Math 145. Codimension

1. Main result and some interesting examples

In class we have seen that the dimension theory of an affine variety (irreducible!) is linked to the structure of the function field in the sense that $\dim Z = \mathrm{trdeg}_k(k(Z))$. In particular, we proved that all strictly increasing chains of irreducible closed subsets of Z have length uniformly bounded by $1+\mathrm{trdeg}_k(k(Z))$, with some such chain achieving this maximal length. But to make a sufficiently robust geometric theory of dimension, we need the following result (to be proved in the next section).

Theorem 1.1. *For every maximal chain of irreducible closed sets*

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Z$$

in an affine variety Z (so Z_0 is a point, by maximality), necessarily $n = \dim Z$.

In this theorem, "maximal" means (since Z_0 is a point and Z is irreducible) that the chain cannot be made longer by inserting an irreducible closed set strictly between some Z_i and Z_{i+1}. Since $\dim Z_0 = 0$ and $\dim Z_{i+1} > \dim Z_i$ for all i (as for any strict inclusion between irreducible closed subsets of k^N), we have

$$\dim Z_i = \sum_{1 \le j \le i} (\dim Z_j - \dim Z_{j-1})$$

with each of the i differences an integer ≥ 1. But for $i = n$ we have $\dim Z_n = \dim Z = n$, so each of the differences $\dim Z_j - \dim Z_{j-1}$ must be exactly 1 and hence $\dim Z_i = i$ for all i.

In other words, for a maximal chain of irreducible closed sets every jump involves an increase of exactly 1 in the dimension. For example, in a 3-dimensional affine variety Z any maximal chain of irreducible closed subsets must be a point in an irreducible curve in an irreducible surface in Z. This has the following important corollary.

Corollary 1.2. *If Z' is an affine variety of dimension d' and $Z \subseteq Z'$ is an irreducible closed subset of dimension $d \le d'$ then every maximal chain of irreducible closed sets beginning at Z and ending at Z' has the form*

$$Z = Z_d \subsetneq \cdots \subsetneq Z_{d'} = Z'$$

with $\dim Z_i = i$ for every $d \le i \le d'$.

Proof. By maximality of such a chain, if we append on the left a maximal chain contained in Z then we get a maximal chain in Z', so by the Theorem applied to Z' this new chain (which begins at a point) must have $1 + \dim Z'$ terms and $\dim Z_i = i$ for all i. ∎

In view of this corollary, we have a good notion of *codimension* in the irreducible case:

Definition 1.3. For an affine variety Z' and an irreducible closed subset Z, the *codimension* $c = \mathrm{codim}_{Z'}(Z)$ of Z in Z' is the unique integer c such that every maximal chain of irreducible closed sets beginning at Z and ending at Z' has $c+1$ terms. Equivalently, $c = \dim Z' - \dim Z$.

There is a reasonable definition of codimension without irreducibility hypotheses (i.e., allowing Z or Z' to be reducible), but it is not as geometrically significant as in the irreducible case, so we won't discuss it.

We end this introductory section with some instructive examples. Inspired by linear algebra, it is natural to wonder if we can define codimension $c = \mathrm{codim}_{Z'}(Z)$ in terms of "minimal number of equations needed to cut out Z inside Z'". This can be interpreted in two reasonable ways. Since $k[Z] = k[Z']/J$ for a radical ideal J, we can consider the minimal number of generators of the ideal J *or* the minimal number of generators of *some* ideal $I \subset k[Z']$ such that $\mathrm{rad}(I) = J$. This latter condition is a weaker requirement (as we do not specify which I to use), but even for this it turns out that working with the number of equations does *not* give the right notion in general.

The "problem" is that we are working too globally. It turns out that in a suitable "local" sense (in the Zariski topology) one can always find a set of c "local equations" that define Z as a subset of Z' near an arbitrary chosen point $z \in Z$, but the proof rests on much deeper work in the dimension theory of local noetherian rings. For cutting out the entirety of Z in Z', there are counterexamples if we try to use only c equations. We now describe two such counterexample, but we omit the justifications (which require techniques in commutative algebra beyond the level of this course).

Our first counterexample arises from the theory of elliptic curves (but the next example will be more geometrically compelling).

Example 1.4. Consider the irreducible plane curve $Z' = \{y^2 = x^3 + 17\}$ in k^2 for an algebraically closed field k of characteristic 0. The point $Z = \{(-2, 3)\}$ in Z' has codimension 1, but it can be shown that there is *no* element $f \in k[Z']$ such that the zero locus of f on Z' is Z. The key input to this is the fact that $(-2, 3)$ has infinite order in the "group law" on an elliptic curve arising from Z', a fact which uses non-obvious results in the arithmetic theory of elliptic curves.

Our next counterexample (due to Hartshorne) works in any characteristic, involving a surface in 4-space. Roughly speaking, we consider a surface S obtained from the plane by identifying the points $(0, 0)$ and $(0, 1)$. More rigorously:

Example 1.5. Consider the k-subalgebra

$$A = \{f \in k[t, u] \mid f(0, 0) = f(0, 1)\} \subset k[t, u].$$

(The equation defining A corresponds to the geometric idea of identifying the points $(0, 0)$ and $(0, 1)$.) I claim that A is the k-subalgebra generated by 4 elements:

$$t, tu, u(u-1) = u^2 - u, u^2(u-1) = u^3 - u^2.$$

Geometrically, this means that A is the coordinate ring of a surface in affine 4-space.

Clearly the 4 indicated elements of $k[t, u]$ lie in A. To prove that they generate A as a k-algebra, consider an arbitrary element $f \in A$. Since any $n > 1$ has the form $2a + 3b$ with integers $a, b \geq 0$, we can use the elements $u^2 - u$ and $u^3 - u^2$ in our list to write f in the form

$$f = h(t, u^2 - u, u^3 - u^2) + ug(t)$$

for some $h \in k[x, y, z]$. Likewise, $ug(t) = cu + tuG(t)$ for some $c \in k$ and $G \in k[t]$. Hence, we have expressed f as an element of $k[t, tu, u(u-1), u^2(u-1)]$ up to adding an element of the form cu. But $cu \in A$ precisely when $f \in A$ (as $t, tu, u(u-1), u^2(u-1) \in A$), and $cu \in A$ if and only if $c = 0$, so $f \in A$ if and only if $c = 0$. Thus, the asserted list of k-algebra generators of A really does work.

To summarize, we see that there is a surjective map $\pi : k[x, y, z, w] \twoheadrightarrow A$ via

$$x \mapsto t, y \mapsto u(u-1), z \mapsto tu, w \mapsto u^2(u-1).$$

The kernel $P := \ker \pi$ is a prime ideal (since the quotient A is a domain), and clearly

$$xw - yz, x^2y - z(z-x), y^3 - w(w-y) \in P. \tag{1}$$

In more geometric terms, since the defining inclusion $A \hookrightarrow k[t, u]$ is injective, we see that the polynomial map $f : k^2 \to k^4$ defined by

$$f : (t, u) \mapsto (t, u(u-1), tu, u^2(u-1))$$

has image contained in $\underline{Z}(P)$ and *dense* in $\underline{Z}(P)$ (since $k[x, y, z, w]/P = A \hookrightarrow k[t, u]$ is injective). Since the injective map $k[x, y, z, w]/P = A \to k[t, u]$ is module-finite (e.g., $t \in A$ and $u^2 - u \in A$), the geometric map $k^2 \to \underline{Z}(P)$ is finite *surjective*, so $\dim \underline{Z}(P) = \dim k^2 = 2$. Thus, $Z := \underline{Z}(P)$ is an irreducible surface in k^4; it has codimension 2.

The elements in (1) vanish on Z, and they do cut out Z set-theoretically; i.e., their common zero locus in k^4 is $Z = f(k^2)$. Indeed, consider a point (x_0, y_0, z_0, w_0) that satisfies all three relations

$$x_0w_0 = y_0z_0, x_0^2y_0 = z_0(z_0 - x_0), y_0^3 = w_0(w_0 - y_0).$$

We seek $(t_0, u_0) \in k^2$ such that

$$(x_0, y_0, z_0, w_0) = (t_0, u_0(u_0 - 1), t_0u_0, u_0^2(u_0 - 1)).$$

The "easy" case is when $x_0 \neq 0$, in which case we define $u_0 = z_0/x_0$ and $t_0 = x_0$; this works since

$$u_0(u_0 - 1) = z_0(z_0 - x_0)/x_0^2 = x_0^2y_0/x_0^2 = y_0, u_0^2(u_0 - 1) = u_0y_0 = z_0y_0/x_0 = w_0.$$

Suppose instead that $x_0 = 0$, so clearly $z_0 = 0$ and our point is $(0, y_0, 0, w_0)$ with $y_0^3 = w_0(y_0 - w_0)$. If $y_0 = 0$ then $w_0 = 0$ and so we can take $t_0 = u_0 = 0$. If $y_0 \neq 0$ then we can take $t_0 = 0$ and $u_0 = w_0/y_0$. This completes the proof that Z is the common zero locus of the elements in (1). (Note that we have not addressed whether or not these three elements of P in fact generate P. This is not necessary to know.)

To prove that Z cannot be the set of common zeros of a pair of polynomials in $k[x, y, z, w]$, one has to use deeper techniques from commutative algebra (related to completions and connectedness properties of Cohen-Macaulay rings). This is explained in Example 3.4.2 of Hartshorne's paper "Complete intersections and connectedness".

2. Proof of Theorem 1.1

We shall prove that if Z is an affine variety over an algebraically closed field k then all maximal chains of irreducible closed subsets of Z have length $1 + \dim Z$. (The argument works over any infinite field, but we prefer to use geometric language in some parts rather than pure commutative algebra, hence our need to restrict to an algebraically closed ground field.) We argue by induction on $\dim Z$. If $\dim Z = 0$ then Z is a point and the result is clear. Thus, we may assume that the common value $\dim Z = \operatorname{trdeg}_k(k(Z))$ is positive. Since any maximal chain of irreducible closed subsets in Z ends with $V \subsetneq Z$ where V is maximal among irreducible proper closed subsets of Z, our task is equivalent to showing that if $V \subsetneq Z$ is a maximal irreducible proper closed subset of Z then

$$\dim(V) \stackrel{?}{=} \dim(Z) - 1,$$

as then we can apply dimension induction to conclude. It is equivalent to show that $\operatorname{trdeg}_k(k(V)) = \operatorname{trdeg}_k(k(Z)) - 1$.

We first treat the special case $Z = k^d$ with $d > 0$, and then we will use this case to handle the general case via the Noether normalization theorem. For $Z = k^d$ we claim that the maximal proper irreducible closed subsets are precisely the irreducible hypersurfaces $V = \underline{Z}(f)$ for an irreducible $f \in k[Z] = k[T_1, \dots, T_d]$. Indeed, if P is any nonzero prime ideal of this polynomial ring then it contains a nonzero polynomial and thus (by primality) contains one of its irreducible factors f. That is, P contains (f), so the prime ideals (f) for irreducible $f \in k[T_1, \dots, T_d]$ are precisely the minimal nonzero primes of this polynomial ring. This yields the asserted description of the maximal irreducible proper closed subsets of k^d. Our task in this special case is to show that $\dim(\underline{Z}(f)) = d - 1$.

By relabeling variables we can assume that f involves T_d, so

$$f = a_n(T_1, \dots, T_{d-1})T_d^n + \cdots \in k[T_1, \dots, T_{d-1}][T_d]$$

with $n > 0$, the omitted terms of lower degree in T_d, and $a_n \in k[T_1, \dots, T_{d-1}]$ nonzero. Since f is irreducible and *involves* T_n, it is easy to see (check!) that f does not divide any nonzero element of $k[T_1, \dots, T_{d-1}]$. Thus, the natural map

$$k[T_1, \dots, T_{d-1}] \to k[T_1, \dots, T_d]/(f) = k[\underline{Z}(f)]$$

between domains is *injective*, yet the induced map of fraction fields

$$k(T_1, \dots, T_{d-1}) \to k(\underline{Z}(f))$$

is finite algebraic since the element $T_d \in k(\underline{Z}(f))$ satisfies the positive-degree algebraic relation over $k(T_1, \dots, T_{d-1})$ given by the condition $f = 0$. Thus, by additivity of transcendence degree in towers of finitely generated field extensions,

$$\dim(\underline{Z}(f)) = \operatorname{trdeg}_k(k(\underline{Z}(f))) = \operatorname{trdeg}_k(k(T_1, \dots, T_{d-1})) = d - 1,$$

as desired.

Now we consider the general case with $d = \dim(Z) > 0$. By Noether normalization there is a finite surjection $f : Z \to k^d$, so the image $V' = f(V) \subseteq k^d$ is an irreducible closed subset and $V \to V'$ is a finite surjection. Thus, $\dim V' = \dim V < \dim Z = d$, so $V' \neq k^d$. It suffices to show that $\dim(V') = d - 1$, so in view of the special case just treated above it suffices to show that V' is maximal as an irreducible proper closed subset of k^d. Recall that V is maximal as an irreducible proper closed subset of Z, by hypothesis. It therefore suffices to apply the following general result to $Z \to k^d$.

Proposition 2.1 (weak going-down theorem)**.** *Let $f : Z \to Z'$ be a finite surjective map between affine varieties over an algebraically closed field k. Assume that $k[Z']$ is an integrally* *closed domain, and let $V \subseteq Z$ be a maximal proper irreducible closed subset. The irreducible closed image $V' = f(V) \subseteq Z'$ is a maximal proper irreducible closed subset of Z'.*

This proposition is true without the "integrally closed" condition once Theorem 1.1 is *proved*, as one sees by then simply considering dimensions (as the hypothesis is then saying $\dim V = \dim Z - 1$ and the conclusion is saying $\dim V' = \dim Z' - 1$, and we know that dimension does not change under finite surjective maps between irreducible affine algebraic sets).

Proof. Let d denote the common dimension of Z and Z' (which we may assume to be positive). Since $\dim(V') = \dim(V) < d$, we certainly have that $V' \neq Z'$. We assume to the contrary that there is an irreducible closed subset W' strictly between V' and Z'. In terms of commutative algebra with the domain $A = k[Z]$ and its integrally closed k-subalgebra $A' = k[Z']$, $P' = \underline{I}(V')$ is a prime ideal of A' strictly containing the nonzero prime ideal $Q' = \underline{I}(W')$ of A' and by construction of V' we know that $P = \underline{I}(V)$ is a prime ideal of A lying over P' (i.e., $P \cap A' = P'$, which is to say $A'/P' \hookrightarrow A/P$). Also, by hypothesis P is minimal as a nonzero prime ideal of A (expressing the maximality hypothesis on V in Z). From this situation we shall deduce a contradiction.

Let $W = f^{-1}(W')$, a closed (possibly reducible!) subset of Z that maps onto W'. By the Nullstellensatz, the ideal $\underline{I}(W) \subset k[Z] = A$ is $\mathrm{rad}(Q'A)$. The finiteness of $W \twoheadrightarrow W'$ implies that each irreducible component of W is finite onto an irreducible closed subset of W', which in turn has dimension $< d$, so the irreducible components of W have dimension $< d$. Thus, these components are all distinct from Z. But V is an irreducible closed subset of W, hence it lies in one of the irreducible components of W, yet V is maximal as an irreducible closed subset of Z! Since we have just seen that all irreducible components of W are distinct from Z, this forces V to exhaust an irreducible component of W that contains it; i.e., V *is* an irreducible component of W. But $f(V) = V' \neq W' = f(W)$, so $V \neq W$. In other words, W must be *reducible*.

Consider the irreducible components $\{V = W_1, \dots, W_s\}$ of W. Since $W_2 \cup \cdots \cup W_s$ is a proper closed subset of W, removing it leaves a *non-empty* open subset U of W that is contained in the irreducible V, so U is also irreducible. A base for the topology of W consists of "hypersurface complements" $W_a = W \cap \{a \neq 0\}$ for $a \in A = k[Z]$, so some W_a is contained in U. In particular, W_a is irreducible and must coincide with V_a. By the Rabinowitz trick, W_a is homeomorphic to the affine algebraic set associated to the finitely generated k-algebra

$$k[W]_a = (A/\underline{I}(W))_a = (A/\mathrm{rad}(Q'A))_a \simeq A_a/\mathrm{rad}(Q'A_a),$$

where the final equality is HW2, Exercise 3(iii). By the irreducibility this ring is a *domain.* But the ideal P of $A = k[Z]$ that cuts out $V \subset Z$ vanishes at all points of the non-empty $V_a = W_a$, so $a \notin P$ and hence PA_a is a *prime* ideal of A_a containing $Q'A_a$, so it contains $\mathrm{rad}(Q'A_a)$. We conclude that $PA_a/\mathrm{rad}(Q'A_a)$ is a prime ideal of the domain $k[W]_a$ and yet vanishes at all of its "points", so this is the ideal (0). That is, $PA_a = \mathrm{rad}(Q'A_a)$. In particular, every $h \in P$ satisfies $h^n \in Q'A_a$ for some $n > 0$, so some $a^m h^n \in Q'A$ for some $m, n > 0$.

Since P contains P', to get a contradiction we may take $h = a' \in P' - Q'$. (Such an a' exists since Q' is strictly contained in P'.) Thus, $\xi := a^m a'^n \in Q'A$. Recall from class that this implies that ξ is a root of a monic polynomial

$$H(T) = T^N + q'_{N-1}T^{N-1} + \cdots + q'_0 \in A'[T]$$

with $q'_j \in Q'$. Now we bring out a special feature of integrally closed domains (such as A'):

Lemma 2.2. *Let A' be an integrally closed domain, and F' its fraction field. For any finite extension F/F' and $\xi \in F$ that is integral over A', the minimal polynomial of ξ over F' lies in $A'[T]$ and it generates the ideal of elements of $A'[T]$ that vanish on ξ.*

This lemma asserts the striking fact that for integrally closed A', the concept of minimal polynomial works well over A' even though $A'[T]$ is not a PID (when A' is not a field).

Proof. We may replace F with its normal closure over F' so that F is normal over F'. Hence, the minimal polynomial $g \in F'[T]$ of ξ splits over F and $\mathrm{Aut}(F/F')$ transitively permutes the roots. But ξ is integral over A', so all of the roots are integral over A'. Their elementary symmetric functions are the coefficients of g in F' yet must be integral over A' (as integrality is preserved under sums and products, such as in the formation of elementary symmetric functions), so by the integral closedness of A' in F' we conclude that the coefficients of g do indeed lie in A'.

Suppose now that $h \in A'[T]$ vanishes at ξ, so $g|h$ in $F'[T]$. But g is monic over A' and $h \in A'[T]$, so if we write $h = gG$ for some necessarily monic $G \in F'[T]$ then G must lie in $A'[T]$ because otherwise we choose the maximal-degree term $c'T^r$ in G with coefficient c' not in A' and compare h and gG in degree $r + \deg(g)$. ∎

We conclude from the lemma that the minimal polynomial $h(T)$ of $\xi = a^m a'^n$ over $k(Z')$ lies in $A'[T]$ and divides $H(T) = T^N + q'_{N-1}T^{N-1} + \cdots + q'_0$ in $A'[T]$. Passing to the quotient modulo Q', it follows that $h \bmod Q'$ divides $H \bmod Q' = T^N$ in $(A'/Q')[T] = k[W'][T]$. Working in $k(W')[T]$, the factor $h \bmod Q'$ of T^N must be a power of T, yet h is monic in $A'[T]$, so we conclude that all lower-degree coefficients of h *also* lies in Q' (just like for H).

The minimal polynomial of $a^m = \xi/a'^n$ over $k(Z')$ is obtained by h by dividing the lower-degree coefficients of h by suitable powers of a', yet these coefficients also all lie in A' since the element a^m of the module-finite A'-algebra A must be integral over A' (by the preceding Lemma). The lower-degree A'-coefficients of this minimal polynomial wind up in Q' *after* multiplying by a suitable power of a' (to recover h), yet $a' \notin Q'$ by design, so primality of Q' then implies that the lower-degree coefficients of the minimal polynomial of a^m all lie in Q'! That is, we have a monic relation

$$(a^m)^N + c'_{N-1}(a^m)^{N-1} + \cdots + c'_0 = 0$$

with all $c'_j \in Q'$, so $(a^m)^N \in Q'A \subset P'A \subset P$. Primality of P then forces $a \in P$. But we saw early on from our choice of a that in fact $a \notin P$! This contradiction completes the proof of Proposition 2.1 (and hence of Theorem 1.1). ∎

Remark 2.3. The idea behind the preceding proof can be expressed in another way which illuminates the role of integral closedness. Rather than showing that a maximal proper irreducible closed set $V \subset Z$ maps onto a maximal proper irreducible closed set $V' \subset Z'$ (with Z and Z' irreducible affine algebraic sets), we can just as well try to show that if $V' \subsetneq W' \subset Z'$ is a strict containment between general irreducible closed sets in Z' and if V

is an irreducible closed set of Z that lies over V' in the sense that it maps onto V' then is V contained in an irreducible closed set W that maps onto W' (so $V \subsetneq W \subseteq Z$ if W' lies strictly between V' and Z')? *This* formulation of the problem turns out to be false when $k[Z']$ is not integrally closed.

Here is a nice counterexample. Consider the map $f : k^2 \to k^3$ defined by $(x, y) \mapsto (x(x-1), x^2(x-1), y)$. The image consists of the points $(u, v, y) \in k^3$ for which $v^2 - uv - u^3 = 0$ (as we see by setting $x = v/u$ when $u \neq 0$), so if we define

$$Z' = \{(u, v, y) \in k^3 \mid v^2 - uv - u^3 = 0\}$$

then it is easy to check that Z' is an irreducible surface and $f : k^2 \to Z'$ is a finite map (since $x^2 - x$ and y lie in the coordinate ring $k[Z'] \subset k[x, y]$). In fact, the module-finite inclusion $k[Z'] \hookrightarrow k[x, y]$ induces an equality of fraction fields since $y \in k[Z']$ and $x = v/u$ with $u, v \in k[Z']$, so $k[x, y]$ is the integral closure of $k[Z']$ in its fraction field and it is strictly larger (e.g., $x \notin k[Z']$). That is, $k[Z']$ is not integrally closed.

Geometrically, f carries both lines $L_0 = \{x = 0\}$ and $L_1 = \{x = 1\}$ onto the y-axis $L = \{u = v = 0\} \subset Z'$ in k^3, with $f^{-1}(L) = L_0 \cup L_1$. Away from L the restricted map $k^2 - (L_0 \cup L_1) \to Z' - L$ is an isomorphism between these basic affine open sets (i.e., the associated map of coordinate rings $k[x, y]_{x(x-1)} \to k[Z']_{x-y}$ is an isomorphism), so we visualize Z' as the result of making the plane k^2 pass through itself along a single line L.

Consider the diagonal line $\Delta = \{x = y\}$ in k^2 which meets $L_0 = \{x = 0\}$ in $(0, 0)$ and meets $L_1 = \{x = 1\}$ in $(1, 1)$. The image $C' = f(\Delta) \subset Z'$ is an irreducible closed set in Z' of dimension 1 that meets the common image L of L_0 and L_1 in the points $P = (0, 0, 0)$ and $Q = (0, 0, 1)$. Visually, C' is a curve in Z' that "wraps around" the surface Z', passing through the line of singularities L at the points P and Q. In particular, the preimage $f^{-1}(C') = \Delta \cup \{(1, 0)\} \cup \{(0, 1)\}$ is a disjoint union of the diagonal Δ and two isolated points $(1, 0)$ and $(0, 1)$. Thus, if we consider the irreducible closed set $V' = \{P\}$ in C' and choose the irreducible closed set $V = \{(1, 0)\}$ over V' then there is *no* irreducible closed set C in $Z = k^2$ that contains V' and maps onto the irreducible closed set C' that contains V'. Indeed, since C' is irreducible of dimension 1 then any such C would have to be irreducible of dimension 1 and yet lie in $f^{-1}(C')$ which is a disjoint union of Δ and two isolated points. That is, the only possibility for C is Δ, yet this does not contain $V = \{(1, 0)\}$!