Math 113 Homework 1 Solutions

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Exercise 1.A.2. Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

Proof. We can use the definition of complex multiplication, we have

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^2 &= \frac{-1+\sqrt{3}i}{2} \times \frac{-1+\sqrt{3}i}{2} \\ &= \frac{1}{4} - \frac{3}{4} + \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) i \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i = \frac{-1-\sqrt{3}i}{2}\end{aligned}$$

Thus

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{-1+\sqrt{3}i}{2} \times \frac{-1-\sqrt{3}i}{2} \\ &= \left(\frac{1}{4} + \frac{3}{4}\right) + \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) i \\ &= 1\end{aligned}$$

□

Exercise 1A.10. Find two distinct square roots of i.

Proof. We first explain how we can find the square roots (students did not have to write this part down). If $a + bi$ is a square root of i, this means a and b are real numbers such that

$$(a + bi)^2 = i$$

Then

$$\begin{aligned}i &= (a + bi)^2 \\ &= (a^2 - b^2) + (ab + ab)i \\ &= a^2 - b^2 + 2abi\end{aligned}$$

Since $0 + 1 \cdot i = (a^2 - b^2) + 2abi$, we conclude that $a^2 = b^2$ and $2ab = 1$. The equation $a^2 = b^2$ implies that $a = b$ or $a = -b$. However, if $a = -b$, then we would have $2ab = -2b^2 = 1$, which is impossible because b is a real number.

So we know we must have $a = b$. The equation $2ab = 1$ now becomes $2b^2 = 1$, which leads to $a = b = \pm\frac{\sqrt{2}}{2}$. Hence the only two possibilities for square roots of i are:

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \text{ and } -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i.$$

Finally, we square each of the numbers we found, to check that they really are square roots of i.

$$\begin{aligned}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2 &= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\times\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\\ &= \frac{1}{2}-\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\right)i\\ &= i\end{aligned}$$

$$\begin{aligned}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^2 &= \left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\times\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\\ &= \frac{1}{2}-\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\right)i\\ &= i\end{aligned}$$

Thus the two numbers above are indeed square roots of i. □

Exercise 1A.10. Find $x \in \mathbb{R}^4$ such that

$$(4,-3,1,7)+2x=(5,9,-6,8)$$

Proof. (We first explain how to find the vector x; students did not need to write up this part.) Subtracting $(4,-3,1,7)$ from both sides of the equations above gives

$$2x=(1,12,-7,1)$$

Multiplying both sides of the equation above by $\frac{1}{2}$ gives

$$x=(\tfrac{1}{2},6,-\tfrac{7}{2},\tfrac{1}{2})$$

We now check that this vector does indeed satisfy the desired equation.

$$\begin{aligned}(4,-3,1,7)+2x &= (4,-3,1,7)+2\cdot(\frac{1}{2},6,-\frac{7}{2},\frac{1}{2})\\ &= (4,-3,1,7)+(1,12,-7,1)\\ &= (5,9,-6,8)\end{aligned}$$

Therefore $x=(\frac{1}{2},6,-\frac{7}{2},\frac{1}{2})$ satisfies the original equation. □

Exercise 1B.1. Prove that $-(-v)=v$ for every $v \in V$.

Proof. For any $v \in V$, by definition of additive inverse of $-v$, we have $(-v)+(-(-v))=0$.

Adding v on both sides of the equation gives

$$v+(-v)+(-(-v))=v+0 \tag{$*$}$$

By definition of the additive inverse of v we know that $v+(-v)=0$, so the left side of the equation $(*)$ equals $0+(-(-v))$. By commutativity, this equals $(-(-v))+0$. Finally, this equals $-(-v)$ by definition of additive identity.

Meanwhile, the right side of $(*)$ equals v by definition of additive identity. Therefore, the equality $(*)$ implies $-(-v)=v$. □

Exercise 1B.2. Suppose $a \in \mathbb{F}, v \in V$ and $av=0$. Prove that $a=0$ or $v=0$.

Proof. We want to prove either $a = 0$ or $v = 0$. If $a = 0$, then the claim is satisfied. Therefore suppose $a \neq 0$. Multiplying both sides of the assumed equation $av = 0$ by $\frac{1}{a}$, we have

$$\tfrac{1}{a}av = \tfrac{1}{a}0$$

The associative property shows that the left side of the equation above equals $1v$, which equals v by definition of multiplicative identity. The right side of the equation above equals 0 by Proposition 1.30 in the textbook. Thus $v = 0$, completing the proof. □

Exercise 1C.4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. Let

$$U = \{f \in \mathbb{R}^{[0,1]} \colon f \text{ is continuous and } \textstyle\int_0^1 f = b\}$$

Recall that the zero element in $\mathbb{R}^{[0,1]}$ is the "zero function" $z \colon [0, 1] \to \mathbb{R}$ defined by $z(x) = 0$ for all $x \in [0, 1]$.

If U is a subspace of $\mathbb{R}^{[0,1]}$, then the zero element is in U. This means that z is continuous and $\int_0^1 z = b$. However we know that $\int_0^1 z = \int_0^1 0 = 0$, so the only way that $\int_0^1 z = b$ can happen is if $b = 0$. (Comment from TC: this shows that when $b \neq 0$, the set U violates the first property of a subspace, because it does not contain the zero element. It turns out that it also violates the second and third properties of a subspace, i.e. it is not closed under addition or multiplication. However we do not need to consider these properties; once we've shown it violates one property, it's not a subspace.)

Conversely, suppose $b = 0$, so the set U is

$$U = \{f \in \mathbb{R}^{[0,1]} \colon f \text{ is continuous and } \textstyle\int_0^1 f = 0\}.$$

Our goal is to show that U is a subspace.

First, the zero function z is continuous and $\int_0^1 z = \int_0^1 0 = 0$, so $z \in U$.

For any $c \in \mathbb{R}$, $f, g \in U$, we know $f + g$ and cf are continuous functions (we were told this on the homework sheet). We also need the properties of integration that $\int_0^1 (f + g) = \int_0^1 f + \int_0^1 g$ and $\int_0^1 (cf) = c \int_0^1 f$. (TC: this should have been on the homework sheet also.) Therefore:

$$\begin{aligned} \int_0^1 (f+g) &= \int_0^1 f + \int_0^1 g \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \int_0^1 (cf) &= c \int_0^1 f \\ &= c \cdot 0 \\ &= 0 \end{aligned}$$

Therefore $f + g \in U$ and $cf \in U$, showing that U is closed under addition and scalar multiplication. We conclude that U is a subspace of $\mathbb{R}^{[0,1]}$ □

Exercise 1.C.20. Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Proof. Let

$$W = \{(a, 0, b, 0) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}.$$

First we check W is a subspace of $\mathbb{F}^4$. The zero element 0 in the vector space $\mathbb{F}^4$ is the vector $(0, 0, 0, 0)$, thus $0 \in W$.

For any $c \in \mathbb{F}$, any $m = (m_1, 0, m_2, 0) \in W$ and any $n = (n_1, 0, n_2, 0) \in W$, we have:

$$m + n = (m_1 + n_1, 0, m_2 + n_2, 0)$$

and

$$c \cdot m = (cm_1, 0, cm_2, 0)$$

Therefore $m + n \in W$ and $c \cdot m \in W$. Therefore W is a subspace of $\mathbb{F}^4$.

Next, we check that $U \oplus W = \mathbb{F}^4$. First we will show that $U + W = \mathbb{F}^4$, then we will check that $U + W = U \oplus W$ using Prop. 1.45 (the Direct Sum Test for Two Subspaces).

1. For any $b = (b_1, b_2, b_3, b_4) \in \mathbb{F}^4$, let us define vectors $b_U = (b_2, b_2, b_4, b_4) \in U$ and $b_W = (b_1 - b_2, 0, b_3 - b_4, 0) \in W$. Adding these we see that $b = b_U + b_W$, so $b \in U + W$. This shows that every vector in $\mathbb{F}^4$ is in $U + W$, so $U + W = \mathbb{F}^4$.

2. Proposition 1.45 says that to check that $U + W = U \oplus W$, we just need to check that the intersection $U \cap W$ is $\{0\}$. So assume that $a = (a_1, a_2, a_3, a_4) \in U \cap W$ lies in both U and W. We have $a_1 = a_2$, $a_3 = a_4$ since $a \in U$, and $a_2 = 0$, $a_4 = 0$ since $a \in W$. This implies

$$a_1 = a_2 = a_3 = a_4 = 0,$$

so $a = 0$. This shows that $U \cap W = \{0\}$, so by Proposition 1.45 we know $U + W = U \oplus W$ is a direct sum.

□

Exercise 1.C.24. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Proof. 1. First, we check that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$. As above, the zero element of $\mathbb{R}^{\mathbb{R}}$ is the zero function $z \colon \mathbb{R} \to \mathbb{R}$ defined by $z(x) = 0$ for all $x \in \mathbb{R}$.

1.1. $z \in U_e$ since $z(-x) = 0 = z(x)$ for every $x \in \mathbb{R}$. Similarly, $z \in U_o$ since $z(-x) = 0 = -0 = -z(x)$ for every $x \in \mathbb{R}$.

1.2. For any $r \in \mathbb{R}, f, g \in U_e$, we know

$$(r \cdot f)(-x) = r \cdot (f(-x)) = r \cdot (f(x)) = (r \cdot f)(x)$$

since f is even. This shows that $r \cdot f$ is even, i.e. that $r \cdot f \in U_e$. Similarly

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$$

since f and g are even, showing that $f + g \in U_e$. This proves that U_e is a subspace of $\mathbb{R}^{\mathbb{R}}$.

1.3. For any $r \in \mathbb{R}, f, g \in U_o$, we know

$$(r \cdot f)(-x) = r \cdot (f(-x)) = r \cdot (-f(x)) = -(r \cdot f)(x)$$

since f is odd, showing that $r \cdot f \in U_o$. Similarly

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x)+g(x)) = -(f+g)(x)$$

since f and g are odd, showing that $f + g \in U_o$. This proves that U_o is a subspace of $\mathbb{R}^{\mathbb{R}}$.

We will use Prop. 1.45 to show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. We first show that $U_e \cap U_o = \{z\}$ (remembering that z is the zero element in $\mathbb{R}^{\mathbb{R}}$), and then show that $U_e + U_o = \mathbb{R}^{\mathbb{R}}$.

2. Assume that $f \in U_e \cap U_o$, or in other words that the function f is *both* even and odd. Then for any $x \in \mathbb{R}$ we have

$$f(x) = f(-x)$$

since f is even, but also that

$$f(-x) = -f(x)$$

since f is odd. Together this means that $-f(x) = f(x)$, which implies that $f(x) = 0$. But if $f(x) = 0$ for all $x \in \mathbb{R}$, then f is the zero function z. This shows $U_e \cap U_o = \{z\}$.

3. For any $g \in \mathbb{R}^{\mathbb{R}}$, define functions g_e and g_o by

$$g_e(x) = \frac{g(x) + g(-x)}{2}$$

and

$$g_o(x) = \frac{g(x) - g(-x)}{2}$$

We claim that $g_e \in U_e$ and $g_o \in U_o$. First we check that g_e is even:

$$\begin{aligned} g_e(-x) &= \frac{g(-x) + g(x)}{2} \\ &= \frac{g(x) + g(-x)}{2} = g_e(x). \end{aligned}$$

Next we check that g_o is odd:

$$\begin{aligned} g_o(-x) &= \frac{g(-x) - g(x)}{2} \\ &= \frac{-g(x) + g(-x)}{2} \\ &= -\frac{g(x) - g(-x)}{2} \\ &= -g_o(x). \end{aligned}$$

We also have

$$g_e + g_o(x) = \frac{g(-x) + g(x)}{2} + \frac{g(-x) - g(x)}{2} = g(x),$$

so $g_e + g_o = g$. This shows that every function g can be written as the sum of an even function g_e and an odd function g_o, proving that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.

By Prop. 1.45, we conclude that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. □

Question 1. Let S be a set, and let U be the vector space over $\mathbb{F}$. Recall that U^S is the set of functions $f : S \to U$. Given functions $f, g \in U^S$ and $a \in \mathbb{F}$, we define $f + g \in U^S$ and $a \cdot f \in U^S$ by

$$(f + g)(x) = f(x) + g(x)$$
$$a \cdot f(x) = a \cdot (f(x))$$

Prove that U^S is a vector space over $\mathbb{F}$.

Proof. We need to go through the definition of vector space and check the following conditions.
1. $u + v = v + u$ for all $u, v \in U^S$.
2. $u + (v + w) = (u + v) + w$ and $(ab)v = a(bv)$ for all $u, v, w \in U^S$ and all $a, b \in \mathbb{F}$
3. There exists an element $0 \in U^S$, called the zero vector, such that $v + 0 = v$ for all $v \in U^S$
4. For every $v \in U^S$, there exists $w \in U^S$ such that $v + w = 0$
5. $1 \cdot v = v$ for all $v \in U^S$
6. $a \cdot (u + v) = a \cdot u + a \cdot v$ and $(a + b) \cdot v = a \cdot v + b \cdot v$ for all $a, b \in F$ and all $u, v \in U^S$

1. By definition of the sum of elements in U^S, we have:

$$\begin{aligned} (u + v)(x) &= u(x) + v(x) \\ &= v(x) + u(x) \\ &= (v + u)(x) \end{aligned}$$

The second equality holds by the commutativity property of the vector space U.
Therefore $u + v = v + u$ for all $u, v \in U^S$.
2. By definition of the sum of elements in U^S, we have:

$$\begin{aligned} (u + (v + w))(x) &= u(x) + (v + w)(x) \\ &= u(x) + (v(x) + w(x)) \end{aligned}$$

The associativity property of the vector space U shows that:

$$\begin{aligned} (u(x) + (v(x) + w(x)) &= (u(x) + v(x)) + w(x) \\ &= (u + v)(x) + w(x) \\ &= ((u + v) + w)(x) \end{aligned}$$

Therefore $(u+(v+w))(x) = ((u+v)+w)(x)$. So we have $u+(v+w) = (u+v)+w$.
By definition of scalar multiplication in U^S, we have:

$$(a \cdot (b \cdot v))(x) = a \cdot ((b \cdot v)(x)) = a \cdot (b \cdot v(x))$$

The associativity property of the vector space U shows that:

$$\begin{aligned} a \cdot (b \cdot v(x)) &= (a \cdot b) \cdot v(x) \\ &= ((a \cdot b) \cdot v)(x) \end{aligned}$$

Therefore $(a \cdot (b \cdot v))(x) = ((a \cdot b) \cdot v)(x)$. So we have $a \cdot (b \cdot v) = (a \cdot b) \cdot v$.

3. The zero element in U^S is the function $z\colon S \to U$ defined by $z(x) = 0$ for all $x \in S$. For every function $v \in U^S$, we have $(v+z)(x) = v(x)+z(x) = v(x)+0 = v(x)$ since 0 is the additive identity in U. Therefore $v + z = v$ for all $v \in U^S$.

4. For every $v \in U^S$, the additive inverse in U^S is the function $\tilde{v}\colon S \to U$ defined by $\tilde{v}(x) = -v(x)$ for all $x \in S$. We have $(v+\tilde{v})(x) = v(x)+\tilde{v}(x) = v(x)+(-v(x)) = 0$ since $-v(x)$ is the additive inverse of $v(x) \in U$. Therefore $v + \tilde{v} = 0$, so $\tilde{v}$ is the additive inverse of $v \in U^S$.

5. For all $v \in U^S$, $(1 \cdot v)(x) = 1 \cdot v(x) = v(x)$. Therefore $1 \cdot v = v$ for all $v \in U^S$.

6. By definition of addition and scalar multiplication in U^S, we have:

$$\begin{aligned}(a \cdot (u + v))(x) &= a \cdot ((u + v)(x)) \\ &= a \cdot (u(x) + v(x))\end{aligned}$$

The distributive property of the vector space U shows that:

$$\begin{aligned}a \cdot (u(x) + v(x)) &= a \cdot u(x) + a \cdot v(x) \\ &= (a \cdot u)(x) + (a \cdot v)(x) \\ &= (a \cdot u + a \cdot v)(x)\end{aligned}$$

Therefore $(a \cdot (u + v))(x) = (a \cdot u + a \cdot v)(x)$, so we have $a \cdot (u + v) = a \cdot u + a \cdot v$.

Similarly, by definition of addition and scalar multiplication in U^S, we have:

$$((a + b) \cdot v)(x) = (a + b) \cdot v(x)$$

The distributive property of the vector space U shows that:

$$\begin{aligned}(a + b) \cdot v(x) &= a \cdot v(x) + b \cdot v(x) \\ &= (a \cdot v)(x) + (b \cdot v)(x) \\ &= (a \cdot v + b \cdot v)(x)\end{aligned}$$

Therefore $((a + b) \cdot v)(x) = (a \cdot v + b \cdot v)(x)$, so we have $(a + b) \cdot v = a \cdot v + b \cdot v$.

The statements above show that U^S satisfies all the conditions of a vector space (see Definition 1.19 from our textbook), so U^S is a vector space over $\mathbb{F}$. □

Question 2. Let $U_1 = \{(a, 0, 0)|a \in \mathbb{F}\}$ and $U_2 = \{(b, b, 0)|b \in \mathbb{F}\}$. There are both subsets of $\mathbb{F}^3$.

a) Prove that U_1 and U_2 are subspaces of $\mathbb{F}^3$

b) Prove that $U_1 + U_2 = \{(x, y, 0|x, y \in \mathbb{F})\}$

Proof. a.1) The zero element in $\mathbb{F}^3$ is a vector z defined by $z = (0, 0, 0)$. Therefore $z = (0, 0, 0) \in U_1$. For any $r \in \mathbb{F}$ and $a_1 = (x_1, 0, 0)$, $a_2 = (x_2, 0, 0) \in U_1$, we have $r \cdot a_1 = (r \cdot x_1, 0, 0)$, $a_1 + a_2 = (x_1 + x_2, 0, 0)$, therefore $r \cdot a_1 \in U_1$ and $a_1 + a_2 \in U_1$. Hence U_1 is a subspace of $\mathbb{F}^3$.

a.2) $z = (0, 0, 0) \in U_2$, so U_2 contains the zero element. For any $t \in \mathbb{F}$ and $b_1 = (y_1, y_1, 0)$, $b_2 = (y_2, y_2, 0) \in U_2$, we have $z \cdot b_1 = (r \cdot y_1, r \cdot y_1, 0)$, $b_1 + b_2 = (y_1 + y_2, y_1 + y_2, 0)$, therefore $r \cdot b_1 \in U_2$ and $b_1 + b_2 \in U_2$. Hence U_2 is a subspace of $\mathbb{F}^3$.

b) Generally speaking, if we are asked to prove that two sets A and B are equal, the most common way is to show that $A \subset B$ and $B \subset A$. So now let us denote the set $\{(x, y, 0)|x, y \in \mathbb{F}\}$ by W.

First, we will show that $U_1 + U_2 \subset W$. For any $a = (x, 0, 0) \in U_1$ and $b = (y, y, 0) \in U_2$, we have $a + b = (x + y, y, 0) \in U$, so $U_1 + U_2 \subset W$.

Second,we will show that $W \subset U_1 + U_2$, for any $c = (x, y, 0) \in W$, we can write c as $c = c_1 + c_2$, here $c_1 = (x - y, 0, 0) \in U_1$ and $c_2 = (y, y, 0) \in U_2$. Hence $W \subset U_1 + U_2$.

Combining the two statements above, we have proved that $U_1+U_2 = \{(x, y, 0)|x, y \in F\}$. □

Question 3. Let V be a vector space, and let U_1 and U_2 be subspaces of V.

a) Their intersection $U_1 \cap U_2$ consists of all vectors that belongs to *both* subspaces:

$$U_1 \cap U_2 = \{v \in V|v \in U_1 \textbf{ and } v \in U_2\}$$

Prove that $U_1 \cap U_2$ is always a subspace of V.

b) Their union $U_1 \cup U_2$ consists of all vectors that belong to *either* subspace:

$$U_1 \cup U_2 = \{v \in V|v \in U_1 \textbf{ or } v \in U_2\}$$

Prove that $U_1 \cup U_2$ is a subspace of V *if and only if* one subspace in contained in the other

Proof. a) U_1, U_2 are both subspaces of V, so $0 \in U_1 \cap U_2$. For any $r \in \mathbb{F}$ and $a_1, a_2 \in U_1 \cap U_2$, we know $r \cdot a_1 \in U_1$ and $r \cdot a_1 \in U_2$ (this is because U_1, U_2 are both vector spaces). We also have $a_1 + a_2 \in U_1$, $a_1 + a_2 \in U_2$, hence $r \cdot a_1 \in U_1 \cap U_2$ and $a_1 + a_2 \in U_1 \cap U_2$. Therefore $U_1 \cap U_2$ is a subspace of V.

b) Suppose $U_1 \subset U_2$, then $U_1 \cup U_2 = U_2$, so $U_1 \cup U_2$ is a subspace of V. We can use the same argument to show that if $U_2 \subset U_1$, then $U_1 \cup U_2 = U_1$ is a subspace of V.

On the other hand, if $U_1 \cup U_2$ is a subspace, we assume $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ and try to get a contradiction. Since $U_1 \not\subset U_2$, we can find an element $u_1 \in U_1$ but $u_1 \notin U_2$ (otherwise it means every element in U_1 belongs to U_2, so $U_1 \subset U_2$, which contradicts with our assumption), and meanwhile we can find an element $u_2 \in U_2$ but $u_2 \notin U_1$ (otherwise it means every element in U_2 belongs to U_1, so $U_2 \subset U_1$, which contradicts with our assumption).

Since this element u_1 is in U_1, we know $u_1 \in U_1 \cup U_2$; similarly, since $u_2 \in U_2$, we know $u_2 \in U_1 \cup U_2$. Because we have assumed that $U_1 + U_2$ is a vector space, this implies that$u_1 + u_2 \in U_1 \cup U_2$. So $u_1 + u_2$ either equals some $a \in U_1$ or some $b \in U_2$.

If $u_1 + u_2 = a \in U_1$, then adding both sides of the equation by $-u_1$, by definition of the additive inverse and additive identity, we have

$$u_2 = -u_1 + u_1 + u_2 = a - u_1 \in U_1.$$

But this contradicts the definition of u_2 as an element that is *not* in U_1.

If $u_1 + u_2 = b \in U_2$, then adding both sides of the equation by $-u_2$, by definition of the additive inverse and additive identity, we have $u_1 = -u_2 + u_2 + u_1 = b - u_2 \in U_2$, which contradicts the definition of u_1 as an element that is *not* in U_2.

Summarizing the two statements above, we have a contradiction because $u_1 + u_2 \in U_1 \cup U_2$, but it does not belong to either U_1 or U_2. Therefore our assumption that neither subspace was contained in the other must have been wrong.

□

Question 4. Let $U_1 = \{(-a, a, 0)|a \in \mathbb{F}\}$, let $U_2 = \{(0, b, -b)|b \in \mathbb{F}\}$, and let $U_3 = \{(c, 0, -c)|c \in \mathbb{F}\}$. These are all subspaces of $\mathbb{F}^3$(you may assume this without proof).

a) Describe the subspaces $U_1 + U_2 + U_3$ by filling in the blank by an equation involving x, y, and z:

$$U_1 + U_2 + U_3 = \{(x, y, z) \in \mathbb{F}^3 | \_\_\_\_\_\_ \}$$

b) Let $W = U_1 + U_2 + U_3$. Is W the direct sum of U_1, U_2, and U_3? Prove or disprove.

Proof. a) We fill in the blank with "$x + y + z = 0''$. In other words, denote the set $\{(x, y, z) \in \mathbb{F}^3|x + y + z = 0\}$ by V; we claim that $U_1 + U_2 + U_3 = V$.

a.1) First we will prove that $U_1 + U_2 + U_3 \subset V$. For any

$$u_1 = (a, -a, 0) \in U_1,\ u_2 = (0, b, -b) \in U_2,\ u_3 = (c, 0, -c) \in U_3,$$

their sum is

$$u_1 + u_2 + u_3 = (a + c, -a + b, -b - c).$$

This vector which satisfies $(a+c)+(-a+b)+(-b-c) = 0$, therefore $u_1+u_2+u_3 \in V$. Hence $U_1 + U_2 + U_3 \subset V$.

a.2) Next we will prove that $V \subset U_1 + U_2 + U_3$. For any $v = (x, y, z) \in V$, we know that $z = -x - y$. So $v = (0, 0, 0) + (0, -y, y) + (x, 0, -x)$. Here $(0, 0, 0) \in U_1$, $(0, -y, y) \in U_2$, $(x, 0, -x) \in U_3$, so $v \in U_1 + U_2 + U_3$. Therefore $V \subset U_1 + U_2 + U_3$.

b) No. Fix any $a \in \mathbb{F}$ which satisfies $a \neq 0$. We have

$$(0, 0, 0) = (-a, a, 0) + (0, -a, a) + (a, 0, -a)$$

. Each of these three vectors is nonzero, but $(-a, a, 0) \in U_1$, $(0, -a, a) \in U_2$, $(a, 0, -a) \in U_3$.

Hence we have more than one way to write 0 as a sum $u_1 + u_2 + u_3$, where each $u_i \in U_i$ (besides the "trivial" way $0 = 0+0+0$). By Prop. 1.44 from our textbook, this proves that $U_1 + U_2 + U_3$ is not a direct sum.

□

Question 5. Let U be the following subset of $\mathbb{F}^\infty$:

$$U = \{(v_1, v_2, v_3 ...) \in \mathbb{F}^\infty | v_{i+3} = v_i \text{ for all i } \}$$

Prove that U is a subspace of $\mathbb{F}^\infty$.

Proof. The zero element $z = (z_1, z_2, z_3, ...) \in \mathbb{F}^\infty$ is defined by $z_i = 0$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, $z_{i+3} = 0 = z_i$, so $z \in U$.

For any $x = (x_1, x_2, x_3, ...) \in U$, $y = (y_1, y_2, y_3, ...) \in U$, and any $r \in \mathbb{F}$, we have

$$r \cdot x = (r \cdot x_1, r \cdot x_2, r \cdot x_3, ...) \text{ and } x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ...).$$

Since x and y are in U, we know that $x_{i+3} = x_i$ and $y_{i+3} = y_i$, so

$$r \cdot x_{i+3} = r \cdot x_i \text{ and } x_{i+3} + y_{i+3} = x_i + y_i.$$

This shows that $r \cdot x \in U$ and $x + y \in U$, so U is a subspace of $\mathbb{F}^\infty$. □

Question 6. Say that a sequence $v = (v_1, v_2, v_3 ...) \in \mathbb{F}^\infty$ is *periodic* if there exists some positive number $k \in \mathbb{N}$ such that $v_{i+k} = v_i$ for all i. Let W be the set of all periodic sequences. Is W a subspace of $\mathbb{F}^\infty$? Prove or disprove.

Proof. Yes, W is a subspace of $\mathbb{F}^\infty$. Let us say that a sequence v is "k-periodic" if $v_{i+k} = v_i$ for all i.

The zero element $z = (z_1, z_2, z_3, ...) \in \mathbb{F}^\infty$ is defined by $z_i = 0$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, $z_{i+1} = 0 = z_i$, so z is 1-periodic. Therefore $z \in W$.

For any $x = (x_1, x_2, x_3...) \in W$, $y = (y_1, y_2, y_3...) \in W$, and any $r \in \mathbb{F}$, assume that x is k-periodic and y is m-periodic. This means $x_{i+k} = x_i$ for all i, and $y_{i+m} = y_i$ for all i. Then we have:

$r \cdot x = (r \cdot x_1, r \cdot x_2, r \cdot x_3, ...)$ and $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ...)$.

Since

$$r \cdot x_{i+k} = r \cdot x_i \quad ,$$

we see that $r \cdot x$ is k-periodic. Hence $r \cdot x \in W$.

The more difficult part is to show that $x + y$ is periodic. The difficulty comes because $x + y$ might not be k-periodic or m-periodic. However, it turns out that it is (km)-periodic, as we now prove. For any $i \in \mathbb{N}$, we have

$$\begin{aligned} x_i &= x_{i+k} \\ &= x_{i+k+k} \\ &= x_{i+3k} \\ &= ... \\ &= x_{i+mk} \end{aligned}$$

where we quoted the k-periodicity of x multiple times to go from each line to the next (m times in total). Similarly,

$$\begin{aligned} y_i &= y_{i+m} \\ &= y_{i+2m} \\ &= y_{i+3m} \\ &= ... \\ &= y_{i+km} \end{aligned}$$

since y is m-periodic. Together, these imply that

$$x_{i+km} + y_{i+km} = x_i + y_i.$$

This shows that $x + y$ is (km)-periodic, so $x + y \in W$. We conclude that W is a subspace of F^∞. $\square$