerations.

For any $n \geq 1, m \geq 1$ consider the natural map $S : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{(n+1)(m+1)-1}$ defined by

$$([x_0, \dots, x_n], [y_0, \dots, y_m]) \mapsto [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m].$$

This is easily seen to be a well-defined morphism; it is called the *Segre morphism.* We now show that S is an isomorphism onto the closed subset of the target defined by the quadric relations $Z_{i'j'}Z_{ij} = Z_{i'j}Z_{ij'}$ for $0 \leq i, i' \leq n$ and $0 \leq j, j' \leq m$ (called the *Plücker relations*); in particular, $\mathbf{P}^n \times \mathbf{P}^m$ *is* a projective variety! Classically S was used to *define* the algebro-geometric structure on $\mathbf{P}^n \times \mathbf{P}^m$, without emphasis on the actual mapping property of a product (which is cumbersome to express in terms of the image of the S)! This might look ad hoc (and be hard to use) if we hadn't recognized that $S(\mathbf{P}^n \times \mathbf{P}^m)$ does indeed have the universal mapping property of a product.

It is clear that the image of S satisfies the Plücker relations, so we have to show that all points $[z_{ij}]$ satisfying the Plücker relations are in the image of S, and moreover that S is an isomorphism onto this closed set with its closed subspace structure. Any point $[z_{ij}]$ has *some* $z_{i_0j_0} \neq 0$. Thus, for $x_i = z_{ij_0}/z_{i_0j_0}$ and $y_j = z_{i_0j}/z_{i_0j_0}$ we see via the Plücker relations that

$$S([x_i], [y_j]) = [z_{ij}/z_{i_0j_0}] = [z_{ij}].$$

Hence, S has the asserted closed image. Moreover, if $S([x_i],[y_j])$ lands in the open set $Z_{i_0j_0} \neq 0$ then necessarily $x_{i_0}, y_{j_0} \neq 0$, so

$$x_i/x_{i_0} = (x_i y_{j_0})/(x_{i_0} y_{j_0}) = z_{ij_0}/z_{i_0j_0},\ \ y_j/y_{j_0} = (x_{i_0} y_j)/(x_{i_0} y_{j_0}) = z_{i_0j}/z_{i_0j_0}.$$

This shows that S is bijective onto its closed image.

To prove that S induces an isomorphism onto the closed subspace structure on its image, the problem is two-fold: homeomorphism on topological spaces, and then compatibility of structure sheaves. These problems are both *local on the target*: it suffices to work over the affine open set $\{Z_{i_0j_0} \neq 0\}$ in the target and its (affine!) open preimage $\{X_{i_0} \neq 0\} \times \{Y_{j_0} \neq 0\}$ in the source, showing that the map induced by S between these corresponds to a surjection between coordinate rings. This map between coordinate rings is

$$S^* : k[Z_{ij}/Z_{i_0j_0}] \to k[X_0/X_{i_0}, \dots, X_n/X_{i_0}, Y_0/Y_{j_0}, \dots, Y_m/Y_{j_0}]$$

defined by $Z_{ij}/Z_{i_0j_0} \mapsto (X_i/X_{i_0})(Y_j/Y_{j_0})$ for all $(i,j) \neq (i_0,j_0)$, though we *do* allow $i = i_0$ when $j \neq j_0$ and $j = j_0$ when $i \neq i_0$! In particular, $Z_{i_0j}/Z_{i_0j_0} \mapsto Y_j/Y_{j_0}$ and $Z_{ij_0}/Z_{i_0j_0} \mapsto X_i/X_{i_0}$. The surjectivity is thereby proved.

3. Product maps, diagonal, and separatedness

For any maps $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$ between abstract algebraic sets, the universal property of the product provides a unique morphism of abstract algebraic sets

$$(f_1, f_2) : Y \to X_1 \times X_2$$

whose composition with the respective projections to X_1 and X_2 are f_1 and f_2. In other words, this map on underlying sets is $y \mapsto (f_1(y), f_2(y))$. The point of the preceding general business is to ensure that this natural-looking map of sets really is a morphism relative to a suitable abstract algebraic set structure on the product set $X_1 \times X_2$.

In the special case $Y = X_1 = X_2 = X$ and $f_1 = f_2 = \mathrm{id}_X$, we get the *diagonal* morphism

$$\Delta_X : X \to X \times X$$

defined by $x \mapsto (x,x)$ for any abstract algebraic set X. Keep in mind that $X \times X$ generally does *not* have the product topology. In ordinary topology, when using product topologies there is the familiar exercise that a topological space T is *Hausdorff* if and only if the diagonal $T \subset T \times T$ is closed relative to the product topology. In algebraic geometry this notion is still important, but it takes on a different meaning unrelated to the Hausdorff condition since the Zariski topology on a product is generally not the product topology!

Definition 3.1. An abstract algebraic set X is *separated* if the diagonal $\Delta_X(X)$ is closed in the Zariski topology on $X \times X$.

For example, if $X = \mathbf{A}^n$ then X is separated: the diagonal is the locus $t_1 = t_{n+1}, \dots, t_n = t_{2n}$ in $\mathbf{A}^n \times \mathbf{A}^n = \mathbf{A}^{2n}$. To make a non-separated example, one carries out the algebro-geometric analogue of the standard non-Hausdorff topological space (line with double origin) by gluing $\mathbf{A}^1$ to itself along $\mathbf{A}^1 - \{0\}$ using the identity map. The fact that this gluing really is non-separated (which is an entirely different assertion than the non-Hausdorff property in the ordinary topological setting since we are not using product topologies) follows from the criterion in Proposition 3.3 below, due to the fact that the punctured diagonal in $\mathbf{A}^2$ (with $(0,0)$ removed) is not a Zariski-closed subset.

Proposition 3.2. *Every open and closed subspace of a separated abstract algebraic set is separated. In particular, any affine abstract algebraic set is separated, as is an open subspace of an affine abstract algebraic set.*

Proof. Let X be a separated abstract algebraic set. For any injective map of abstract algebraic sets $f : Y \to X$, consider the product map $f \times f : Y \to Y \to X$ which arises from the mapping property of products applied to the two composite maps

$$Y \times Y \rightrightarrows Y \xrightarrow{f} X.$$

On underlying sets this is $(y, y') \mapsto (f(y), f(y'))$, so by injectivity of f we see $(f \times f)^{-1}(\Delta_X(X)) = \Delta_Y(Y)$. Since $f \times f$ is continuous and $\Delta_X(X)$ is closed in $X \times X$, $\Delta_Y(Y)$ is closed in $Y \times Y$. ∎

For spaces built by gluing, we have the following useful separatedness criterion.

Proposition 3.3. *Let X be an abstract algebraic set and $\{X_i\}$ an open cover. Let $X_{ij} = X_i \cap X_j$. Then X is separated if and only if each X_i is separated and for $i \neq j$ the subset subset $\Delta(X_{ij}) \subset X_i \times X_j$ is closed.*

Proof. The product $X \times X$ has an open cover given by the products $X_i \times X_j$. Thus, $\Delta(X)$ is closed in $X \times X$ if and only if it meets each open subset $X_i \times X_j$ in a closed subset. For $i = j$ we have $\Delta(X) \cap (X_i \times X_i) = \Delta(X_i)$, so its closedness in $X_i \times X_i$ is the separatedness of each X_i. For $i \neq j$,

$$\Delta(X) \cap (X_i \times X_j) = \mathrm{image}(\Delta(X_{ij}) \to X_i \times X_j).$$

∎

Corollary 3.4. *Every projective space $\mathbf{P}^n$ is separated.*

Proof. We apply Proposition 3.3 to the open cover of $\mathbf{P}^n$ given by the affine opens $U_i = \{x_i \neq 0\}$. Each of these is separated, so the problem is to show that if $i \neq j$ then $U_i \cap U_j$ has Zariski-closed image under its diagonal embedding into $U_i \times U_j$.

Denote the coordinate ring of U_i as $k[t_{hi}]$ as usual (with $0 \leq h \leq n$ and t_{ii} understood to mean 1), so the open subset $U_i \cap U_j \subset U_i$ is $\{t_{ji} \neq 0\}$ and likewise the open subset $U_i \cap U_j \subset U_j$ is $\{t_{ij} \neq 0\}$. The diagonal image of this overlap in $U_i \times U_j$ corresponds to the locus of points $(u, u') \in U_i \times U_j$ in which $t_{ji}(u) \neq 0$, $t_{ij}(u') \neq 0$ and u corresponds to u' under the transition isomorphism

$$\{\xi \in U_i \,|\, t_{ji}(\xi) \neq 0\} \simeq \{\xi' \in U_j \,|\, t_{ij}(\xi') \neq 0\}.$$

This isomorphism is given by the formula $(t_{hi})_{h \neq i} \mapsto (t_{hi}/t_{ji})_{h \neq i'}$ where as usual $t_{rr} = 1$ for any r. In other words, we are interested in the closedness of the subset

$$\{(u, u') \in U_i \times U_j \,|\, t_{ji}(u) \neq 0, t_{ij}(u') \neq 0, t_{hj}(u') = t_{hi}(u)/t_{ji}(u) \text{ for all } h \neq j\}.$$

The case $h = j$ can certainly be permitted (as it asserts the tautology $1 = t_{ji}(u)/t_{ji}(u)$). The case $h = i$ says $t_{ij}(u') = 1/t_{ji}(u)$, or equivalently $t_{ij}(u')t_{ji}(u) = 1$. This latter equation forces $t_{ij}(u'), t_{ji}(u) \neq 0$, so we arrive at the description

$$\{(u, u') \in U_i \times U_j \,|\, t_{hj}(u')t_{ji}(u) = t_{hi}(u) \text{ for all } h\}.$$

This is visibly Zariski-closed in $U_i \times U_j$. (In the special case $n = 1$ and $i = 0, j = 1$ this is the Zariski-closed hyperbola $xy = 1$ in $U_0 \times U_1 = \mathbf{A}^2$.) ∎

4. Rational maps

Our interest in separatedness is that it rescues a very important application of the Hausdorff property from ordinary topology, now within the context of the Zariski topology in algebraic geometry (where the spaces are typically very non-Hausdorff):

Proposition 4.1. *Let X and Y be abstract algebraic sets. If two morphisms $f, g : X \rightrightarrows Y$ coincide on a dense subset and Y is separated then $f = g$.*

The conclusion fails without separatedness; the two natural inclusions of the affine line into the non-separated "line with a double origin" coincide on $\mathbf{A}^1 - \{0\}$ but are not the same map (they send 0 to different places in the target).

Proof. Consider the morphism $(f, g) : X \to Y \times Y$ given on underlying sets by $x \mapsto (f(x), g(x))$. To show $f = g$ is to say that this factors through the diagonal $\Delta = \Delta_Y(Y) \subset Y \times Y$, or equivalently that the preimage $(f, g)^{-1}(\Delta)$ coincides with X. By hypothesis this preimage contains a dense subset of X, yet it is also closed since Δ is closed in $Y \times Y$ (due to the separatedness hypothesis!). Thus, this preimage must indeed equal X. ■

We will be interested in working with locally closed subsets of projective spaces, so the preceding proposition becomes very important to ensure that morphisms defined on a dense open subset of the source have at most one extension to the entire space. Here is the key fact which we need:

Corollary 4.2. *Let X, Y be abstract algebraic sets with Y separated. Let $\{U_i\}$ be a collection of dense open subsets of X and $f_i : U_i \to Y$ morphisms such that f_i and f_j coincide on a dense subset of $U_i \cap U_j$ for all i and j. Then f_i and f_j coincide on the entirety of $U_i \cap U_j$ for all (i, j) and the f_i uniquely "glue" to define a morphism $\cup_i U_i \to Y$ which recovers f_i on U_i for all i.*

Proof. The preceding proposition applied to the restrictions of f_i and f_j to $U_i \cap U_j$. implies the equality on this entire overlap. Then we can glue the maps topologically as continuous maps, and check that this glued map is a morphism of ringed spaces since such a property for a *continuous* map is local on the source (e.g., it suffices to check it on each U_i separately). ■

Example 4.3. Let $f : U \to Y$ be a morphism defined on a dense open subset U of an abstract algebraic set X, and assume Y is separated. It is natural to seek a "maximal domain of definition" for f. That is, we consider open subsets $U' \subset X$ containing U (so U' is dense in X) such that f extends to a morphism $f' : U' \to Y$ (necessarily uniquely determined if it exists, since U is dense in U' and Y is separated).

The key point is that there is exactly one "maximal" such U' in the sense that it contains all others. Indeed, if we let (U_i, f_i) be the set of *all* open subsets of X containing U such that f extends to a morphism $f_i : U_i \to Y$ then by the preceding corollary these f_i's glue to a morphism $f' : U' = \cup_i U_i \to Y$. Hence, (U', f') is also in the collection, yet (U_i, f_i) was taken to be the set of *all* such pairs. Hence, (U', f') is indeed the unique maximal pair as claimed.

In general, a *rational map* from an abstract algebraic set X to a separated abstract algebraic set Y is an equivalence class of pairs (U, f) where $U \subset X$ is dense open and $f : U \to Y$ is a morphism, with $(U, f) \sim (V, g)$ precisely when f and g agree on a dense open subset of $U \cap V$. This formulation makes it easy to check that this really is an *equivalence relation* (the point being transitivity, for which we use that an intersection of dense open subsets is dense open, so there is no need to keep track of "triple overlap compatibility"!). The preceding example shows that within any such equivalence class there is a unique member (U, f) for which U is as big as possible, and we then call U the *domain of definition* of the rational map.

For example, if X is irreducible then a rational map (U, f) to the affine line is nothing more or less than an element $f \in \mathscr{O}_X(U)$ for a non-empty open subset U (check!), in which case the *domain of definition* is exactly the domain of definition of this regular function in the sense of Exercise 1 in HW3 (which treated the affine case). In class we will see how to interpret rational maps in terms of function fields, which makes it very easy to construct lots of rational maps. The hard part in practice is to nail down exactly the domain of definition of a rational map (e.g., see Exercise 3(iii) in HW4). We illustrate the subtleties with an interesting class of examples:

Choose homogenous $f_0, \dots, f_m \in k[T_0, \dots, T_n]$ not all zero such that the nonzero f_j all have the same degree, and let φ_f be the rational map from $\mathbf{P}_k^n$ to $\mathbf{P}_k^m$ given by $[f_0, \dots, f_m]$. (with $n, m \geq 1$). The common zero locus $\underline{Z}(f)$ of the f_j's is a *proper* closed subset of $\mathbf{P}_k^n$, so the domain of definition of φ_f contains the non-empty open set $\mathbf{P}_k^n - \underline{Z}(f)$ (as we saw in class).

The question is: what is the domain of definition of φ_f? There are hidden subtleties because if the f_j's all have a common non-constant factor h (necessarily homogenous, by Proposition 1.1 in the handout on Bezout's Theorem), say $f_j = hg_j$, then $[g_0, \dots, g_m]$ is another expression for the same rational map (since scaling by $h(x) \neq 0$ is harmless on points in $\mathbf{P}_k^m$!) yet the open set $\mathbf{P}_k^n - \underline{Z}(g)$ is generally larger than $\mathbf{P}_k^n - \underline{Z}(f)$ (as we don't need to remove the entire zero locus of h). By stripping away the homogenous greatest common factor of the f_j's in this way, we bring ourselves to the situation in which the GCD is 1. It is natural to wonder if this GCD obstruction is the only one to being in the domain of definition of this map. The answer is affirmative:

Theorem 4.4. *If the greatest common divisor of the f_j's is 1, then the rational map φ_f has domain of definition exactly $\mathbf{P}_k^n - \underline{Z}(f)$.*

This result fails completely for more general projective varieties in place of $\mathbf{P}^n$.

Proof. Choose $x \in \mathbf{P}_k^n$ at which all of the f_j's vanish (if there is no such x, there's nothing to show). In particular, each nonzero f_j has a common *positive* degree (though some f_j's might be 0). Assume that φ_f can be defined at x. We seek a contradiction.

For clarity of notation, let $\{U_i^{(n)}\}$ denote the standard open affine spaces covering $\mathbf{P}^n$ and $\{U_j^{(m)}\}$ the same for $\mathbf{P}^m$. By a projective linear change of coordinates, we may assume $x = [1, 0, \dots, 0]$ and $\varphi_f(x) = [1, 0, \dots, 0]$, with $\varphi_f((U_0^{(n)})_b) \subseteq U_0^{(m)}$, where $b \in k[T_i/T_0]$ is non-vanishing at 0 (the point corresponding to x in $U_0^{(n)} \simeq \mathbf{A}_k^n$). Note in particular that $f_0 \neq 0$, for otherwise the rational map φ_f would take *some* non-empty open (not necessarily containing x!) into the hyperplane locus $H = \{T_0 = 0\} \subseteq \mathbf{P}_k^m$, in which case the entire domain of definition of φ_f would have to map under φ_f into H (as the preimage of H would be a dense closed subset of the domain of definition), contradicting that $\varphi_f(x) = [1, 0, \dots, 0]$.

Identifying $U_0^{(n)}$, $U_0^{(m)}$ with $\mathbf{A}_k^n$, $\mathbf{A}_k^m$ as usual, the map $\varphi_f : (U_0^{(n)})_b \to U_0^{(m)}$ is given by an m-tuple of elements a_j/b^N with $a_j \in k[T_i/T_0]$. Replace b by b^N without loss of generality. Since $f_0 \neq 0$, the ratios $f_j/f_0 \in k(T_i/T_0)$ make sense and define φ_f from some non-empty open in $U_0^{(n)}$ into $U_0^{(m)}$.

We conclude that for each $1 \leq j \leq m$, the rational functions a_j/b and f_j/f_0 on $U_0^{(n)} \simeq \mathbf{A}_k^n$ coincide on a non-empty open, whence coincide as elements in the function field. That is, $a_j/b = f_j/f_0$ inside of the fraction field of $k[T_i/T_0]$. If we let A_j and B denote T_0-homogenizations of a_j and b (with the same common degree), then $a_j/b = A_j/B$, so upon injecting the fraction field of $k[T_i/T_0]$ into the fraction field of $k[T_0, \dots, T_n]$ in the natural way, we have $A_j/B = f_j/f_0$, so $f_0 | Bf_j$ for all $j > 0$. But we chose the f's to have no common non-constant factor, so every irreducible factor of the *non-zero* positive degree homogenous f_0 must divide B. But $f_0(x) = 0$, so we conclude that $B(x) = 0$. But B was constructed as a T_0-homogenization of b, with $b(0) \neq 0$, while $x = [1, 0, \dots, 0]$ forces $b(0) = B(x)$. Since $B(x) = 0$, we have a contradiction. ■