Math 121. Uniqueness of algebraic closure

Let k be a field, and $\overline{k}/k$ a choice of algebraic closure. As a first step in the direction of proving that $\overline{k}$ is "unique up to (non-unique) isomorphism", we prove:

Lemma 0.1. *Let L/k be an algebraic extension, and L'/L another algebraic extension. There is a k-embedding $i : L \hookrightarrow \overline{k}$, and once i is picked there exists a k-embedding $L' \hookrightarrow \overline{k}$ extending i.*

Proof. Since an embedding $i : L \hookrightarrow \overline{k}$ realizes the algebraically closed $\overline{k}$ as an algebraic extension of L (and hence as an algebraic closure of L), by renaming the base field as L it suffices to just prove the first part: any algebraic extension admits an embedding into a specified algebraic closure.

Define Σ to be the set of pairs (k', i) where $k' \subseteq L$ is an intermediate extension over k and $i : k' \hookrightarrow \overline{k}$ is a k-embedding. Using the inclusion $i_0 : k \hookrightarrow \overline{k}$ that comes along with the data of how $\overline{k}$ is realized as an algebraic closure of k, we see that $(k, i_0) \in \Sigma$, so Σ is non-empty. We wish to apply Zorn's Lemma, where we define a partial ordering on Σ by the condition that $(k', i') \leq (k'', i'')$ if $k' \subseteq k''$ inside of L and $i''|_{k'} = i'$. It is a simple exercise in gluing set maps to see that the hypothesis of Zorn's Lemma is satisfied, so there exists a maximal element $(K, i) \in \Sigma$.

We just have to show $K = L$. Pick $x \in L$, so x is algebraic over K (as it is algebraic over k). If $f_x \in K[T]$ is the minimal polynomial of x, then $K(x) \simeq K[T]/f_x$. Using $i : K \hookrightarrow \overline{k}$ realizes $\overline{k}$ as an algebraic closure of K, so $f_x \in K[T]$ has a root in $\overline{k}$. Pick such a root, say r, and then we define $K[T] \to \overline{k}$ by using i on the coefficients K and sending T to r. This map kills f_x, and hence factors through the quotient to define a map of fields $K[T]/f_x \hookrightarrow \overline{k}$ extending i. Composing this with the isomorphism $K(x) \simeq K[T]/f_x$ therefore defines an element $(K(x), i') \in \Sigma$ which dominates (K, i). By maximality, this forces $(K(x), i') = (K, i)$, or in other words $K(x) = K$ as subfields of L. This holds for all $x \in L$, and says exactly $x \in K$. Thus, $L = K$, as desired. ■

As an application of the lemma, we get the "uniqueness" of algebraic closures:

Theorem 0.2. *Let $\overline{k}_1$ and $\overline{k}_2$ be two algebraic closures of k. Then there exists an isomorphism $\overline{k}_1 \simeq \overline{k}_2$ over k.*

Beware that the isomorphism in the theorem is nearly always highly non-unique (it can be composed with any k-automorphism of $\overline{k}_2$, of which there are many in general). Thus, one should *never* write $\overline{k}_1 = \overline{k}_2$; *always* keep track of the choice of isomorphism. In particular, always speak of *an* algebraic closure rather than *the* algebraic closure; there is no "preferred" algebraic closure except in cases when there are no non-trivial automorphisms over k (which happens for fields which have the property of being "separably closed", a notion we'll encounter later).

Proof. By the lemma, applied to $L = \overline{k}_1$ (algebraic over k) and $\overline{k} = \overline{k}_2$ (an algebraically closed field equipped with a structure of algebraic extension of k), there exists a k-embedding $i : \overline{k}_1 \hookrightarrow \overline{k}_2$. Since $\overline{k}_1$ is algebraic over k and $\overline{k}_2$ is algebraically closed, it follows that the k-embedding i realizes $\overline{k}_2$ as an algebraic extension of $\overline{k}_1$. But an algebraically closed field (such as $\overline{k}_1$) admits no non-trivial algebraic extensions, so the map i is forced to be an isomorphism. More concretely, any $y \in \overline{k}_2$ is a root of an irreducible monic $f \in k[T]$, and $f = \prod(T - r_j)$ in $\overline{k}_1[T]$ since $\overline{k}_1$ is algebraically closed, so applying i shows that the $i(r_j)$'s exhaust the roots of f in $\overline{k}_2$. Thus, $y = i(r_j)$ for some j, so indeed i is surjective. ■