Math 145. Dimension theory for locally closed subsets

Recall that the *dimension* of a topological space X (for applications to algebraic geometry) is defined to be the supremum over all $n \geq 0$ such that X contains a strictly increasing chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

with each Z_i an irreducible closed subset of X. We have seen in class that if Y is a subspace of X then $\dim Y \leq \dim X$. In this handout, we address the behavior of dimension on "reasonable" subspaces of X.

Let $Y \subset X$ be a subset that is *locally closed* in the following sense: for all $y \in Y$ there exists an open set $U_y \subset X$ containing y such that $Y \bigcap U_y$ is closed in U_y. For visualization purposes, the basic example to keep in mind is to begin with a closet set $Z \subseteq X$ and to remove a closed subset of Z. This gives *all* examples. That is, a subset Y of X is locally closed if and only if $Y = Z \bigcap U$ for a closed set Z in X and an open set U in X. Indeed, the implication "$\Leftarrow$" is clear, and for the converse we note that $U := \bigcup_{y \in Y} U_y$ contains Y as a closed subset (since $U - Y$ meets each open U_y in the subset $U_y - (Y \cap U_y)$ that is open in U_y, and an arbitrary union of open subsets is open). Thus, $Y = Z \bigcap U$ for a closed set Z in X as desired. Locally closed sets arise very often in algebraic geometry: we may begin with a closed set Z in k^n, and then pass to the open set $Z \cap \{f \neq 0\}$ for some $f \in k[x_1, \dots, x_n]$.

Another way to express the "locally closed" condition is that Y is open in its closure $\overline{Y}$. Indeed, since the formation of closures in topological spaces commutes with intersecting against an open subset (check!), if $Y = Z \cap U$ for closed Z in X and open U in X then $\overline{Y} \cap U$ is the closure of Y in U, which is simply Y. That is, $Y = U \cap \overline{Y}$ is open in $\overline{Y}$. Conversely, if Y is open in $\overline{Y}$ then $Y = Z \bigcap U$ where $Z = \overline{Y}$ is closed in X and U is some open subset of X.

Our main goal is to prove:

Theorem 0.1. *If Y is locally closed in an affine algebraic set X then $\dim Y = \dim \overline{Y}$.*

To prove this, we will use the result (proved in class) that in the irreducible case, $\dim X = \mathrm{trdeg}_k k(X)$ where $k(X)$ denotes the fraction field of the coordinate ring $k[X]$ (a domain since X is irreducible).

First note that we can replace X by $\overline{Y}$, bringing us to the case where $Y = U$ is a *dense open* subset of X. In such cases we seek to prove that $\dim U = \dim X$. The first key step is to pass to the case when X is irreducible, as follows. If $\{X_1, \dots, X_m\}$ is the set of irreducible components of X then $Z_1 = X_2 \cup \dots \cup X_m$ is closed in X and not equal to X (as $X_1 \not\subset Z_1$), so $U \cap (X - Z_1)$ is a *non-empty* open subset of X_1 since U is dense in X and $X - Z_1$ is a non-empty open subset of X. Hence, $U \cap X_1$ is also a non-empty open subset of X_1. Likewise, $U \cap X_i$ is a non-empty open subset of X_i for all i.

Open sets in irreducible spaces are dense, so each $U \cap X_i$ is irreducible (as its closure X_i in X is irreducible) and $U \cap X_i$ is not contained in $U \cap X_{i'}$ whenever $i \neq i'$ (as their respective closures X_i and $X_{i'}$ in X are distinct). Hence, $\{U \cap X_i\}$ is the finite set of irreducible components of the noetherian topological space U, so $\dim U = \max \dim(U \cap X_i)$. Likewise, $\dim X = \max \dim X_i$. It therefore suffices to show that $\dim(U \cap X_i) = \dim X_i$ for each i, so we may now assume that X is irreducible. Thus, *every* non-empty open subset of X is dense.

The next step is to pass to a more concrete U. Pick $x_0 \in U$, and let J be the ideal in the coordinate ring $k[X]$ of X corresponding to the closed set $X - U$ in X. There must be some $f \in J$ that is non-vanishing at x_0 (since $x \notin X - U = \underline{Z}(J)$), so $X_f := \{x \in X \mid f(x) \neq 0\} = X - \underline{Z}(f)$ is an open subset of X that contains x_0 (so it is non-empty) and is contained in U (since $f \in J$). That is,

$$x_0 \in X_f \subseteq U \subseteq X,$$

so $\dim X_f \leq \dim U \leq \dim X$ and hence it suffices to prove that $\dim X_f = \dim X$. That is, we can replace U with X_f for some nonzero f in the coordinate ring $k[X]$.

Now comes the key point. Recall that we are granting the link between dimension theory of affine algebraic sets and transcendence degree: $\dim X = \text{trdeg}_k k(X)$. We shall prove likewise that $\dim X_f = \text{trdeg}_k k(X)$. Although X_f is open in X and generally not closed, the Rabinowitz trick that arose in the proof of the Nullstellensatz will enable us to *homeomorphically* identify the irreducible open set X_f with an affine algebraic set whose function field is also $k(X)$ (so we would be done).

Consider the affine space k^N (with coordinates $x_1, \ldots, x_N$) that contains X as $\underline{Z}(J)$ for a radical ideal J. Since X is irreducible, $J = P$ is even prime. The Rabinowitz trick identifies X_f as a set with the affine algebraic set X' in k^{N+1} (with coordinates $x_1, \ldots, x_N, t$) defined by killing J and $tf(x) - 1$. Explicitly, the continuous projection map $k^{N+1} \to k^N$ to the first N coordinates restricts to a continuous map $X' \to X_f$ that is visibly bijective with inverse $x \mapsto (x, 1/f(x))$.

Grant for a moment that this set-theoretic inverse map $X_f \to X'$ is also continuous. We conclude that X_f is (naturally) homeomorphic to X', so by the topological nature of the definition of dimension we see that $\dim X_f = \dim X' = \text{trdeg}(k(X'))$. But the coordinate ring $k[X']$ is k-isomorphic to $k[X][t]/(tf - 1) \simeq k[X]_f$, so $k[X']$ is a domain whose fraction field is $\text{Frac}(k[X]_f) = \text{Frac}(k[X]) = k(X)$. Hence, comparing transcendence degrees over k then gives that $\dim X_f = \dim X' = \text{trdeg}_k k(X) = \dim X$, so we would be done. It remains (granting the equality $\dim X = \text{trdeg}_k(k(X))$ for irreducible X) to show that the inverse map $X_f \to X' \subset k^{N+1}$ is continuous. We shall show that the preimage of any closed set in k^{N+1} is closed in X_f (which will give the result, since the topology on X' is defined to make it closed in k^{N+1}.

Since every closed set in k^{N+1} is an intersection of finitely many hypersurfaces, it suffices to show that for any $h \in k[x_1, \ldots, x_N, t]$ the preimage in X_f of $\underline{Z}(h) \subset k[X']$ is closed in X_f. This preimage consists of points $x \in k^N$ such that $h(x, 1/f(x)) \neq 0$. If h has degree d then we can write

$$h(c_1, \ldots, c_n, 1/f(c)) = \frac{H(c_1, \ldots, c_n)}{f(c)^d}$$

where H is an auxiliary polynomial in $k[x_1, \ldots, x_n]$, so the visibly closed subset $\underline{Z}(H) \subset k^N$ is the preimage in X' of $\underline{Z}(h) \subset X'$.