SEMICLASSICAL SECOND MICROLOCAL PROPAGATION OF REGULARITY AND INTEGRABLE SYSTEMS

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Abstract. We develop a second-microlocal calculus of pseudodifferential operators in the semiclassical setting. These operators test for Lagrangian regularity of semiclassical families of distributions on a manifold X with respect to a Lagrangian submanifold of T^*X. The construction of the calculus, closely analogous to one performed by Bony in the setting of homogeneous Lagrangians, proceeds via the consideration of a model case, that of the zero section of $T^*\mathbb{R}^n$, and conjugation by appropriate Fourier integral operators. We prove a propagation theorem for the associated wavefront set analogous to Hörmander's theorem for operators of real principal type.

As an application, we consider the propagation of Lagrangian regularity on invariant tori for quasimodes (e.g. eigenfunctions) of an operator with completely integrable classical hamiltonian. We prove a secondary propagation result for second wavefront set which implies that even in the (extreme) case of Lagrangian tori with all frequencies rational, provided a nondegeneracy assumption holds, Lagrangian regularity either spreads to fill out a whole torus or holds nowhere locally on it.

1. Introduction

1.1. **Second microlocalization on a Lagrangian.** One purpose of the calculus of pseudodifferential operators is to test distributions for regularity. In the case of the semiclassical calculus, regularity is measured by powers of the semiclassical parameter h; if u_h is a family of distributions as $h \downarrow 0$, one can, following [8], define a "frequency set" or (as we will refer to it here) "semiclassical wavefront set" inside the cotangent bundle T^*X of the underlying manifold X, by decreeing that $p \notin \mathrm{WF}_h(u_h)$ if and only if for arbitrary k and $A_1, \ldots, A_k \in \Psi_h^{-\infty}(X)$, microsupported sufficiently close to p, we have $h^{-k} A_1 \ldots A_k u_h \in L^2$, uniformly as $h \downarrow 0$ (this uniformity will henceforth be tacit). Here $\Psi_h^{-\infty}(X)$ stands for the algebra of smoothing semiclassical pseudodifferential operators, of order 0 in h (thus uniformly bounded on L^2); see Section 2 for the details of the notation. This "oscillatory testing" definition is quite flexible, and illustrates the role of semiclassical pseudodifferential operators as test operators for regularity relative to L^2. With $\mathrm{WF}_h(u_h)$ also defined for points at "fiber infinity" on the cotangent bundle, i.e. on $S^*X = (T^*X \setminus o)/\mathbb{R}^+$, we have $\mathrm{WF}_h(u_h) = \emptyset$ if and only if $u_h \in h^\infty L^2_{\mathrm{loc}}$.

Many distributions arising in the theory of PDE are, of course, not $O(h^\infty)$ (or, in the conventional, homogeneous, theory, not smooth); a great many of the examples that arise in practice, however, turn out to be regular in a different way: they are *Lagrangian distributions,* associated to a Lagrangian submanifold $\mathcal{L} \subset T^*X$. We

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may characterize these distributions again by an "iterated regularity" criterion: if for all k and all $A_1, \dots A_k \in \Psi_h^1(X)$, with $\sigma_h(A_i) \equiv 0$ on $\mathcal{L}$

$$h^{-k} A_1 \dots A_k u_h \in L^2,$$

we say that u is a *Lagrangian distribution* with respect to $\mathcal{L}$. This characterization, analogous to the Melrose-Hörmander characterization of ordinary (i.e. homogeneous, or non-semiclassical) Lagrangian distributions, is equivalent to the statement that u_h has an oscillatory integral representation as a sum of terms of the form

$$\int a(x, \theta, h) e^{i\phi(x,\theta)/h} \, d\theta,$$

where ϕ parametrizes the Lagrangian $\mathcal{L}$ appropriately (see, for instance, [12] in the classical case, and [1] or [20] for an account of semiclassical Lagrangian distributions). We may, by limiting the microsupport of the test operators A_i, somewhat refine this description of Lagrangian regularity to be *local* on $\mathcal{L}$. It remains, however, somewhat crude: it turns out to be quite natural to test more finely, with semiclassical pseudodifferential operators whose principal symbols are allowed to be singular at $\mathcal{L}$ in such a way as to be smooth on the manifold obtained by performing *real blowup* on $\mathcal{L}$ inside T^*X, i.e. by introducing polar coordinates about it. The resulting symbols localize not only on $\mathcal{L}$ itself, but more finely, in $SN(\mathcal{L})$, the spherical normal bundle. (We may, by using the symplectic structure, identify $SN(\mathcal{L})$ with $S(\mathcal{L})$, the unit sphere bundle of $T(\mathcal{L})$, but we will not adopt this notation.) The resulting pseudodifferential calculus is said to be *second microlocal*; there is an associated wavefront set in $SN(\mathcal{L})$ whose absence (together with absence of ordinary semiclassical wavefront set on $T^*X \backslash \mathcal{L}$) is equivalent to u_h being a semiclassical Lagrangian distribution. A helpful analogy is that second-microlocal wavefront set in $SN(\mathcal{L})$ is to failure of local Lagrangian regularity on $\mathcal{L}$ as ordinary wavefront set is to failure of local regularity on X, better known as singular support.

The first part of this paper is devoted to the construction of the semiclassical second microlocal calculus for a Lagrangian in T^*X, and an enumeration of its properties. Other instances of second microlocalization abound in the literature, although we know of none existing in the semiclassical case, with respect to a Lagrangian. Our approach stays fairly close to that adopted by Bony [3] in the classical case of homogeneous Lagrangians, and to that of Sjöstrand-Zworski [20], who construct a semiclassical second microlocal calculus adapted to hypersurfaces in T^*X.

1.2. **An application to quasimodes of integrable Hamiltonians.** As an example of the power of second microlocal techniques in the description of Lagrangian regularity, in the second part of the paper we consider quasimodes of certain operators[1] with real principal symbol with *completely integrable* Hamilton flow. Quasimodes are solutions to

$$P_h u_h \in h^k L^2$$

for some k (the order of the quasimode); eigenfunctions of Schrödinger operators are of course motivating examples. We further assume that the foliation of the phsae space is (locally, at least) given by *compact* invariant tori; these tori are Lagrangian. (See [2] for an account of the theory of integrable systems.) The Hamilton

[1] In addition to the condition that the principal symbol be real, we also require a hypothesis on the subprincipal symbol: see §5 for details.

flow on a Lagrangian torus $\mathcal{L}$ is given by quasi-periodic motion with respect to a set of frequencies $\overline{\omega}_1, \dots, \overline{\omega}_n$. It was shown in [21] that local Lagrangian regularity on $\mathcal{L}$ propagates along Hamilton flow, hence if all frequencies are irrationally related, it fills out the torus. (The set on which local Lagrangian regularity holds is open.) Thus Lagrangian regularity is, on one of these irrational tori, an "all or nothing" proposition: it obtains either everywhere or nowhere on $\mathcal{L}$. In [21], the opposite extreme case was also considered: Lagrangian tori on which $\overline{\omega}_i/\overline{\omega}_j \in \mathbb{Q}$ for each i, j. Local Lagrangian regularity must occur on unions of closed orbits, but in this case, these orbits need not fill out the torus. It was shown, however, that in the presence of a standard nondegeneracy hypothesis ("isoenergetic nondegeneracy"), local Lagrangian regularity propagates in one additional way: to fill in small tubes of bicharacteristics. This apparently mysterious and ungeometric propagation phenomenon is elucidated here. We study the propagation of *second microlocal* regularity on $SN(\mathcal{L})$, and find that it is invariant under two separate flows: the Hamilton flow lifted to $SN(\mathcal{L})$ from the blowdown map to T^*X, and a second flow given by the next-order jets of the Hamilton flow near $\mathcal{L} \subset T^*X$. This leads, in the case considered in [21], to the "all or nothing" condition also holding for local Lagrangian regularity on nondegenerate rational invariant tori: once again either the distribution is Lagrangian on the whole of $\mathcal{L}$ or nowhere locally Lagrangian on it.

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2. The Calculus

Let X denote a manifold without boundary. We adopt the convention that $\Psi_h^{m,k}(X) = h^{-k}\Psi_h^m(X)$ is the space of semiclassical pseudodifferential operators on X of differential order m, hence given locally by semiclassical quantization of symbols lying in $h^{-k}\mathcal{C}^\infty([0,1)_h; S^m(T^*X))$. However, we almost exclusively work microlocally in a compact subset of $T^*X \times 0 \subset T^*X \times [0,1)$, so the differential order, corresponding to the behavior of total symbols at infinity in the fibers of the cotangent bundle, is irrelevant for us, hence we also let

$$\tilde{\Psi}_h^k(X) \subset h^{-k}\Psi_h^{-\infty}(X) = \Psi_h^{-\infty,k}(X).$$

be the subalgebra consisting of ps.d.o's with total symbols compactly supported in the fibers of T^*X plus symbols in $h^\infty\mathcal{C}^\infty([0,1); S^{-\infty}(T^*X))$. (For accounts of the semiclassical pseudodifferential calculus, see, for instance, [15, 4, 7]). We will suppress the h-dependence of families of operators (writing P instead of P_h) of distributions (writing u instead of u_h).

Let $\mathcal{L} \subset T^*X$ be a Lagrangian submanifold with the restriction of the bundle projection to $\mathcal{L}$ being proper. We will define a calculus of pseudodifferential operators $\Psi_{2,h}(X;\mathcal{L})$ associated to $\mathcal{L}$ with the following properties.

(i) $\Psi_{2,h}^{*,*}(X;\mathcal{L})$ is a calculus: it is a bi-filtered algebra of operators $A = A_h : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ with properly supported Schwartz kernels, closed under adjoints and asymptotic summation: if $A_j \in \Psi_{2,h}^{m-j,l}(X;\mathcal{L})$ then there exists $A \in \Psi_{2,h}^{m,l}(X;\mathcal{L})$ such that $A - \sum_{j=0}^{N-1} A_j \in \Psi_{2,h}^{m-N,l}(X;\mathcal{L})$ for all N.

(ii) There is a principal symbol map

$${}^2\sigma_{m,l} : \Psi_{2,h}^{m,l}(X;\mathcal{L}) \to \mathcal{A}_{\mathrm{cl}}^{m-l}(\mathsf{S}_0),$$

where $\mathsf{S}_0 = [T^*X, \mathcal{L}]$ denotes the *real blowup* of $\mathcal{L}$ as a submanifold of T^*X given by introducing normal coordinates about $\mathcal{L}$ (see [18] for extensive discussion or [16] for a brief account) and where

$$\mathcal{A}^{m-l}_{\mathrm{cl}}(\mathsf{S}_0)$$

is the space of classical conormal distributions with respect to $SN(\mathcal{L})$, the spherical normal bundle of $\mathcal{L}$, which is canonically identified with $\partial \mathsf{S}_0$: if $\tilde{\rho}_{\mathrm{ff}}$ *is a boundary defining function for this face*, then[2]

$$\mathcal{A}^r_{\mathrm{cl}}(\mathsf{S}_0) \equiv \tilde{\rho}^r_{\mathrm{ff}} \mathcal{C}^\infty_c(\mathsf{S}_0).$$

For brevity, we will let

$$S^r(\mathsf{S}_0) = \mathcal{A}^{-r}_{\mathrm{cl}}(\mathsf{S}_0).$$

The map ${}^2\sigma$ is a $*$-algebra homomorphism, and fits into the short exact sequence

$$0 \to \Psi^{m-1,l}_{2,h}(X; \mathcal{L}) \to \Psi^{m,l}_{2,h}(X; \mathcal{L}) \overset{{}^2\sigma_{m,l}}{\to} S^{l-m}(\mathsf{S}_0) \to 0. \tag{1}$$

We remark in particular that the vanishing of the symbol only reduces the order in one of the two indices.

(iii) There is a quantization map

$$\mathrm{Op} : \mathcal{A}^{-m,-l}_{\mathrm{cl}}(\mathsf{S}) \to \Psi^{m,l}_{2,h}(X; \mathcal{L}),$$

where

$$\mathsf{S} = [T^*X \times [0, 1); \mathcal{L} \times 0]$$

can be thought of as the space of "total symbols" of two-pseudors. Note that S_0, the space on which principal symbols live, is one of the boundary faces of the manifold with corners S. Here again $\mathcal{A}_{\mathrm{cl}}$ refers to (compactly supported) classical conormal distributions, i.e. multiples of boundary defining functions times smooth functions on the manifold with corners S; the indices $-m, -l$ refer to the orders at the front face of the blowup and the side face (i.e. the lift of S_0) respectively.

For brevity, we will let

$$S^{m,l}(\mathsf{S}) = \mathcal{A}^{-m,-l}_{\mathrm{cl}}(\mathsf{S}).$$

Since one boundary face of S is S_0, if $a \in S^{l-m}(\mathsf{S}_0)$ we may extend it to an element of $S^{0,l-m}(\mathsf{S})$, and multiply by h^{-m} to obtain $\tilde{a} \in S^{m,l}(\mathsf{S})$. This we may quantize and obtain of course

$${}^2\sigma_{m,l}(\mathrm{Op}(\tilde{a})) = a.$$

(iv) If $a \in S^{m,l}(\mathsf{S})$, let $\mathrm{WF}'(\mathrm{Op}(a))$ be defined as $\mathrm{esssupp}(a) \subset \mathsf{S}_0$, where $\mathrm{esssupp}(a)^c$ is the set of points in $\mathsf{S}_0 \subset \mathsf{S}$ which have a neighborhood in which a vanishes to infinite order *at* S_0. Then WF' in fact well-defined on $\Psi^{*,*}_{2,h}(X; \mathcal{L})$, and

$$\mathrm{WF}'(A + B) \subset \mathrm{WF}'(A) \cup \mathrm{WF}'(B),\ \ \mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B).$$

For $A \in \Psi^{m,l}_{2,h}(X; \mathcal{L})$, $\mathrm{WF}'(A) = \emptyset$ if and only if $A \in \Psi^{-\infty,l}_{2,h}(X; \mathcal{L})$.

[2]The requirement of compact support which we have built into this definition is convenient but not strictly necessary; see Remark 2.1.

(v) If $A \in \Psi^{k,l}_{2,h}(X;\mathcal{L})$, $B \in \Psi^{k',l'}_{2,h}(X;\mathcal{L})$ then

$$^2\sigma_{k+k'-1,l+l'}(i[A,B]) = \{^2\sigma_{k,l}(A), {}^2\sigma_{k',l'}(B)\}$$

where the Poisson bracket on the right hand side is computed with respect to the symplectic form on S_0 lifted from the symplectic form on T^*X.

(vi) There is a microlocal parametrix near elliptic points: if $p \in \operatorname{ell}(A)$, $A \in \Psi^{m,l}_{2,h}(X;\mathcal{L})$ then there exist $B \in \Psi^{-m,-l}_{2,h}(X;\mathcal{L})$, $E, F \in \Psi^{0,0}_{2,h}(X;\mathcal{L})$ such that $p \notin \mathrm{WF}'(E)$, $p \notin \mathrm{WF}'(F)$, and $AB = I + E$, $BA = I + F$, where[3]

$$\operatorname{ell}(A) = \{p \in \mathsf{S}_0 : \ (\tilde\rho_{\mathrm{ff}}^{l-m}\, {}^2\sigma_{m,l}(A))(p) \neq 0\} \subset \mathsf{S}_0.$$

(vii) If $A \in \Psi^{m,m}_{2,h}(X;\mathcal{L})$ then $A : h^k L^2 \to h^{k-m} L^2$ for all k.

If $A \in \Psi^{-\infty,l}_{2,h}(X,\mathcal{L})$ then

$$A : L^2(X) \to I^{\infty}_{(-l)}(\mathcal{L}),$$

where, for $k \in \mathbb{N}$,

$$I^k_{(s)}(\mathcal{L}) = \{u : h^{-j-s} A_1 \dots A_j u \in L^2 \ \forall A_i \in \Psi^1_h(X),\ \sigma(A_i)\restriction_{\mathcal{L}} = 0, j \leq k\}$$

(hence $I^{\infty}_{(-l)}(\mathcal{L})$ is, by definition, a space of semiclassical Lagrangian distributions). For general $k \in \mathbb{R}$, $I^k_{(s)}(\mathcal{L})$ is defined by interpolation and duality.

More generally, if $A \in \Psi^{m,l}_{2,h}(X,\mathcal{L})$ and $m, k \in \mathbb{N}$, then

$$A : I^k_{(s)}(\mathcal{L}) \to I^{k-m}_{(s-l)}(\mathcal{L}).$$

for each k.

The distributions in $I^{-\infty}_{(s)} = \bigcup_k I^k_{(s)}$, are called non-focusing relative to $h^{-s}L^2$ at $\mathcal{L}$ (of order $-k$, if they are in $I^k_{(s)}$) in [17].

(viii) For $u \in I^{-\infty}_{(l)}$, there is an associated wavefront set, ${}^2\mathrm{WF}^{m,l}\, u \subset \mathsf{S}_0$, defined by

$$(\mathrm{WF}^{m,l}\, u)^c = \bigcup \{\operatorname{ell}(A) : A \in \Psi^{m,l}_{2,h}, Au \in L^2\}.$$

$\mathrm{WF}^{\infty,l}(u) = \emptyset$ if and only if $h^{-l}u$ is an L^2-based semiclassical Lagrangian distribution with respect to $\mathcal{L}$ (as defined above in vii).

Moreover, if $A \in \Psi^{m',l'}_{2,h}(X;\mathcal{L})$ then

$$\mathrm{WF}^{m-m',l-l'}(Au) \subset \mathrm{WF}^{m,l}(u).$$

Away from $SN(\mathcal{L})$, this wavefront set just reduces to the usual semiclassical wavefront set:

$${}^2\mathrm{WF}^{m,l}(u) \backslash SN(\mathcal{L}) = \mathrm{WF}^m_h(u) \backslash \mathcal{L},$$

where we have identified the complement of the front face of $[T^*X;\mathcal{L}]$ with $T^*X\backslash\mathcal{L}$ in the natural way.

[3]Note that the (non-)vanishing of $(\tilde\rho_{\mathrm{ff}}^{l-m}\, {}^2\sigma_{m,l}(A))(p)$ is independent of the choice of the defining function $\tilde\rho_{\mathrm{ff}}$ of the front face of S_0, so one may reasonably write ${}^2\sigma_{m,l}(A)(p) \neq 0$, meaning $(\tilde\rho_{\mathrm{ff}}^{l-m}\, {}^2\sigma_{m,l}(A))(p) \neq 0$.

(ix) The smoothing semiclassical calculus lies inside $\Psi_{2,h}(X;\mathcal{L})$: $\tilde{\Psi}_h^k(X) \subset \Psi_{2,h}^{k,k}(X;\mathcal{L})$; if $A \in \tilde{\Psi}_h^k(X)$,

$$^2\sigma_{k,k}(A) = \beta^*\sigma_h(A),\ \mathrm{WF}'(A) = \beta^{-1}(\mathrm{WF}'_h(A)),$$

(hence $^2\sigma_{k,k}(A)$ is independent of the fiber variables of $SN(\mathcal{L})$); here $\beta : [T^*X;\mathcal{L}] \to T^*X$ is the blowdown map, σ_h is the semiclassical principal symbol, and WF'_h is the semiclassical operator wave front set.

If $A \in \tilde{\Psi}_h^k(X)$ and $\sigma_h(A) = 0$ on $\mathcal{L}$, then

$$A \in \Psi_{2,h}^{k,k-1}(X).$$

(x) If $Q \in \tilde{\Psi}_h^{m'}(X)$ and $\mathrm{WF}'_h(Q) \cap \mathcal{L} = \emptyset$ then for all $A \in \Psi_{2,h}^{m,l}(X;\mathcal{L})$, $QA, AQ \in \tilde{\Psi}_h^{m+m'}(X)$, $\mathrm{WF}'(QA), \mathrm{WF}'(AQ) \subset \mathrm{WF}'(Q)$, i.e. microlocally away from $\mathcal{L}$, $\Psi_{2,h}(X;\mathcal{L})$ is just $\tilde{\Psi}_h(X)$.

(xi) If $Q, Q' \in \tilde{\Psi}_h^0(X)$ and $\mathrm{WF}'(Q) \cap \mathrm{WF}'(Q') = \emptyset$, then for $A \in \Psi_{2,h}^{m,l}(X;\mathcal{L})$, $QAQ' \in \Psi_{2,h}^{-\infty,l}(X;\mathcal{L})$, i.e. $\Psi_{2,h}(X;\mathcal{L})$ is 2-microlocal, but not microlocal at $\mathcal{L}$: *Lagrangian* singularities can spread along $\mathcal{L}$.

The most important case is the model case, where $\mathcal{L}$ is the zero section of T^*X. We give the detailed construction arguments in this case: the definition is in Definition 3.11, while the precise location of the proofs of the properties listed above is given after the proof of Lemma 3.18. The general definition is given in Definition 3.20, and the properties are briefly discussed afterwards.

Remark 2.1. While we chose to exclude diagonal singularities for elements of $\Psi_{2,h}(X;\mathcal{L})$ because this is irrelevant for most considerations here, and because it would require an additional filtration, principal symbol, etc., the properties listed easily allow one to define a new space of operators,

$$\Psi_{2,h}^{m,k,l}(X;\mathcal{L}) = \Psi_h^{m,k}(X) + \Psi_{2,h}^{k,l}(X;\mathcal{L}), \tag{2}$$

and deduce the analogues of all listed properties. In particular, note that if $A \in \Psi_h^{m,k'}(X)$ and $B \in \Psi_{2,h}^{k,l}(X;\mathcal{L})$ then choosing $Q \in \tilde{\Psi}_h^0(X) \subset \Psi_h^{-\infty,0}(X)$ with $\mathrm{WF}'(\mathrm{Id}-Q) \cap \mathcal{L} = \emptyset$ and $\mathrm{WF}'(\mathrm{Id}-Q) \cap \mathrm{WF}'(B) = \emptyset$ then $B = QB + (\mathrm{Id}-Q)B$, $(\mathrm{Id}-Q)B = B - QB \in \Psi_h^{-\infty,-\infty}(X)$ in view of the composition formula, so $A(\mathrm{Id}-Q)B \in \Psi_h^{-\infty,-\infty}(X)$, while $AQB = (AQ)B \in \Psi_{2,h}^{k+k',l+k'}(X)$ since $AQ \in \tilde{\Psi}_h^{-\infty,k'}(X)$, so we deduce that $AB \in \Psi_{2,h}^{k+k',l+k'}(X)$.

3. The Model Case and the Construction of the Calculus

We construct $\Psi_{2,h}(X;\mathcal{L})$ by constructing it first in the model case of the zero section in $\mathbb{R}^n$ (i.e. $\mathcal{L} = o \subset T^*\mathbb{R}^n$) and verifying its properties, concluding with invariance under semiclassical FIOs preserving the zero section.

Recall that in the case at hand, our "total symbol space" is defined as

$$\mathsf{S} = [(T^*\mathbb{R}^n) \times [0,1); o \times 0],$$

while the "principal symbol space" is the side face (the lift of $T^*\mathbb{R}^n \times 0$), which can be identified with

$$\mathsf{S}_0 = [T^*\mathbb{R}^n; o].$$

Let ρ_{sf} and ρ_{ff} denote boundary defining functions for the side and front faces of this blown-up space. The space of symbols with which we will be primarily concerned will be

$$S^{m,l}(\mathsf{S}) = \rho_{\mathrm{sf}}^{-m}\rho_{\mathrm{ff}}^{-l}\mathcal{C}_c^\infty(\mathsf{S})$$

It is also sometimes useful to consider the space of symbols which are Schwartz at 'fiber infinity',

$$\dot{S}^{m,l}(\mathsf{S}) = \rho_{\mathrm{sf}}^{-m}\rho_{\mathrm{ff}}^{-l}\mathcal{S}(\mathsf{S});$$

here $\mathcal{S}(\mathsf{S})$ stands for the space of Schwartz functions on S, i.e. elements of $\mathcal{C}^\infty(\mathsf{S})$, which near infinity in $T^*X\times[0,1)$ (where the blow-up of the zero section can be ignored) decay rapidly together with all derivatives corresponding to the vector bundle structure (recall that Schwartz functions on a vector bundle are well-defined).

Explicitly, in local coordinates $x=(x_1,\dots,x_n)$ in some open set $\mathcal{U}\subset X$ and canonical dual coordinates $\xi=(\xi_1,\dots,\xi_n)$, coordinates on $T^*X\times[0,1)_h$ are given by x,ξ,h, and $o\times 0$ is given by $\xi=0$, $h=0$. Coordinates on S near the corner (given by the intersection of the front face with the lift of the boundary, $h=0$), where $|\xi_k|>\epsilon|\xi_j|$ for $j\neq k$, are given by $x, h/|\xi_k|, |\xi_k|$ and $\xi_j/|\xi_k|$ $(j\neq k)$, while $x,\Xi=\xi/h$ and h are valid coordinates in a neighborhood of the interior of the front face. Alternatively, near the corner, one can use polar coordinates, $x, h/|\xi|, |\xi|$ and $\xi/|\xi|\in\mathbb{S}^{n-1}$. Locally $|\xi|$ is then a defining function for ff, and $h/|\xi|$ is a defining function for sf. Thus, a typical example of an element of $S^{m,l}(\mathsf{S})$ is a function of the form $h^{-m}|\xi|^{-l+m}a(x,h/|\xi|,|\xi|,\xi/|\xi|)$, $a\in\mathcal{C}_c^\infty(\mathcal{U}\times[0,\infty)\times[0,\infty)\times\mathbb{S}^{n-1})$.

Slightly more globally in the fibers of the cotangent bundle (but locally in $\mathcal{U}$), one can use $\langle\xi/h\rangle^{-1}=(h/\xi)(1+(h/|\xi|)^2)^{-1/2}$ as the defining function for sf, which is now a non-vanishing smooth function in the interior of ff, so $h\langle\xi/h\rangle$ can be taken as the defining function of ff. A straightforward calculation shows that $a\in\dot{S}^{-\infty,l}(\mathsf{S})$ if and only if $b(x,\Xi,h)=h^l a(x,h\Xi,h)\in\mathcal{C}^\infty(\mathcal{U}\times[0,1)_h;\mathcal{S}(\mathbb{R}^n_\Xi))$, with $\mathcal{S}$ standing for the space of Schwartz functions. Indeed, this merely requires noting that $\Xi_j\partial_{\Xi_k}=\xi_j\partial_{\xi_k}$, and the rapid decay in Ξ fibers corresponds bounds by $C_N\langle\Xi\rangle^{-N}=C_N\langle\xi/h\rangle^{-N}=C_N\rho_{\mathrm{sf}}^N$.

Let ${}^h\mathrm{Op}_{\mathrm{l}}$, ${}^h\mathrm{Op}_{\mathrm{r}}$ and ${}^h\mathrm{Op}_{\mathrm{W}}$ denote left-, right-, and Weyl-semiclassical quantization maps on $\mathbb{R}^n$, i.e. for $a\in S^{m,l}(\mathsf{S})$,

$$\begin{aligned}
{}^h\mathrm{Op}_{\mathrm{l}}(a) &= \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)a(x,\xi,h)\,d\xi,\\
{}^h\mathrm{Op}_{\mathrm{r}}(a) &= \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)a(y,\xi,h)\,d\xi,\\
{}^h\mathrm{Op}_{\mathrm{W}}(a) &= \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)a((x+y)/2,\xi,h)\,d\xi,
\end{aligned}$$

where χ is a cutoff properly supported near the diagonal (used to obtain proper supports), identically 1 in a smaller neighborhood of the diagonal, e.g. $\chi=\chi_0(|x-y|^2)$, $\chi_0\in\mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near 0. Note that the allowed singularity of a at $\xi=h=0$ does not cause any problem in defining the integral for $h>0$. More generally, if

$$a\in S^{m,l}([\mathbb{R}^{2n}_{xy}\times\mathbb{R}^n_\xi\times[0,1)_h;\mathbb{R}^{2n}\times\{0\}\times\{0\}])=\mathcal{C}^\infty(\mathbb{R}^n_x;S^{m,l}(\mathsf{S}_{y,\xi,h})),$$

we write

$$I(a)=\frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)a(x,y,\xi,h)\,d\xi.$$

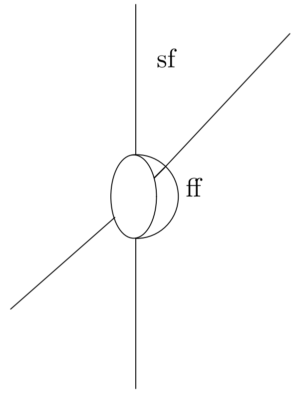

FIGURE 1. The total symbol space S, in the case $n = 2$ and $\mathcal{L} = 0$, with base variables omitted. The front face of the blowup is labeled ff. The side face, labeled sf, is the space S_0 on which principal symbols are defined, and is canonically diffeomorphic to $[T^*X; o]$ The boundary sphere of this side face is diffeomorphic to $SN(o)$.

Definition 3.1. If $a \in S^{m,l}(\mathsf{S})$, let $\operatorname{esssupp}(a) = \operatorname{esssupp}_l(a)$ be the subset of S_0 defined as follows:

$$\operatorname{esssupp}(a)^c = \{p \in \mathsf{S}_0 : \ \exists \phi \in \mathcal{C}^\infty(\mathsf{S}),\ \phi(p) \neq 0,\ \phi a \in \bigcap_{m' \in \mathbb{R}} S^{m',l}(\mathsf{S})\},$$

i.e. a point p is *not* in $\operatorname{esssupp}(a)$ is p has a neighborhood *in* S in which a vanishes to infinite order *at* S_0. We usually suppress the subscript l in the notation.

We give a manifestly invariant definition of the residual operators in our calculus: they are powers of h times families of smoothing operators, with conormal regularity in h :

Definition 3.2. For each $l \in \mathbb{R}$, let

$$\mathcal{R}^l = \{R : \mathcal{C}^{-\infty}(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) : \left\| h^l (h\partial_h)^\alpha \langle\Delta\rangle^\beta R \langle\Delta\rangle^\gamma u \right\| \leq C_{\alpha\beta\gamma} \|u\| \ \forall \alpha, \beta, \gamma\}.$$

Let $\mathcal{R} = \bigcup_{l \in \mathbb{R}} \mathcal{R}^l$. We further assume that all operators in $\mathcal{R}$ have properly supported Schwartz kernels. (Norms are with respect to L^2.)

An alternate characterization is as follows. We let $\kappa(\cdot)$ denote the Schwartz kernel of an operator.

Lemma 3.3. *$R \in \mathcal{R}^l$ if and only if*

$$\sup \left| h^l (h\partial_h)^\alpha \partial_x^\beta \partial_y^\gamma \kappa(R)(x, y, h) \right| \leq C_{\alpha\beta\gamma}. \tag{3}$$

Proof. Certainly if (3) does hold, we obtain a uniform estimate on operator norms as required by Definition 3.2, as indeed we may estimate Hilbert-Schmidt norms of R in terms of the estimates (3) and the size of the support. Conversely, Definition 3.2 tells us that $h^l (h\partial_h)^\alpha R : H^{-s} \to H^{s+|\beta|}$ for any desired $s \in \mathbb{R}$ and multiindex β; taking $s > n/2$ and using Sobolev imbedding gives $\partial_x^\beta \partial_y^\gamma h^l (h\partial_h)^\alpha \kappa(R) \in L^\infty$ (uniformly in h). □

As mentioned above this definition generalizes immediately to a manifold X without boundary:

Definition 3.4. $R \in \mathcal{R}^l(X)$ if it has a properly supported Schwartz kernel on $X \times X \times [0,1)$, satisfying (3) in local coordinates x, resp. y, or equivalently, that for all k and all compactly supported vector fields $V_1, \dots, V_k \in \mathcal{V}(X \times X \times [0,1))$ tangent to $h = 0$, there exists C such that $|h^l V_1 \dots V_k \kappa(R)| \leq C$.

Returning to $\mathbb{R}^n$, we now show that the quantizations of the "residual" symbols in

$$S^{-\infty,l}(\mathsf{S}) \equiv \bigcap_{m \in \mathbb{R}} S^{m,l}(\mathsf{S})$$

lie in the space of residual operators $\mathcal{R}^l$.

Lemma 3.5. ${}^h\mathrm{Op}_\mathrm{l}, {}^h\mathrm{Op}_\mathrm{r}, {}^h\mathrm{Op}_\mathrm{W}$ *map* $S^{-\infty}(\mathsf{S})$ *into* $\mathcal{R}$, *and* I *maps* $\mathcal{C}^\infty(\mathbb{R}^n; S^{-\infty}(\mathsf{S}))$ *into* $\mathcal{R}$.

Proof. We have

$$S^{-\infty,l}(\mathsf{S}) = \{a : a \in \rho_{\mathrm{sf}}^\infty \rho_{\mathrm{ff}}^{-l} \mathcal{C}_c^\infty(\mathsf{S})\}.$$

Then the kernel of the quantization of a is given by

$${}^h\mathrm{Op}_\mathrm{l}(a) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\xi/h} \chi(x,y) a(x,\xi,h)\, d\xi.$$

Owing to its rapid vanishing at sf, we find that a is classical conormal (i.e. a power of a boundary defining function times a $\mathcal{C}^\infty$ function) on the space obtained from S by blowing down sf, i.e. by introducing new variables $\Xi = \xi/h$ instead of ξ; we can write $a(x, h\Xi, h) = \tilde{a}(x, \Xi, h) h^{-l}$ where $\tilde{a}$ is $\mathcal{C}^\infty$ and vanishing rapidly as $\Xi \to \infty$. Hence

$$\begin{aligned} {}^h\mathrm{Op}_\mathrm{l}(a) &= \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\xi/h} \chi(x,y) a(x,\xi,h)\, d\xi \\ &= \frac{h^{-l}}{(2\pi)^n} \int e^{i(x-y)\cdot\Xi} \chi(x,y) \tilde{a}(x,\Xi,h)\, d\Xi = h^{-l} \chi(x,y) (\mathcal{F}^{-1}\tilde{a})(x, x-y, h), \end{aligned}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform in the second argument of $\tilde{a}$ (i.e. in Ξ), and this is just h^{-l} times a family of smoothing operators with parameter h. In particular (3) is easily verified, hence ${}^h\mathrm{Op}_\mathrm{l}(a)$ lies in $\mathcal{R}^l$. Analogous arguments hold for ${}^h\mathrm{Op}_\mathrm{r}, {}^h\mathrm{Op}_\mathrm{W}$ and I. □

In fact, we have the following slight strengthening:

Lemma 3.6. *If* $A \in \mathrm{Op}_\bullet(S^{m,l}(\mathsf{S}))$, *and for all* N *there exists* $a_N \in S^{m-N,l}$ *such that* $A = \mathrm{Op}_\bullet(a_N)$, *then* $A \in \mathcal{R}^l$, *where* $\bullet$ *can be l,r, or W.*

Proof. We prove the lemma for the case of ${}^h\mathrm{Op}_\mathrm{l}$; the other cases are analogous.

As

$$\begin{aligned} &(h\partial_h)^\alpha \partial_x^\beta \partial_y^\gamma\, {}^h\mathrm{Op}_\mathrm{l}(a_N) \\ &= \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\xi/h} (h\partial_h - i(x-y)\cdot\xi/h)^\alpha (\partial_x + i\xi/h)^\beta \\ &\qquad (\partial_y - i\xi/h)^\gamma (\chi(x,y) a_N(x,\xi,h))\, d\xi, \end{aligned}$$

the conclusion follows from choosing N large enough such that $\langle \xi/h \rangle^{|\alpha|+|\beta|+|\gamma|} a_N \in S^{l,l}(\mathsf{S})$, which in turn is possible as $\langle \xi/h \rangle = (1 + |\xi/h|^2)^{1/2}$ is the reciprocal of a defining function of sf as described at the beginning of the section. □

Lemma 3.7. *If $a \in \mathcal{C}^\infty(\mathbb{R}^n; S^{m,l}(\mathsf{S}))$, then there exist $a_l, a_r, a_W \in S^{m,l}(\mathsf{S})$ and $R_l, R_r, R_W \in \mathcal{R}^l$ such that*

$$I(a) = {}^h\mathrm{Op}_\mathrm{l}(a_l) + R_l = {}^h\mathrm{Op}_\mathrm{r}(a_r) + R_r = {}^h\mathrm{Op}_\mathrm{W}(a_W) + R_W,$$

and

$$a|_{y=x} - a_l, a|_{x=y} - a_r, a|_{x=y} - a_W \in S^{m-1,l}(\mathsf{S}).$$

In particular, we may change from left- to right-quantization and vice-versa: if $a \in S^{m,l}(\mathsf{S})$; then there exist $b, b' \in S^{m,l}(\mathsf{S})$ and $R, R' \in \mathcal{R}^l$ such that

$${}^h\mathrm{Op}_\mathrm{l}(a) = {}^h\mathrm{Op}_\mathrm{r}(b) + R, \quad {}^h\mathrm{Op}_\mathrm{r}(a) = {}^h\mathrm{Op}_\mathrm{l}(b') + R'.$$

Moreover, $\operatorname{esssupp} a_l = \operatorname{esssupp} a_r = \operatorname{esssupp} a_W$.

Proof. To begin, we prove that for $a \in \mathcal{C}^\infty(\mathbb{R}^n; S^{m,l}(\mathsf{S}))$, $I(a) = {}^h\mathrm{Op}_\mathrm{r}(b) + R$, $b \in S^{m,l}(\mathsf{S})$, $R \in \mathcal{R}^l$. The statement about ${}^h\mathrm{Op}_\mathrm{l}$ follows the same way reversing the role of x and y below.

Using a partition of unity, we may decompose a into pieces supported on the lifts to S of the set $\{\xi_j \neq 0\} \subset (T^*\mathbb{R}^n \times [0,1))$ for various values of j. By symmetry, it will suffice to deal with the term supported on $\xi_1 \neq 0$. On this region, we may take as coordinates in S the functions $\Xi = \xi'/\xi_1$, $H = h/\xi_1$, and ξ_1. Thus, ξ_1 is locally a defining function for ff and H for sf. We may Taylor expand in x around y,

$$a(x, y, \xi, h) \sim \sum \frac{1}{\alpha!}(x-y)^\alpha (\partial_x^\alpha a)(y, y, \xi, h))$$

Now in the variables Ξ, ξ_1, H, we have

$$\begin{aligned} h\partial_{\xi_1} &= H(\xi_1\partial_{\xi_1} - H\partial_H - \Xi\cdot\partial_\Xi),\\ h\partial_{\xi'} &= H\partial_\Xi,\\ h\partial_h &= H\partial_H, \end{aligned} \tag{4}$$

hence by our symbolic assumptions on a,

$$(h\partial_\xi)^\alpha(\partial_x^\alpha a)(x, x, \xi, h)) \in H^{-m+|\alpha|}\xi_1^{-l}\mathcal{C}^\infty(\mathbb{R}^n \times \mathsf{S})$$

near the corner $H = \xi_1 = 0$, hence these terms may be Borel summed to some $a_r \in H^{-m}\xi_1^{-l}\mathcal{C}^\infty(\mathbb{R}^n \times \mathsf{S})$.

For any $N \in \mathbb{N}$, by integrating by parts, we have

$$I(a) = \sum_{|\alpha|<N} \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)\frac{1}{\alpha!}(x-y)^\alpha(\partial_x^\alpha a)(y, y, \xi, h)\,d\xi + R_N' \tag{5}$$

$$= \sum_{|\alpha|<N} C_\alpha \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)\frac{1}{\alpha!}((h\partial_\xi)^\alpha\partial_x^\alpha a)(y, y, \xi, h)\,d\xi + R_N' \tag{6}$$

$$= {}^h\mathrm{Op}_\mathrm{r}(a_r) + R_N \tag{7}$$

where R_N and R_N' both have the form

$$\begin{aligned} I&\left(\sum_{|\alpha|=N}(x-y)^\alpha r_\alpha'\right)\\ &= \sum_{|\alpha|=N}\frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\chi(x,y)(h\partial_\xi)^\alpha(r_\alpha'(x, y, \xi, h))\,d\xi \end{aligned} \tag{8}$$

with $r'_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n \times S^{-N,l}(\mathsf{S}))$. We thus have

$$I(a) - {}^h\mathrm{Op_r}(a_r) \in I(\mathcal{C}^\infty(\mathbb{R}^n; S^{-N,l}(\mathsf{S})))$$

for all $N \in \mathbb{N}$. By Lemma 3.6 we obtain the desired result.

The statement about ${}^h\mathrm{Op_W}$ can be proved similarly, writing

$$a(x, y, \xi, h) = \tilde{a}((x + y)/2, (x - y)/2, \xi, h),$$

i.e. $\tilde{a}(w, z, \xi, h) = a(z + w, z - w, \xi, h)$, and expanding $\tilde{a}$ in Taylor series in $z = (x - y)/2$ around 0, so

$$a(x, y, \xi, h) \sim \sum \frac{1}{\alpha!} \left(\frac{x - y}{2}\right)^\alpha ((\partial_x + \partial_y)^\alpha a) \left(\frac{x + y}{2}, \frac{x + y}{2}, \xi, h\right).$$

Finally, the statements about $\operatorname{esssupp} a_l$, etc., follows for e.g. if $a(x, y, \xi, h) = a_l(y, \xi, h)$, the terms $a_{r,\alpha} = \frac{1}{\alpha!}(h\partial_\xi)^\alpha \partial_x^\alpha(a_l)(y, \xi, h)$ in the asymptotic expansion for a_r all have $\operatorname{esssupp} a_{r,\alpha} \subset \operatorname{esssupp} a_l$. $\square$

Lemma 3.8. *If $a \in \mathcal{C}^\infty(\mathbb{R}^n; S^{m,l}(\mathsf{S}))$ then for $\phi \in \mathcal{C}^\infty(X^2 \times [0, 1))$ with support disjoint from $\mathrm{diag} \times \{0\}$, $\phi I(a) \in \mathcal{R}^l$.*

Proof. As $I(a) \in \mathcal{C}^\infty(X^2 \times (0, 1))$, we may assume that $\operatorname{supp} \phi$ is disjoint from $\mathrm{diag} \times [0, 1)$, hence $|x - y| > \epsilon$ on $\operatorname{supp} \phi$. Then for all N,

$$\begin{aligned} I(a) &= \frac{1}{(2\pi h)^n} \int \frac{h^{2N}}{|x - y|^{2N}} \Delta_\xi^N e^{i(x-y)\cdot\xi/h} \chi(x, y) a(x, y, \xi, h)\, d\xi \\ &= \frac{1}{(2\pi h)^n} \int \frac{1}{|x - y|^{2N}} e^{i(x-y)\cdot\xi/h} \chi(x, y) (h^{2N} \Delta_\xi^N a)(x, y, \xi, h)\, d\xi \end{aligned}$$

We can assume, using a partition of unity as above, that a is supported in the lift of the set where $\xi_1 \neq 0$. Then (4) shows that $h^{2N}\Delta_\xi^N a \in S^{m-2N,l}(\mathsf{S})$. Choosing N sufficiently large, depending on α, β, γ, it follows immediately (cf. the proof of Lemma 3.6) that

$$|h^l (h\partial h)^\alpha \partial_x^\beta \partial_y^\gamma I(a)| \leq C,$$

for each derivative at most gives an additional factor of H^{-1} in growth. $\square$

The proof of this lemma can in fact be extended to show that ${}^h\mathrm{Op_l}(a)$ determines a modulo $S^{-\infty,l}(\mathsf{S})$:

Lemma 3.9. *For $a \in S^{m,l}(\mathsf{S})$, let κ denote the Schwartz kernel of ${}^h\mathrm{Op_l}(a)$. Then*

$$a(x, \xi, h) - (\mathcal{F}_z \kappa(x, x - z, h))(\xi) \in \dot{S}^{-\infty,l}(\mathsf{S}),$$

where $\dot{S}$ denotes the space of symbols rapidly decreasing at infinity (rather than compactly supported) in S. In particular, modulo $S^{-\infty,l}$, ${}^h\mathrm{Op_l}(a)$ determines a.[4]

Analogous statements hold for ${}^h\mathrm{Op_r}(a)$ and ${}^h\mathrm{Op_W}(a)$ as well.

Proof. For $a \in S^{m,l}(\mathsf{S})$, let

$$K(x, z, h) = ({}^h\mathcal{F}_\xi^{-1} a(x, \xi, h))(z) \equiv (2\pi h)^{-n} \int e^{iz\cdot\xi/h} a(x, \xi, h)\, d\xi$$

[4]Here $\tilde{a}(x, \xi, h) = (\mathcal{F}_z \kappa(x, x - z, h))(\xi) \in \mathcal{C}^\infty(T^*\mathbb{R}^n \times (0, 1))$, so one may indeed think of it as a function, as indicated by the notation. However, one must use the blown-up coordinates on $[T^*\mathbb{R}^n \times [0, 1); o \times 0]$ in order to realize a as a polyhomogeneous function, hence in order to evaluate $\rho_{\mathrm{sf}}^m \rho_{\mathrm{ff}}^l \tilde{a}$ at the front face.

be the semiclassical inverse Fourier transform of a in ξ, so

$$a(x,\xi,h) = ({}^h\mathcal{F}_z K(x,z,h))(\xi) = (\mathcal{F}_z K(x,z,h))(\xi/h) = \int e^{-iz\cdot\xi/h} K(x,z,h)\,dz.$$

Then

$$\kappa(x, x-z, h) = \chi(x, x-z) K(x,z,h),$$

hence

$$r \equiv a(x,\xi,h) - (\mathcal{F}_z \kappa(x,x-z,h))(\xi) = \mathcal{F}_z\left((1-\chi(x,x-z))K(x,z,h)\right)(\xi/h);$$

we need to show that this lies in $S^{-\infty,l}(\mathsf{S})$.

The proof of the preceding lemma shows that $(1-\chi(x,y))K(x,y,h)$ is Schwartz in $x-y$, smooth in x, conormal in h of order l, i.e.

$$h^l (h\partial_h)^s z^\alpha D_z^\beta D_x^\gamma (1-\chi(x,x-z))K(x,z,h)$$

is bounded for all s,α,β,γ, so its (non-semiclassical) Fourier transform in $z = x-y$,

$$\begin{aligned}\tilde r(x,\Xi,h) &= \left(\mathcal{F}_z (1-\chi(x,x-z))K(x,z,h)\right)(\Xi)\\ &= \int e^{-i\Xi\cdot z}(1-\chi(x,x-z))K(x,z,h)\,dz\end{aligned}$$

satisfies

$$h^l (h\partial_h)^s \Xi^\alpha D_\Xi^\beta D_x^\gamma \tilde r(x,\Xi,h) \in L^\infty$$

for all s,α,β,γ. Thus, $\tilde r \in \mathcal{C}^\infty(\mathbb{R}^n_x \times [0,1)_h; \mathcal{S}(\mathbb{R}^n_\Xi))$, so as remarked at the beginning of the section,

$$r(x,\xi,h) = \tilde r(x,\xi/h,h) \in \dot S^{-\infty,l}(\mathsf{S}). \quad \square$$

We now prove diffeomorphism invariance.

Lemma 3.10. *If $F : U \to U'$ is a diffeomorphism, $G = F^{-1}$, $U, U' \subset \mathbb{R}^n$, and $A = {}^h\mathrm{Op}_l(a)$, $a \in S^{m,l}(\mathsf{S})$, with the Schwartz kernel of A supported in U' then $F^*AG^* = {}^h\mathrm{Op}_l(b) + R$ for some $b \in S^{m,l}(\mathsf{S})$, $R \in \mathcal{R}^l$.*

Moreover, $b - (G^\sharp)^* a \in S^{m-1,l}(\mathsf{S})$, where $G^\sharp : T^*U \to T^*U'$ is the induced pull-back of one-forms, and $\operatorname{esssupp} b = G^\sharp(\operatorname{esssupp} a)$.

For the Weyl quantization, and with A acting on half-densities, the analogous statement holds with the improvement $b - (G^\sharp)^* a \in S^{m-2,l}(\mathsf{S})$.

Proof. We follow the usual proof of the diffeomorphism invariance formula.

Note first that $\mathcal{R}^l$ is certainly invariant under pullbacks by diffeomorphisms, and a partition of unity, with an element identically 1 near the diagonal, allows us to assume that K_A is supported in a prescribed neighborhood of the diagonal.

The Schwartz kernel $K_B(x,y)|dy|$ of $B = F^*AG^*$ is $(F\times F)^*(K_A(\tilde x,\tilde y)|d\tilde y|)$, i.e.

$$\begin{aligned}K_B(x,y) &= K_A(F(x),F(y))|\det F(y)|\\ &= (2\pi h)^{-n}\int e^{i(F(x)-F(y))\cdot\xi/h} a(F(x),\xi)|\det F(y)|\,d\xi.\end{aligned}$$

Now, $F_i(x) - F_i(y) = \sum_j (x_j - y_j) T_{ij}(x,y) = T(x,y)(x-y)$ by Taylor's theorem, with $T_{ij}(x,x) = \partial_j F_i(x)$, so $T(x,x)$ invertible. Thus, T is invertible in a neighborhood of the diagonal; we take this as the prescribed neighborhood mentioned above. Then, with $\eta = T^t(x,y)\xi$,

$$\begin{aligned}&K_B(x,y)\\ &= (2\pi h)^{-n}\int e^{i(x-y)\cdot\eta/h} a(F(x),(T^t)^{-1}(x,y)\eta)|\det T(x,y)|^{-1}|\det F(y)|\,d\eta.\end{aligned}$$

By Lemma 3.7, this is of the form ${}^h\mathrm{Op}_l(b) + R'$ for some $b \in S^{m,l}(\mathsf{S})$, $R' \in \mathcal{R}^l$, provided that we show that $a(\dots) \in \mathcal{C}^\infty(\mathbb{R}^n; S^{m,l}(\mathsf{S}))$, which in turn is immediate.

For the Weyl quantization, acting on half-densities, the Schwartz kernel

$$K_B(x,y)|dx|^{1/2}|dy|^{1/2}$$

of $B = F^*AG^*$ is $(F \times F)^*(K_A(\tilde{x},\tilde{y})|d\tilde{x}|^{1/2}|d\tilde{y}|^{1/2})$,

$$K_B(x,y) = K_A(F(x),F(y))|\det F(x)|^{1/2}|\det F(y)|^{1/2}.$$

Now we use Taylor's theorem for F around $(x+y)/2$, so $F_i(x) - F_i(y) = \sum_{ij}(x_j - y_j)T_{ij}(x,y)$ with $T_{ij}(x,y) - T_{ij}((x+y)/2,(x+y)/2) = O(|x-y|^2)$, and $(F(x) + F(y))/2 = F((x+y)/2) + O(|x-y|^2)$, with an analogous statement for the product of the determinants, to obtain the improved result. □

In view of the diffeomorphism invariance and Lemma 3.8, we can naturally define 2-microlocal operators on manifolds, associated to the 0-section.

Definition 3.11. Let

$$\Psi^{m,l}_{2,h}(\mathbb{R}^n; o) = \{{}^h\mathrm{Op}_l(a) : a \in S^{m,l}(\mathsf{S})\} + \mathcal{R}^l.$$

If X is a manifold without boundary, let $\Psi^{m,l}_{2,h}(X,o)$ consist of operators A with properly supported Schwartz kernels[5] $K_A \in \mathcal{C}^{-\infty}(X \times X \times [0,1); \pi_R^*\Omega X)$, such that for any coordinate neighborhood U of $p \in X$, and any $\phi,\psi \in \mathcal{C}^\infty_c(U)$, A satisfies $\phi A \psi \in \Psi^{m,l}_{2,h}(\mathbb{R}^n; o)$, while if $\phi,\psi \in \mathcal{C}^\infty_c(X)$ with disjoint support then $\phi A \psi \in \mathcal{R}^l(X)$.

Remark 3.12. Directly from the definition,

$$\tilde{\Psi}^m_h(\mathbb{R}^n) = h^{-m}\tilde{\Psi}^0_h(\mathbb{R}^n) \subset \Psi^{m,m}_{2,h}(\mathbb{R}^n),$$

with the relationship between total symbols, modulo $S^{-\infty,m}(\mathsf{S})$, for, say, left-quantization, given by the pullback under the blow-down map

$$\beta : [T^*\mathbb{R}^n \times [0,1); o \times 0].$$

Lemma 3.13. $\mathcal{R}^l\big(\Psi^{m',l'}_{2,h}(\mathbb{R}^n; o)\big) \subset \mathcal{R}^{l+l'}$, $\big(\Psi^{m',l'}_{2,h}(\mathbb{R}^n; o)\big)\mathcal{R}^l \subset \mathcal{R}^{l+l'}$.

Proof. It suffices by Lemma 3.7 to show that if $a \in S^{m,l}(\mathsf{S})$ and $R \in \mathcal{R}^{l'}$ then

$$R \circ {}^h\mathrm{Op}_r(a) \in \mathcal{R}^{l+l'}, \tag{9}$$

as the rest of the statement will follow by taking adjoints. To show (9), we begin by showing that $h^{l+l'}R \circ {}^h\mathrm{Op}_r(a)$ is bounded on L^2 (uniformly in h). If m is negative, the uniform L^2-boundedness of $h^l\,{}^h\mathrm{Op}_r(a)$ follows from Calderón-Vaillancourt,[6] so it suffices to consider the case $m > 0$. In that case, let k be an integer greater than m. We again split a up into pieces and employ local coordinates as in the proof of Lemma 3.7; thus, using the fact that

$$\frac{h}{\xi_1}D_{y_1}e^{i(y-z)\cdot\xi/h} = e^{i(y-z)\cdot\xi/h},$$

[5]Here $\pi_R : X \times X \times [0,1) \to X$ is the projection to the second factor of X, ΩX the density bundle.

[6]Note that we use the fact that h lifts to S to be $H\xi_1$ in the local coordinates of the proof of Lemma 3.7, so that h^l times a symbol in $S^{m,l}(\mathsf{S})$ is bounded.

we may write

$$R \circ {}^h\mathrm{Op_r}(a) = \frac{1}{(2\pi h)^n} \int K(x,y,h) a(z,\xi,h) \left(\frac{h}{\xi_1} D_{y_1}\right)^k e^{i(y-z)\cdot\xi/h}\, d\xi\, dy,$$

where K is the kernel of R. We now integrate by parts in y_1, to obtain

$$R \circ {}^h\mathrm{Op_r}(a) = \frac{1}{(2\pi h)^n} \int (D_{z_1})^k K(x,y,h) a(z,\xi,h) \left(\frac{h}{\xi_1}\right)^k e^{i(y-z)\cdot\xi/h}\, d\xi\, dy;$$

noting that $h^l(h/\xi_1)^k a \in S^{0,0}(\mathsf{S})$ and that K is smooth in z, we again obtain L^2 boundedness by Calderón-Vaillancourt.

To finish the proof, it suffices to show that $h^{l+l'}(h\partial_h)^\alpha D^\beta R \circ {}^h\mathrm{Op_r}(a) D^\gamma$ are also L^2 bounded. The follows from stable regularity of the kernel of R under $h\partial_h$ and D_x, and from a further integration by parts, since

$$D_z e^{i(y-z)\cdot\xi/h} = -D_y e^{i(y-z)\cdot\xi/h},$$

and y-derivatives falling on K may also be absorbed without loss. □

Theorem 3.14. *$\Psi_{2,h}(\mathbb{R}^n;o)$ and $\Psi_{2,h}(X,o)$ are bi-filtered $*$-algebras, with $\mathcal{R}$ a filtered two-sided ideal.*

Proof. By localization we immediately reduce the general case to $\mathbb{R}^n$.

That $\Psi_{2,h}(\mathbb{R}^n;o)$ is closed under adjoints follows from our ability to exchange left and right quantization, as proved above, together with the fact that the residual calculus is closed under adjoints.

To prove that the calculus is closed under composition, it suffices (using Lemmas 3.13 and 3.7) to show that if we take $a \in S^{m,l}(\mathsf{S})$ and $b \in S^{m',l'}(\mathsf{S})$ then ${}^h\mathrm{Op_l}(a) \circ {}^h\mathrm{Op_r}(b) \in \Psi_{2,h}^{m+m',l+l'}(\mathbb{R}^n;o) + \mathcal{R}^{l+l'}$. We have

$$\begin{aligned} {}^h\mathrm{Op_l}(a) \circ {}^h\mathrm{Op_r}(b) &= \frac{1}{(2\pi h)^{2n}} \int a(x,\xi,h) b(y,\eta,h) e^{i(x-w)\cdot\xi/h} e^{i(w-y)\cdot\eta/h}\, d\xi\, d\eta\, dw \\ &= \frac{1}{(2\pi h)^n} \int a(x,\xi,h) b(y,\xi,h) e^{i(x-y)\cdot\xi/h}\, d\xi \end{aligned}$$

Lemma 3.7 permits us to rewrite this expression as a left quantization of a symbol in $S^{m+m',l+l'}(\mathsf{S})$ plus a term in $\mathcal{R}^{l+l'}$.

The ideal property of $\mathcal{R}$ is immediate from Lemma 3.13. □

We now discuss the definition and properties of the principal symbol map. If $A \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;o)$ is given by

$$A = {}^h\mathrm{Op_l}(a) + R, \quad R \in \mathcal{R}^l,$$

we define[7]

$${}^2\sigma(A) = (h^m a) \restriction_{\mathrm{sf}} \in S^{l-m}(\mathrm{sf}) = S^{l-m}(\mathsf{S}_0).$$

[7] As h is a globally well-defined function on S, we do not need to introduce a line bundle to take care of this renormalization; this is in contrast with the case when one wishes to define the usual principal symbol as a function on the cosphere bundle, but a line bundle appears unavoidably in the definition.

As usual, we may write this in terms of the kernel of ${}^h\mathrm{Op}_\mathrm{l}(a)$ in terms of Fourier transform:[8]

$$(\rho_{\mathrm{ff}}^l a)\restriction_{\mathrm{sf}}(x,\rho,\hat\xi) = \lim_{h\downarrow 0} \rho^l \mathcal{F}_z(\kappa({}^h\mathrm{Op}_\mathrm{l}(a))(x, x-z))(\rho\hat\xi);$$

here we have identified sf with $\mathsf{S}_0 = [T^*\mathbb{R}^n; o]$, and $(x,\rho,\hat\xi)$ are coordinates in this space, hence $\rho = |\xi|, \hat\xi = \xi/|\xi|$; we use κ to denote the Schwartz kernel of an operator.

Note that ${}^2\sigma(A)$ is not a priori well-defined owing to the presence of the term in $\mathcal{R}$ in our definition of the calculus, but Lemma 3.9 shows that in fact it is. Also, directly from the definition of $\Psi_{2,h}^{m,l}(X,o)$, ${}^2\sigma(A)$ can be defined by localization (i.e. considering $\phi A\phi$, ϕ identically 1 near the point in question) for arbitrary X, and is independent of all choices.

Lemma 3.15. *The principal symbol sequence*

$$0 \to \Psi_{2,h}^{k-1,l}(X;o) \to \Psi_{2,h}^{k,l}(X;o) \xrightarrow{{}^2\sigma_{k,l}} S^{l-k}(\mathsf{S}_0) \to 0,$$

where the map $\Psi_{2,h}^{k-1,l}(X;o) \to \Psi_{2,h}^{k,l}(X;o)$ is inclusion, is exact.

Proof. ${}^2\sigma_{k,l}$ is surjective since ${}^2\sigma_{k,l}({}^h\mathrm{Op}_\mathrm{l}(a)) = a$. ${}^2\sigma_{k,l}(A) = 0$ if $A \in \Psi_{2,h}^{k-1,l}(X;o)$ directly from the definition. Finally, if ${}^2\sigma_{k,l}(A) = 0$ for some $A \in \Psi_{2,h}^{k,l}(X;o)$, then $A - {}^h\mathrm{Op}_\mathrm{l}(a) \in \Psi_{2,h}^{-\infty,l}(X,o)$ for some $a \in S^{k,l}(\mathsf{S})$, with $h^k a|_{\mathrm{sf}} = {}^2\sigma_{k,l}(A)$, so if the latter vanishes, then $h^k a|_{\mathrm{sf}} \in \rho_{\mathrm{sf}} S^{0,l-k}(\mathsf{S})$, hence $a \in S^{k-1,l}(\mathsf{S})$, giving the conclusion. □

Lemma 3.16. *The principal symbol map is a homomorphism, and if $A \in \Psi_{2,h}^{k,l}(X;o)$, $B \in \Psi_{2,h}^{k',l'}(X;o)$ then*

$${}^2\sigma_{k+k'-1,l+l'}(i[A,B]) = \{{}^2\sigma_{k,l}(A), {}^2\sigma_{k',l'}(B)\}$$

where the Poisson bracket is computed with respect to the symplectic form on $[T^*X;o]$ lifted from the symplectic form on T^*X.

This follows from $[A,B] \in \Psi_{2,h}^{k+k'-1,l+l'}(X;o)$ (which in turn follows from

$${}^2\sigma_{k+k',l+l'}([A,B]) = {}^2\sigma_{k,l}(A)\,{}^2\sigma_{k',l'}(B) - {}^2\sigma_{k',l'}(B)\,{}^2\sigma_{k,l}(A) = 0$$

and the exactness of the principal symbol sequence), with the principal symbol in $\Psi_{2,h}^{k+k'-1,l+l'}(X;o)$ calculated by continuity from the region $\xi \neq 0$, where the stated formula follows from a standard argument in the development of the semiclassical calculus.

We now discuss the definition and properties of the operator wave front set. If $A \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;o)$ is given by

$$A = {}^h\mathrm{Op}_\mathrm{l}(a) + R, \quad a \in S^{m,l},\ R \in \mathcal{R}^l,$$

we define

$$\mathrm{WF}'(A) = \mathrm{WF}'_l(A) = \mathrm{esssupp}_l(a) = \mathrm{esssupp}(a).$$

[8]The following formula should be interpreted with a grain of salt: the value of the symbol at $\rho = 0$ (where, indeed, it is of greatest interest) must be obtained from the formula by continuous extension from the case $\rho > 0$, where it makes sense (and equals the ordinary semiclassical symbol).

Again, this is well-defined by Lemma 3.9, and by localization, $\mathrm{WF}'(A)$ is also well-defined for $A \in \Psi_{2,h}^{m,l}(X,o)$; it is a subset of S_0. Directly from the proof of Theorem 3.14, which in turn hinges on the asymptotic expansion given in the proof of Lemma 3.7, we have:

Lemma 3.17. *For $A \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;o)$, $\mathrm{WF}'(A) = \emptyset$ if and only if $A \in \Psi_{2,h}^{-\infty,l}(\mathbb{R}^n;o)$.*

For $A, B \in \Psi_{2,h}(\mathbb{R}^n;o)$, $\mathrm{WF}'(A+B) \subset \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$ and $\mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$.

The notion of the operator wave front set also allows us to show that microlocally away from o, $\Psi_{2,h}^{m,l}(X,o)$ is the same as $\tilde{\Psi}_h^m(X)$. If $Q \in \tilde{\Psi}_h^m(X)$, we let $\mathrm{WF}_h'(Q)$ denote the usual semiclassical wave front set, while treating Q as an element of $\Psi_{2,h}(X;o)$, we have $\mathrm{WF}'(Q) = \beta^{-1}(\mathrm{WF}_h'(Q))$ directly from the definition of WF' and WF_h' as essential support.

Lemma 3.18. *If $Q \in \tilde{\Psi}_h^{m'}(X)$ and $\mathrm{WF}_h'(Q) \cap o = \emptyset$ then for all $A \in \Psi_{2,h}^{m,l}(X,o)$, and QA, $AQ \in \tilde{\Psi}_h^{m+m'}(X)$, $\mathrm{WF}'(QA), \mathrm{WF}'(AQ) \subset \mathrm{WF}'(Q)$.*

Thus, the set of operators $Q \in \tilde{\Psi}_h(X)$ with $\mathrm{WF}_h'(Q) \cap o = \emptyset$ is a two-sided ideal in $\Psi_{2,h}(X,o)$.

Proof. It is straightforward to see the conclusion when A is residual, as

$$(h/\xi_j) D_{x_j} e^{i(x-y)\cdot\xi/h} = e^{i(x-y)\cdot\xi/h},$$

so integration by parts in x, using that $\xi \neq 0$ on $\operatorname{esssupp} q$ (where $Q = {}^h\mathrm{Op}_\mathrm{r}(q)$), shows that, if K is the Schwartz kernel of $R \in \mathcal{R}^l$,

$$R\,{}^h\mathrm{Op}_\mathrm{l}(q)(z,y) = (2\pi h)^{-n} \int K(z,x)\, e^{i(x-y)\cdot\xi/h} q(y,\xi,h)\, d\xi\, dx$$

lies in $h^\infty \mathcal{C}^\infty(X^2 \times [0,1))$. Thus, we may assume that $A = {}^h\mathrm{Op}_\mathrm{l}(a)$, and $Q = {}^h\mathrm{Op}_\mathrm{r}(q)$, $q \in h^{-m'} \mathcal{C}_c^\infty(T^*X \times [0,1))$ with $\operatorname{esssupp} q \cap o = \emptyset$. Then $AQ = I(b)$, $b(x,y,\xi,h) = a(x,\xi,h) q(y,\xi,h)$, as in the proof of Theorem 3.14, and

$$b \in S^{m+m',-\infty}(\mathsf{S}) \subset S^{m+m'}(T^*X \times [0,1)),$$

so $I(b) \in \tilde{\Psi}_h^{m+m'}(X)$. □

We now check that the properties listed in Section 2 hold for $\Psi_{2,h}(X,o)$:

(i) This is Theorem 3.14, plus the observation that if $A_j - {}^h\mathrm{Op}_\mathrm{l}(a_j) \in \mathcal{R}^l$, $a_j \in S^{m-j,l}(\mathsf{S})$, then one can Borel sum the a_j to some $a \in S^{m,l}(\mathsf{S})$ with $a - \sum_{j=0}^{N-1} a_j \in S^{m-N,l}(\mathsf{S})$ for all N.
(ii) This is Lemma 3.16, Lemma 3.15, and the preceding discussion.
(iii) One can take $\mathrm{Op} = {}^h\mathrm{Op}_\mathrm{l}$, for instance.
(iv) This is Lemma 3.17 and the preceding discussion.
(v) This is Lemma 3.16.
(vi) This is a standard consequence of (i)-(iv).
(vii) The uniform boundedness of $\Psi_{2,h}^{m,m}(X,o)$ from $h^k L^2$ to $h^{k-m} L^2$ follows from the uniform boundedness of $\Psi_{2,h}^{0,0}(X,o)$ on L^2, which in turn is a consequence of the corresponding property of $\mathcal{R}^0$ and of the argument of Calderón-Vaillancourt, as noted in the proof of Lemma 3.13.

For $A \in \Psi_{2,h}^{-\infty,l}(X,o)$, $A : L^2(X) \to I_{(-l)}^{\infty}(o)$ follows from the definition of $\mathcal{R}^l$ and $h^{-j}A_1 \dots A_j = (h^{-1}A_1)\dots(h^{-1}A_j)$, $h^{-1}A_i \in \Psi_{2,h}^{1,0}(X,o)$, so $h^{-j}A_1 \dots A_j A \in \Psi_{2,h}^{-\infty,l}(X,o)$, hence maps $L^2(X)$ to $h^{-l}L^2(X)$.

As $\mathcal{M} = \{A \in \tilde{\Psi}_h^1(X) : \ \sigma(A)|_o = 0\}$ is a (locally) finitely generated module over $\tilde{\Psi}_h^0(X)$, with any set of $\mathcal{C}^\infty$ vector fields spanning T_pX for all p giving a set of generators (for vanishing of the principal symbol at o means that $A = q_L(h^{-1}a)$, $a|_o = 0$, so $a = \sum a_j\xi_j$ in local coordinates by Taylor's theorem), closed under commutators, we deduce that for non-negative integers k,

$$I_{(0)}^k(o) = L^\infty((0,1)_h; H^k_{\mathrm{loc}}(X)),$$

hence (by interpolation and duality) in general the same formula still holds. Note also that these spaces are local. In case $X = \mathbb{R}^n$, using the 'large calculus' discussed at the end of Section 2, since $A = (1+\Delta)^{k/2} \in \Psi_{2,h}^{k,k,0}(\mathbb{R}^n;o)$ is elliptic, we have produced an elliptic operator $A \in \Psi_{2,h}^{k,k,0}(\mathbb{R}^n;o)$ such that $u \in I_{(0)}^k$ implies $Au \in L^2$. The elliptic parametrix construction shows the converse, so for all $k \geq 0$ (multiplying by h^s if needed)

$$I_{(s)}^k = \{u \in h^sL^2 : \ \exists \text{ elliptic } A \in \Psi_{2,h}^{k,k,0}(X,o), \ Au \in h^sL^2\}.$$

Given $A \in \Psi_{2,h}^{k,k,0}(X,o)$ elliptic there exists a parametrix $B \in \Psi_{2,h}^{-k,-k,0}(X,o)$, with $BA - \mathrm{Id} = E \in \Psi_{2,h}^{-\infty,-\infty,0}(X,o)$. Thus if $\tilde{A} \in \Psi_{2,h}^{k-m,k-m,0}(X,o)$ is also elliptic then for any $P \in \Psi_{2,h}^{m,l}(X,o)$,

$$\tilde{A}Pu = \tilde{A}P(BA-E)u = (\tilde{A}PB)(Au) - (\tilde{A}PE)u,$$

with $\tilde{A}PB \in \Psi_{2,h}^{0,l}(X,o)$, $\tilde{A}PE \in \Psi_{2,h}^{-\infty,l}(X,o)$ both bounded $h^sL^2 \to h^{s-l}L^2$. As a result, we conclude that $P : I_{(s)}^k(o) \to I_{(s-l)}^{k-m}(o)$, whenever $k, k-m \geq 0$. The general case follows by duality arguments.

(viii) These are standard consequences of the definition of $\mathrm{WF}^{m,l}$, and the properties of $\Psi_{2,h}(X,o)$, in particular (iv), (vi) and (vii).

(ix) See Remark 3.12. The case when $\sigma_h(A) = 0$ on o is evident from the fact that we may write such an operator as $A = {}^h\mathrm{Op}_l(a)$ with $h^{-k}a$ vanishing on $h = \xi = 0$, hence lifting to be in $S^{k,k-1}(\mathsf{S})$.

(x) See Lemma 3.18.

(xi) Follows from (iv).

The following result is necessary in order to transfer our definition from the model case $\mathcal{L} = o$ to the case of a general Lagrangian in an invariant way.

Proposition 3.19. *Let T be a properly supported semiclassical FIO with canonical relation Φ equal to the identity on the zero section, and with T elliptic on $\mathcal{U}$, an open set in $T^*\mathbb{R}^n$; let S be a microlocal parametrix for T on $\mathcal{U}$. Then for $A \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;o)$ with $\mathrm{WF}'(A) \subset \mathcal{U}$,*

$$TAS \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;o),$$

with

$${}^2\sigma(TAS) = (\Phi^{-1})^*\, {}^2\sigma(A),$$

and

$$\mathrm{WF}'(TAS) = \Phi(\mathrm{WF}'(A)).$$

The proof proceeds by deformation to a pseudodifferential operator—cf. section 10 of [7] and [9].

Proof. A priori, we have

$$\Phi(x,\xi) = (X(x,\xi), \Xi(x,\xi))$$

with

$$X_i(x,0) = x_i, \quad \Xi(x,0) = 0,$$

i.e.,

$$\frac{\partial X_i}{\partial x_j}\bigg|_{(x,0)} = \delta_{ij}, \quad \frac{\partial \Xi_i}{\partial x_j}\bigg|_{(x,0)} = 0.$$

As Φ is a symplectomorphism,

$$\sum d\xi_i \wedge dx_i = \sum d\Xi_i \wedge dX_i;$$

evaluating this expression at $\xi = 0$ then yields the further information that

$$\frac{\partial \Xi_i}{\partial \xi_j}\bigg|_{(x,0)} = \delta_{ij},$$

hence

$$(X_i, \Xi_i) = (x_i + O(\xi), \xi_i + O(\xi^2)).$$

Consequently, Φ can be parametrized by a *generating function* (see [2, §47A]) $S(x,\Xi) = x\cdot\Xi + \sum \Xi_i\Xi_j\tilde{S}(x,\Xi)$. In a neighborhood of o, we may thus connect Φ to the identity map via a family of symplectomorphisms Φ_t parametrized by $x\cdot\Xi + t\sum \Xi_i\Xi_j\tilde{S}(x,\Xi)$. Thus, $\Phi_0 = \mathrm{Id}$, $\Phi_1 = \Phi$, and Φ_t fixes the zero section for each t. We connect T to a semiclassical pseudodifferential operator (microlocally near o) via a family T_t of elliptic semiclassical FIOs given by

$$T_t = h^{(-n+N)/2} \int e^{i\phi(t,x,y,\theta)/h} b(t,x,y,\theta;h)\, d\theta, \tag{10}$$

with $T_0 \in \tilde{\Psi}_h^0(\mathbb{R}^n)$, $T_1 = T$, and T_t having the canonical relation Φ_t. Let S_t be a family of parametrices. Then, for $A \in \Psi_{2,h}(\mathbb{R}^n;o)$, we have

$$\frac{d}{dt} T_t A S_t \cong [T_t' S_t, T_t A S_t] \bmod \tilde{\Psi}_h^{-\infty}(\mathbb{R}^n).$$

Hence, setting $A(t) = T_t A S_t$, we have

$$A'(t) \cong [P(t), A(t)] \bmod \tilde{\Psi}_h^{-\infty}(\mathbb{R}^n), \tag{11}$$

with

$$A(0) \in \Psi_{2,h}^{m,l}(\mathbb{R}^n;0)$$

having the desired wavefront and symbol properties, by the properties of the calculus $\Psi_{2,h}$. A priori, we have

$$P(t) = T_t' S_t \in \tilde{\Psi}_h^1(\mathbb{R}^n).$$

However, since the canonical relation of T_t is always the identity on o, we can parametrize the FIOs T_t by phase functions of the form

$$\phi(t,x,\theta) = (x-y)\cdot\theta + O(\theta^2); \tag{12}$$

thus, $\partial_t\phi = O(\theta^2)$. Differentiating (10), we see that there are two terms in T_t', coming from differentiation of the phase ϕ and the amplitude b; the latter gives a term in $T_t' S_t$ lying in $\tilde{\Psi}_h^0(\mathbb{R}^n) \subset \Psi_{2,h}^{0,0}(\mathbb{R}^n;o)$. The former, by (12), has amplitude

$h^{-1}O(\theta^2)b$, i.e. is of the form h^{-1} times an order zero FIO with symbol vanishing to second order on $(o\times o)\cap\operatorname{diag}$. Consequently, the contribution to $T'_t S_t$ from this term is an element of $\tilde\Psi^1_h(\mathbb{R}^n)$ with principal symbol vanishing on the diagonal to second order. By property (ix), one order of vanishing yields

$$P(t)\in\Psi^{1,0}_{2,h}(\mathbb{R}^n;\mathcal{L});$$

the second order of vanishing additionally gives[9]

$$^2\sigma(P(t))=0 \text{ on } SN(\mathcal{L}).$$

Thus the ODE (11) can be solved for $A(t)\in\Psi^{m,l}_{2,h}(\mathbb{R}^n;o)$ order-by-order, with a remainder in $\mathcal{R}^l$ (which can be integrated away). Moreover, the principal symbol on $SN(\mathcal{L})$ is manifestly constant, as the Hamilton vector field of $P(t)$ vanishes there. Thus

$$^2\sigma(TAS)\restriction_{SN(\mathcal{L})}={}^2\sigma(A)\restriction_{SN(\mathcal{L})}.$$

On the other hand, on $\mathsf{S}_0\backslash SN(\mathcal{L})$, the corresponding statement follows from the usual semiclassical Egorov theorem and property (x) of the calculus.

The statement about microsupports is likewise straightforward from the ODE. □

Definition 3.20. Suppose X is a manifold without boundary and $\mathcal{L}$ is a Lagrangian submanifold of T^*X such that the restriction of the bundle projection, $\pi_{\mathcal{L}}:\mathcal{L}\to X$, is proper. We say that a family of operators $A=A_h:\mathcal{C}^\infty(X)\to\mathcal{C}^\infty(X)$, $h\in(0,1)$, with properly supported Schwartz kernel $K_A\in\mathcal{C}^{-\infty}(X\times X\times[0,1);\pi_R^*\Omega X)$, is in $\Psi^{m,l}_{2,h}(X;\mathcal{L})$ if

(1) for $Q\in\tilde\Psi^0_h(X)$ with $\mathrm{WF}'(Q)\cap\mathcal{L}=\emptyset$, $QA, AQ\in\tilde\Psi^m_h(X)$,
(2) for each point $q\in\mathcal{L}$ and neighborhood $\mathcal{U}\subset T^*X$ of q symplectomorphic, via a canonical transformation Φ, to a neighborhood of $q'\in o\subset T^*\mathbb{R}^n$, mapping $\mathcal{L}$ to o, and for each semiclassical Fourier integral operator T with canonical relation Φ elliptic in $\mathcal{U}$, with parametrix S, and for each $Q,Q'\in\tilde\Psi^0_h(X)$ with $\mathrm{WF}'(Q),\mathrm{WF}'(Q')\subset\mathcal{U}$, $TQAQ'S\in\Psi^{m,l}_{2,h}(\mathbb{R}^n;o)$,
(3) if $Q,Q'\in\tilde\Psi^0_h(X)$ with $\mathrm{WF}'(Q)\cap\mathrm{WF}'(Q')=\emptyset$, T, resp T', elliptic semiclassical FIOs mapping a neighborhood of $\mathrm{WF}'(Q)$, resp. $\mathrm{WF}'(Q')$, to a neighborhood of o, and $\mathcal{L}$ to o, with parametrices S, resp. S', then $TQAQ'S'\in\mathcal{R}^l$.

Definition 3.21. (A global quantization map.) Let $\{\mathcal{U}_j:\ j\in J\}$ be an open cover of $\mathcal{L}$ such that

(1) $\bar{\mathcal{U}}_j\subset T^*X$ is compact for each j,
(2) for each $K\subset X$ compact, $\pi^{-1}(K)\cap\mathcal{U}_j=\emptyset$ for all but finitely many j, where $\pi:T^*X\to X$ is the bundle projection.
(3) for each j there is a canonical transformation Φ_j from $\mathcal{U}_j$ to an open set $\mathcal{U}'_j$ in $T^*\mathbb{R}^n$ mapping $\mathcal{L}$ to o, with inverse Ψ_j, and a semiclassical Fourier integral operator T_j elliptic in $\mathcal{U}_j$ with canonical relation Φ_j with parametrix S_j.

(Such an open cover exists because each point in $\mathcal{L}$ has a neighborhood satisfying (1) and (3), and as $\pi_{\mathcal{L}}$ is proper, (2) can be fulfilled as well.) Let $J_*=J\cup\{*\}$ be

[9] One can see this simply by writing the total symbol in the form $\sum h^{-1}\xi_i\xi_j+O(1)$ and lifting it to $[T^*\mathbb{R}^n\times[0,1);o\times 0]$.

a disjoint union, and let $\mathcal{U}_* = T^*X \setminus \mathcal{L}$, so $\{\mathcal{U}_j : j \in J_*\}$ is an open cover of T^*X. Let $\{\chi_j,\ j \in J_*\}$ be a subordinate partition of unity. For $a \in S^{m,l}(\mathsf{S})$, let

$$\mathrm{Op}(a) = \sum_{j \in J_*} S_j\, {}^h\mathrm{Op}_{\mathrm{W}}(\Psi_j^*(\chi_j a))T_j. \tag{13}$$

(Here one could use ${}^h\mathrm{Op}_{\mathrm{l}}$ or ${}^h\mathrm{Op}_{\mathrm{r}}$ instead of ${}^h\mathrm{Op}_{\mathrm{W}}$ to obtain another quantization.)

Definition 3.22. With $(X;\mathcal{L})$, $A \in \Psi^{m,l}_{2,h}(X;\mathcal{L})$ as above, if $O \subset T^*X \setminus \mathcal{L}$, $Q \in \tilde{\Psi}^0_h(X)$, $\mathrm{WF}'(Q) \cap \mathcal{L} = \emptyset$, $\mathrm{WF}'(\mathrm{Id} - Q) \cap O = \emptyset$, then

$${}^2\sigma_{m,l}(A)|_O = \sigma_h(QA)|_O,\ \mathrm{WF}'(A) \cap O = \mathrm{WF}'(QA) \cap O,$$

while if Q, Q', S, T as in (2) of Definition 3.20, O a neighborhood of q such that $O \cap \mathrm{WF}'(\mathrm{Id} - Q) = \emptyset$, $O \cap \mathrm{WF}'(\mathrm{Id} - Q') = \emptyset$, then

$$\begin{aligned} &{}^2\sigma_{m,l}(A)|_O = \Phi^*\big({}^2\sigma_{m,l}(TQAQ'S)|_{\Phi(O)}\big),\\ &\mathrm{WF}'(A) \cap O = \Phi^{-1}(\mathrm{WF}'(TQAQ'S) \cap \Phi(O)). \end{aligned}$$

It is easy to check that in the overlap regions, the various cases give the same classes of operators. For instance, if $Q, Q' \in \tilde{\Psi}^0_h(X)$ with $\mathrm{WF}'(Q), \mathrm{WF}'(Q') \subset \mathcal{U}$ as in (2), but $\mathrm{WF}'(Q) \cap \mathrm{WF}'(Q') = \emptyset$, then taking $G, G' \in \tilde{\Psi}^0_h(X)$ such that $\mathrm{WF}'(G) \cap \mathrm{WF}'(G') = \emptyset$, $\mathrm{WF}'(\mathrm{Id} - G) \cap \mathrm{WF}'(TQS) = \emptyset$, $\mathrm{WF}'(\mathrm{Id} - G') \cap \mathrm{WF}'(TQ'S) = \emptyset$, $TQAQ'S - GTQAQ'SG' \in \tilde{\Psi}^{-\infty}_h(X)$ as $(\mathrm{Id} - G)TQS \in \tilde{\Psi}^{-\infty}_h(X)$ by properties of the standard semiclassical calculus, etc., so $TQAQ'S \in \Psi^{m,l}_{2,h}(\mathbb{R}^n; o)$, and Lemma 3.17 shows that $GTQAQ'SG' \in \mathcal{R}^l$, so $TQAQ'S \in \mathcal{R}^l$ as well. Thus, in the overlap, where both make sense, the cases (2) and (3) are equivalent.

Correspondingly, Op indeed maps into $\Psi_{2,h}(X;\mathcal{L})$:

$$TQS_j\, {}^h\mathrm{Op}_{\mathrm{W}}(\Psi_j^*(\chi_j a))T_jQ'S \in \Psi^{m,l}_{2,h}(\mathbb{R}^n; o),\ TQS_j\, {}^h\mathrm{Op}_{\mathrm{W}}(\Psi_j^*(\chi_j a))T_jQ'S' \in \mathcal{R}^l,$$

as TQS_j and $T_jQ'S$ are semiclassical Fourier integral operators preserving the zero section. It is similarly easy to check that the principal symbol and operator wave front sets are well-defined (one only needs to check on $SN(\mathcal{L})$, as away from this face they agree with the corresponding semiclassical quantities).

The proof of the properties (i)-(xi) follows from the case $\mathcal{L} = o$ using the semiclassical FIOs as in the definition, Proposition 3.19 and Lemma 3.18.

If $\mathcal{L}$ is a torus, there is an improved quantization map (for symbols supported sufficiently close to $\mathcal{L}$) for which the full asymptotic formula for composition is given by the formula from Weyl calculus. First, suppose that $X = \mathbb{T}^n$, and $\mathcal{L}$ is the zero section. Let $\{\phi_i : i \in I\}$ be a partition of unity subordinate to a finite cover of X by coordinate charts (O_i, F_i), $G_i = F_i^{-1}$, such that the transition maps $F_i \circ G_j$ between coordinate charts are all given by translations in $\mathbb{R}^n$, and let $\psi_i \equiv 1$ on a neighborhood of $\operatorname{supp}\phi_i$. Then for $a \in S^{m,l}(\mathsf{S})$, define

$$\mathrm{Op}(a) = \sum_i F_i^*\psi_i\, {}^h\mathrm{Op}_{\mathrm{W}}(\phi_i a)\psi_i G_i^*.$$

For $b \in S^{m,l}(\mathsf{S})$ supported in $T^*_{O_i \cap O_j}X$,

$${}^h\mathrm{Op}_{\mathrm{W}}(b) - (F_j \circ G_i)^*\, {}^h\mathrm{Op}_{\mathrm{W}}(b)(F_i \circ G_j)^* \in \mathcal{R}^l.$$

It follows that for $a \in S^{m,l}(\mathsf{S})$,

$$\mathrm{Op}(a)^* - \mathrm{Op}(\bar{a}) \in \mathcal{R}^l,$$

with the adjoint taken with respect to a translation invariant measure, and the composition formula in the Weyl calculus (cf. [4, Theorem 7.3 et seq.]) holds for the *global* quantization map Op: for $a \in S^{m,l}(\mathsf{S})$, $b \in S^{m',l'}(\mathsf{S})$,

$$\operatorname{Op}(a)\operatorname{Op}(b) = \operatorname{Op}\left(\sum \frac{h^{|\alpha+\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\left((\partial_x^\alpha \partial_\xi^\beta a)(\partial_\xi^\alpha \partial_x^\beta b)\right)\right) + E \tag{14}$$

where the sum is a Borel sum, and $E \in \Psi_{2,h}^{-\infty,l+l'}(\mathbb{T}^n; o)$.

Now, if $\mathcal{L}$ is a Lagrangian torus in T^*X, there may not exist in general a globally defined Fourier integral operator from a neighborhood of $\mathcal{L}$ in T^*X to a neighborhood of the zero section in $T^*\mathbb{T}^n$, even though the underlying canonical relation Φ exists: such a choice may in general be obstructed by both the Maslov bundle and Bohr-Sommerfeld quantization conditions. In fact, as we are conjugating, a multi-valued FIO suffices, as noted by Hitrik and Sjöstrand [9] (see also [5] in the non-semiclassical case). We can phrase this slightly differently, with the notation of Definition 3.21, locally identifying $\mathbb{T}^n$ with $\mathbb{R}^n$, by choosing an open cover of $\mathcal{L}$ by open sets $\mathcal{U}_j$ ($j \in J$) with $\mathcal{U}_k \cap \mathcal{U}_j$ contractible for all $k, j \in J$, and choosing Fourier integral operators T_j associated to the canonical relation $\Phi|_{\mathcal{U}_j}$ mapping from the open subsets $\mathcal{U}_j$ of T^*X to $T^*\mathbb{T}^n$ (mapping $\mathcal{L}$ to the zero section), such that

(1) T_j is unitary (modulo $O(h^\infty)$) on a neighborhood of $\operatorname{supp}\chi_j$, $j \in J$, i.e. $T_j^*T_j - \operatorname{Id} \in \tilde{\Psi}_h^{-\infty}(X)$ microlocally near $\operatorname{supp}\chi_j$ (so we can take $S_j = T_j^*$),
(2) on $\operatorname{supp}\chi_k \cap \operatorname{supp}\chi_j$, $T_k^*T_j \in \tilde{\Psi}_h^0(X)$ is given by multiplication by $c_{kj}e^{i\alpha_{kj}/h}$ for some constants $c_{kj} \neq 0$ and α_{kj} (modulo $O(h^\infty)$) and for $j, k \in J$ (so not necessarily so if $j, k \in J_*$),
(3) and finally, for each $j \in J$, in local coordinates x on $\mathbb{T}^n$ and y on X, in which the measure is $|dx|$, resp. $|dy|$, T_j is given by an oscillatory integral of the form
$$T_j u(x) = h^{-(n+N)/2} \int e^{i\phi(x,y,\theta)/h} t_j(x,y,\theta,h) u(y)\, dy\, d\theta$$
where ϕ parameterizes the Lagrangian corresponding to $\Phi|_{\mathcal{U}_j}$, and $t_j|_{C_\phi \times \{0\}}$ has constant argument, where $C_\phi = \{(x,y,\theta) : d_\theta \phi(x,y,\theta) = 0\}$, so the graph of $\Phi|_{\mathcal{U}_j}$ is $\{((x, d_x\phi(x,y,\theta)), (y, -d_y\phi(x,y,\theta))) :\ (x,y,\theta) \in C_\phi\}$.

Such a choice of the T_j exists (see [9, §2]), and the last condition implies (see [9, Proposition 2.1]) that for $p \in h^{-m}\mathcal{C}_c^\infty(T^*X \times [0,1))$,

$$T_j^* \, {}^h\mathrm{Op_W}((\Phi_{\mathcal{U}_j}^{-1})^* p) T_j - {}^h\mathrm{Op_W}(p) \in \tilde{\Psi}_h^{m-2}(X). \tag{15}$$

Then (13) gives a global quantization map.

As $T_k S_j \in \tilde{\Psi}_h^0(\mathbb{T}^n)$ is given by multiplication by $c_{jk}^{-1} e^{-i\alpha_{jk}/h}$ (modulo $O(h^\infty)$), if $b \in S^{m,l}(\mathsf{S})$ is supported in $\Phi(\mathcal{U}_j) \cap \Phi(\mathcal{U}_k)$, then $S_j \, {}^h\mathrm{Op_W}(b) T_j - S_k \, {}^h\mathrm{Op_W}(b) T_k \in \Psi_{2,h}^{-\infty,l}(X)$, for $T_k S_j \, {}^h\mathrm{Op_W}(b) T_j S_k - {}^h\mathrm{Op_W}(b) \in \mathcal{R}^l$. Thus for $a \in S^{m,l}(\mathsf{S})$, $b \in S^{m',l'}(\mathsf{S})$ with $\operatorname{esssupp} a, \operatorname{esssupp} b \subset T^*X \setminus \operatorname{supp}\chi_*$,

$$\begin{aligned}
\operatorname{Op}(a)\operatorname{Op}(b) &= \sum_{j,k \in J} T_j^* \, {}^h\mathrm{Op_W}(\Psi^*(\chi_j a)) T_j T_k^* \, {}^h\mathrm{Op_W}(\Psi^*(\chi_k b)) T_k \\
&\cong \sum_{j,k \in J} T_j^* T_j T_k^* \, {}^h\mathrm{Op_W}(\Psi^*(\chi_j a)) \, {}^h\mathrm{Op_W}(\Psi^*(\chi_k b)) T_k \\
&\cong \sum_{j,k \in J} T_k^* \, {}^h\mathrm{Op_W}(\Psi^*(\chi_j a)) \, {}^h\mathrm{Op_W}(\Psi^*(\chi_k b)) T_k \text{ modulo } \mathcal{R}^{l+l'},
\end{aligned}$$

using properties (1) and (2) of the T_j, so using the symplectomorphism invariance of the Weyl composition formula,

$$\mathrm{Op}(a)\,\mathrm{Op}(b)=\mathrm{Op}\left(\sum\frac{h^{|\alpha+\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\Big((\partial_x^\alpha\partial_\xi^\beta a)(\partial_\xi^\alpha\partial_x^\beta b)\Big)\right)+E \tag{16}$$

where the sum is a Borel sum, computed in any local coordinates, and $E\in\Psi_{2,h}^{-\infty,l+l'}(X;\mathcal{L})$. Since modulo $\mathcal{R}$ composition is microlocal, it suffices if one of a,b satisfies the support condition.

In addition, if a satisfies the support condition, then $\mathrm{Op}(a)^*-\mathrm{Op}(\bar a)\in\mathcal{R}^l$, so replacing Op by

$$\mathrm{Op}'(a)=\frac12(\mathrm{Op}(a)+\mathrm{Op}(a)^*)+\frac12(\mathrm{Op}(a)-\mathrm{Op}(\bar a)),$$

for real-valued a, $\mathrm{Op}'(a)$ is self-adjoint. Thus, we have the following result:

Proposition 3.23. *Suppose that $\mathcal{L}$ is a Lagrangian torus in T^*X, with $\pi_{\mathcal{L}}:\mathcal{L}\to X$ proper. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{L}$ in T^*X and a quantization map $\mathrm{Op}:S^{m,l}(\mathsf{S})\to\Psi_{2,h}^{m,l}(X;\mathcal{L})$ satisfying all properties listed in Section 2, and such that, in addition,*

(1) *if O is a coordinate chart in X in which the volume form is given by the Euclidean measure, then for $p\in h^{-m}\mathcal{C}_c^\infty(T^*O\times[0,1))$ with $\operatorname{esssupp}p\subset\mathcal{U}$,*

$$\mathrm{Op}(p)-{}^h\mathrm{Op}_{\mathrm{W}}(p)\in\tilde\Psi_h^{m-2}(X). \tag{17}$$

(2) *for $a\in S^{m,l}(\mathsf{S})$, $b\in S^{m',l'}(\mathsf{S})$ with either $\operatorname{esssupp}a\subset\mathcal{U}$ or $\operatorname{esssupp}b\subset\mathcal{U}$, (16) holds,*
(3) *if a is real-valued with $\operatorname{esssupp}a\subset\mathcal{U}$, then $\mathrm{Op}(a)$ is self-adjoint,*
(4) *for any a satisfying $\operatorname{esssupp}a\subset\mathcal{U}$, $\mathrm{Op}(a)^*-\mathrm{Op}(\bar a)\in\mathcal{R}^l$.*

4. Real principal type propagation

Recall that we let S_0 denote the space $[T^*X;\mathcal{L}]$ on which principal symbols of operators live and $\mathsf{S}=[T^*X\times[0,1);\mathcal{L}\times 0]$ the space for total symbols. If $P\in\Psi_{2,h}^{m,l}(X;\mathcal{L})$ has real principal symbol p, its Hamilton vector field $\bar{\mathsf{H}}$ is a vector field on S_0, conormal to $SN(\mathcal{L})=\partial\mathsf{S}_0$. If $\tilde\rho_{\mathrm{ff}}$ is a boundary defining function for $SN(\mathcal{L})\subset\mathsf{S}_0$, then the appropriately rescaled Hamilton vector field

$$\mathsf{H}=\tilde\rho_{\mathrm{ff}}^{\,l-m+1}\bar{\mathsf{H}}$$

is a smooth vector field on S_0, tangent to its boundary, $SN(\mathcal{L})$. In particular, if a point in an orbit of H is in $\partial\mathsf{S}_0$, the whole orbit is in $\partial\mathsf{S}_0$.

The following result is the corresponding real principal type propagation theorem.

Theorem 4.1. *Let $P\in\Psi_{2,h}^{m,l}(X;\mathcal{L})$ have real principal symbol p. If $u\in I_{(r)}^{-\infty}(\mathcal{L})$, $Pu=f$ then*

(I) ${}^2\mathrm{WF}^{k,r}(u)\setminus{}^2\mathrm{WF}^{k-m,r-l}(f)\subset{}^2\Sigma(P)\equiv\{{}^2\sigma(P)=0\}$.
(II) ${}^2\mathrm{WF}^{k,r}(u)\setminus{}^2\mathrm{WF}^{k-m+1,r-l}(f)$ *is invariant under the Hamilton flow of p inside ${}^2\Sigma(P)$.*

Proof. This is just the usual real principal type propagation away from the boundary of S_0, so we only need to consider points at $\partial\mathsf{S}_0$.

The proof of the first part follows from the existence of elliptic parametrices (property (vi) of the calculus).

The proof of the flow invariance follows the outline of Hörmander's classic commutator proof of the propagation of singularities for operators of real principal type [10], hence we give only a sketch here. (See also [19] for an account of essentially the same proof in the setting of a different pseudodifferential calculus.)

Pick $q \in {}^2\Sigma(P) \subset SN(\mathcal{L})$. Let H denote the Hamilton vector field of P, and assume that

$$\exp(r_0\mathsf{H})q \notin {}^2\mathrm{WF}^{\alpha,r}(u) \tag{18}$$

and

$$\exp(t\mathsf{H})q \notin {}^2\mathrm{WF}^{\alpha-1/2,r}(u),\ \exp(t\mathsf{H})q \notin {}^2\mathrm{WF}^{\alpha-m+1,r-l}(f) \quad \text{for } t \in [0,r_0]; \tag{19}$$

we will show that for $r_0 > 0$ sufficiently small (depending on H, but not depending on u) (18)–(19) imply that

$$\exp(t\mathsf{H})q \notin {}^2\mathrm{WF}^{\alpha,r}(u) \quad \text{for } t \in [0,r_0]. \tag{20}$$

We obtain the corresponding result with the interval $[0,r_0]$ in (18) replaced by $[-r_0,0]$ by applying the result to the operator $-P$. If $\alpha < k$, we can then iterate this argument to obtain the desired result.

To prove that (18)–(19) imply (20), let $\chi_0(s) = 0$ for $s \le 0$, $\chi_0(s) = e^{-M/s}$ for $s > 0$ ($M > 0$ to be fixed), let $\chi \ge 0$ be a smooth non-decreasing function on $\mathbb{R}$ with $\chi = 0$ on $(-\infty,0]$ and $\chi = 1$ on $[1,\infty)$, with $\chi^{1/2}$ and $(\chi')^{1/2}$ both smooth. Let ϕ be a cutoff function supported in $(-1,1)$. As H is a vector field tangent to the boundary of S_0, assuming that H does not vanish at $q \in \partial\mathsf{S}_0$ (otherwise there is nothing to prove), we can choose local coordinates $\rho_1,\dots,\rho_{2n}$ on $[T^*X;\mathcal{L}]$ centered at q in which $\mathsf{H} = \partial_{\rho_1}$, and the boundary is defined by $\rho_{2n} = 0$, so we may take $\tilde\rho_{\mathrm{ff}} = \rho_{2n}$. We choose r_0 so that $\exp(t\mathsf{H})q$, $t \in [-r_0,2r_0]$, remains in a compact subset of the coordinate chart given by the ρ_j. Let $\rho' = (\rho_2,\dots,\rho_{2n})$, and set

$$\begin{aligned} a =&\tilde\rho_{\mathrm{ff}}^{-r+l/2+\alpha-(m-1)/2}\phi^2(\lambda^2|\rho'|^2)\chi_0(\lambda\rho_1+1)\chi(\lambda(r_0-\rho_1)-1)\\ &\in \tilde\rho_{\mathrm{ff}}^{-r+l/2+\alpha-(m-1)/2}\mathcal{C}^\infty(\mathsf{S}_0). \end{aligned}$$

Then a has support in the region where we have assumed regularity, provided λ is chosen large enough since $|\rho'| < \lambda^{-1}$, $\rho_1 \ge -\lambda^{-1}$, $\rho_1 \le r_0 - \lambda^{-1}$ on $\operatorname{supp} a$. Let $A \in \Psi_{2,h}^{\alpha-(m-1)/2,r-l/2}(X;\mathcal{L})$ have principal symbol a; recall that the principal symbol of an element of $\Psi_{2,h}^{\alpha-(m-1)/2,r-l/2}(X;\mathcal{L})$ arises by factoring out $h^{-(\alpha-(m-1)/2)}$ from the 'amplitude' a_{tot} (with $A = {}^h\mathrm{Op}_{\mathrm{l}}(a_{\mathrm{tot}})$ locally), explaining the power of $\tilde\rho_{\mathrm{ff}}$ appearing above.

Noting that the weight function $\tilde\rho_{\mathrm{ff}} = \rho_{2n}$ has vanishing derivative along H, we compute

$$\begin{aligned} {}^2\sigma_{2\alpha,2r}(-i(A^*AP - P^*A^*A)) &= {}^2\sigma_{2\alpha,2r}(-i[A^*A,P]) + {}^2\sigma_{2\alpha,2r}(-iA^*A(P-P^*))\\ &= \mathsf{H}a^2 - i\,{}^2\sigma_{m-1,l}(P-P^*)a^2 = b^2 - e^2 + a^2q, \end{aligned}$$

where $b, e \in \tilde\rho_{\mathrm{ff}}^{r-\alpha}\mathcal{C}^\infty(\mathsf{S}_0)$ (arising from symbolic terms in which H has been applied to $\chi_0(\lambda\rho_1+\lambda^{-1})^2$, and $\chi(\lambda(r_0-\rho_1))^2$ respectively), $q \in \tilde\rho_{\mathrm{ff}}^{l-m+1}\mathcal{C}^\infty(\mathsf{S}_0)$ (arising from $P-P^*$). Thus, in view of the principal symbol short exact sequence,

$$-i(A^*AP-P^*A^*A)=B^*B-E^*E+A^*QA+R$$

with $B,E\in\Psi^{\alpha,r}_{2,h}(X;\mathcal{L})$ $Q\in\Psi^{m-1,l}_{2,h}(X;\mathcal{L})$, having $\mathrm{WF}'(B)$, etc., given by $\operatorname{esssupp} b$, etc., ${}^2\sigma_{\alpha,r}(B)=b$, etc., and $R\in\Psi^{2\alpha-1,2r}_{2,h}(X;\mathcal{L})$, with $\mathrm{WF}'(R)\subset\operatorname{esssupp} a$. We thus find that $\mathrm{WF}'(E)$ is contained in $|\rho'|<\lambda^{-1}$, $\rho_1\in[r_0-2\lambda^{-1},r_0-\lambda^{-1}]$, hence (for sufficiently large λ) in the complement of ${}^2\mathrm{WF}^{\alpha,r}(u)$, so $\|Eu\|$ is uniformly bounded, as $\mathrm{WF}'(R)$ is contained in $|\rho'|<\lambda^{-1}$, $\rho_1\in[-\lambda^{-1},r_0-\lambda^{-1}]$, so $|\langle Ru,u\rangle|$ is also uniformly bounded, and B is elliptic on $\exp(t\mathsf{H})q$, for $t\in[0,r_0-2\lambda^{-1}]$, with $a\leq CM\lambda^{-1}b$. Thus

$$\|Bu\|^2\lesssim|\langle Ru,u\rangle|+|\langle Au,Af\rangle|+|\langle Au,Q^*Au\rangle|+\|Eu\|^2.$$

Let $T\in\Psi^{(m-1)/2,l/2}_{2,h}(X;\mathcal{L})$ be elliptic on a neighborhood of $\mathrm{WF}'(A)$ and let T' be a parametrix; we rewrite $|\langle Au,Af\rangle|\lesssim\delta\|TAu\|^2+\frac{1}{4\delta}\|T'Af\|^2$ modulo residual errors; $TA\in\Psi^{\alpha,r}_{2,h}(X,\mathcal{L})$ with principal symbol a smooth non-vanishing multiple of a, so for δ sufficiently small the first term may be absorbed into $\|Bu\|^2$ (as $\sqrt{b^2-c^2a^2}$ is smooth for small $c>0$) modulo a residual term, while the second is uniformly bounded as $h\to 0$ by our assumption (19). On the other hand, $|\langle TAu,T'QAu\rangle|\lesssim\|TAu\|^2+\|T'QAu\|^2$ modulo residual errors, and $TA,T'QA\in\Psi^{\alpha,r}_{2,h}(X,\mathcal{L})$ with principal symbol a smooth non-vanishing multiple of a. For M sufficiently small, both terms can be absorbed into $\|Bu\|^2$ (modulo a term that can be absorbed into R). □

5. Propagation of ^{2}WF on Invariant Tori in Integrable Systems

Let $P\in\tilde\Psi^0_h(X)$. Assume that

$$P \text{ has real principal symbol } p \tag{21}$$

with Hamilton vector field denoted H, and assume that

$$\begin{gathered}\mathsf{H} \text{ is completely integrable in a neighborhood of}\\ \text{a compact Lagrangian invariant torus } \mathcal{L}\subset\{p=0\}.\end{gathered} \tag{22}$$

Let $(I_1,\dots,I_n,\theta_1,\dots,\theta_n)$ be the associated action-angle variables, i.e. symplectic coordinates $I\in\mathbb{R}^n$ (or some open set in $\mathbb{R}^n$), $\theta\in(S^1)^n$ such that $p=p(I)$ is independent of θ. Without loss of generality we may translate the action coordinates so that $\mathcal{L}$ is defined by $I_i=0$ for all $i=1,\dots,n$. Let $\omega_i=\partial p/\partial I_i$ and $\omega_{ij}=\partial^2p/\partial I_i\partial I_j$. Let $\overline{\omega}_i$ and $\overline{\omega}_{ij}$ denote the corresponding quantities restricted to $\mathcal{L}$ (where they are constant). We introduce coordinates on $[T^*X;\mathcal{L}]$ by setting

$$\rho=|I|=\left(\sum I_j^2\right)^{\frac12},\quad \hat I_j=\frac{I_j}{|I|}.$$

The front face of the blown-up space is defined by $\rho=0$ and is canonically identified with the spherical normal bundle $SN(\mathcal{L})$.

The real principal symbol assumption on P means that (with respect to the inner product on $L^2(X)$ given by *any* smooth density on X) $P^*-P\in\tilde\Psi^{-1}_h(X)$, i.e. P is self-adjoint to leading order. It turns out that for our improved result we need at the very least that $\sigma_{h,-1}(P^*-P)$ vanishes at $\mathcal{L}$ with respect to some inner product; unlike the statement that $P^*-P\in\tilde\Psi^{-1}_h(X)$, this depends on the choice of an inner product. So we assume from now on that X has a fixed density ν on it (e.g. a

Riemannian density). This density ν in turn yields a trivialization of the bundle of half-densities on X. As is well-known, this yields a canonically defined *subprincipal symbol* $\operatorname{sub} A$ for a semiclassical pseudodifferential operator $A \in \tilde{\Psi}_h^m(X)$. In our (semiclassical) setting, $\operatorname{sub} P$ can be defined using the Weyl quantization—cf. the improved symbol invariance statement in Lemma 3.10 (see also, for instance, [11] in the non-semiclassical case). To obtain the subprincipal symbol we thus choose a coordinate system in which the Euclidean volume form agrees with the fixed one (this can always be arranged by changing one of the coordinates, while fixing the others); writing $A = {}^h\mathrm{Op}_{\mathrm{W}}(a)$ in these coordinates, we have

$$a = \sigma_h(A) + h\,\mathrm{sub}_h(A) + O(h^2). \tag{23}$$

As ${}^h\mathrm{Op}_{\mathrm{W}}(a)^* = {}^h\mathrm{Op}_{\mathrm{W}}(\bar{a})$ (adjoint taken with respect to the Euclidean volume form), if $\sigma_{h,m}(A)$ is real,

$$\sigma_{h,m-1}(A - A^*) = 2i \operatorname{Im} \mathrm{sub}_h(A),$$

so $A - A^* \in \tilde{\Psi}_h^{m-2}(X)$ if and only if $\mathrm{sub}_h(A)$ is real.

We now impose a weakened self-adjointness condition on P, namely that $P - P^* \in \tilde{\Psi}_h^{-2}(X)$ with respect to the fixed density, i.e. $\mathrm{sub}_h(P)$ is real; we further assume that $\mathrm{sub}_h(P)$ is constant on $\mathcal{L}$:

$$\mathrm{sub}_h(P) \text{ is real on } T^*X, \text{ and it is constant on } \mathcal{L}. \tag{24}$$

In fact, the slightly weaker assumption

$$\mathrm{sub}_h(P) \text{ is real and constant on } \mathcal{L} \tag{25}$$

would suffice; one would need to take care of $P - P^*$ much as in the proof of Theorem 4.1: we assume (24) as it covers the cases of interest.

We recall that as $\mathcal{L}$ is characteristic, we have $P \in \Psi_{2,h}^{0,-1}(X; \mathcal{L})$, and the principal symbol ${}^2\sigma_{0,-1}(P)$ near $SN(\mathcal{L})$ is

$$\sum \overline{\omega}_j I_j + \frac{1}{2} \sum \overline{\omega}_{ij} I_i I_j + O(I^3) = \rho \sum \overline{\omega}_j \hat{I}_j + \frac{\rho^2}{2} \sum \overline{\omega}_{ij} \hat{I}_i \hat{I}_j + O(\rho^3) \tag{26}$$

The Hamilton vector field is thus

$$\mathsf{H} = \sum \overline{\omega}_j \partial_{\theta_j} + \rho \sum \overline{\omega}_{ij} \hat{I}_i \partial_{\theta_j} + \rho^2 \mathsf{H}'$$

with H' smooth on $\mathsf{S}_0 = [T^*X; \mathcal{L}]$ and tangent to the boundary, $SN(\mathcal{L})$.

Theorem 5.1. *Assume that $u \in L^2$ and $Pu \in O(h^\infty)L^2$, where P and $\mathcal{L}$ satisfy* (21),(22),(24). *Then for each k and each $l \leq 0$, ${}^2\mathrm{WF}^{k,l}(u) \cap SN(\mathcal{L})$ is invariant under*

$$\mathsf{H}_1 = \sum \overline{\omega}_j \partial_{\theta_j}. \tag{27}$$

Also, for each k, and each $l \leq -1/2$, ${}^2\mathrm{WF}^{k,l}(u) \cap SN(\mathcal{L})$ is invariant under

$$\mathsf{H}_2 = \sum \overline{\omega}_{ij} \hat{I}_i \partial_{\theta_j} \tag{28}$$

Remark 5.2. In fact, for the first part of the theorem it suffices to adopt the weaker hypotheses that

$${}^2\mathrm{WF}^{k+1,l+1}(Pu) = \emptyset$$

and for the second part

$${}^2\mathrm{WF}^{k+1,l+2}(Pu) = \emptyset;$$

for instance, it certainly suffices to require $Pu \in h^s L^2$ where $s \geq \max(k+1, l+1)$ or $s \geq \max(k+1, l+2)$ rspectively.

The rest of this section will be devoted to a proof.

Invariance under H_1 follows from Theorem 4.1, as $h^{-1}P \in \Psi^{1,0}_{2,h}(X;\mathcal{L})$, and H_1 is the restriction of the Hamilton vector field to $SN(\mathcal{L})$; note that $\mathrm{WF}^{-\infty,l}(u) = \emptyset$ for each $l \leq 0$ as $u \in L^2$.

Invariance under H_2 requires considerable further discussion.

Given $\zeta \in SN(\mathcal{L})$, suppose that we know that $\mathrm{WF}^{k-1/2,l}\, u$ is invariant and that $\zeta \notin {}^2\mathrm{WF}^{k,l}(u)$. We need to show that the H_2 orbit through ζ is disjoint from ${}^2\mathrm{WF}^{k,l}(u)$; the general case can be obtained from this argument by the usual iteration. We know that the closure of the H_1-orbit of ζ is a torus $T_\zeta \subset SN(\mathcal{L})$, and that $T_\zeta \cap {}^2\mathrm{WF}^{k,l}\, u = \emptyset$ by H_1-invariance.

We extend the vector field H_1 to a neighborhood of $\partial \mathsf{S}_0$ using the coordinates (I,θ) as above. Thus, the closure of each orbit of H_1 near $\partial\mathsf{S}_0$ is still a torus. Let H_2 be defined on a neighborhood of $SN(\mathcal{L})$ by

$$\mathsf{H}_2 = \rho^{-1}(\mathsf{H} - \mathsf{H}_1);$$

this naturally agrees with (28) on $SN(\mathcal{L})$. Then $[\mathsf{H}_1, \mathsf{H}_2] = 0$ near $\mathcal{L}$ as the principal symbol of P is independent of θ there, so H, H_1 are linear combination of the ∂_{θ_j} with coefficients depending on I only, and $\rho = |I|$. Note that I is constant along the flow of H_1 and H_2, so for $\delta > 0$ small, the $(\mathsf{H}_1, \mathsf{H}_2)$ joint flow from $\rho < \delta$ stay in this prescribed neighborhood of $SN(\mathcal{L})$ for all times, on which I and θ are thus defined.

Now let $a_0 \in \mathcal{C}^\infty(\mathsf{S}_0)$ be supported in the complement of ${}^2\mathrm{WF}\, u$, but sufficiently close to $SN(\mathcal{L})$ (so that I, etc., are defined on a neighborhood of the $(\mathsf{H}_1, \mathsf{H}_2)$-flowout of $\operatorname{supp} a_0$), with a_0 having smooth square root, and

$$\mathsf{H}_1 a_0 = 0.$$

For $\delta \in (0,1)$ small, let

$$a_1 = -\int_0^\delta (2-s) a_0 \circ \exp(-s\mathsf{H}_2)\, ds$$

so that

$$\rho \mathsf{H}_2(a_1) = \rho \int_0^\delta a_0 \circ \exp(-s\mathsf{H}_2)\, ds + (2-\delta)\rho a_0 \circ \exp(-\delta \mathsf{H}_2) - 2\rho a_0 = \int_0^\delta b_s^2\, ds + c^2 - d^2$$

with b^2, c^2, d^2 real and vanishing to first order at $\rho = 0$, given by the three terms in the middle expression in the displayed formula. Note also that by construction

$$\mathsf{H}_1(a_1) = 0$$

since H_1 and H_2 commute.

Now let

$$a = h^{-2k} |I|^{-(2l+1)+2k} a_1, \tag{29}$$

and let Op be the quantization given in Proposition 3.23 corresponding to the local symplectomorphism $\Phi = (\theta, I)$ near $\mathcal{L}$, mapping a neighborhood of $\mathcal{L}$ to a neighborhood of the zero section of $\mathbb{T}^n$. Thus

$$A = \mathrm{Op}(a) \in \Psi^{2k,2l+1}_{2,h}(X)$$

is selfadjoint, with

$$^2\sigma(A) = h^{-2k}|I|^{-(2l+1)+2k}a_1,$$

and if $P = \mathrm{Op}(p)$ microlocally near $\mathcal{U}$, $p \in \mathcal{C}_c^\infty(T^*X \times [0,1))$, then

$$\begin{aligned} i((h^{-1}P)^*A - A(h^{-1}P) &\cong ih^{-1}(\mathrm{Op}(\bar{p})\,\mathrm{Op}(a) - \mathrm{Op}(a)\,\mathrm{Op}(p)) \\ &\cong \mathrm{Op}\left(\sum \frac{h^{|\alpha+\beta|-1}(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!}\Big((\partial_\theta^\alpha\partial_I^\beta\bar{p})(\partial_I^\alpha\partial_\theta^\beta a) - (\partial_\theta^\alpha\partial_I^\beta a)(\partial_I^\alpha\partial_\theta^\beta p)\Big)\right) \\ &\qquad \text{modulo } \mathcal{R}^{2l+1}. \end{aligned} \tag{30}$$

Here we used the symplectomorphism invariance of the Weyl composition formula in order to utilize the action-angle coordinates (which simply undoes the already used symplectomorphism invariance in obtaining (16)).

Note that by (17) and (23), if $P = \mathrm{Op}(p)$, $p = p_0 + hp_1' + O(h^2)$ then

$$\begin{aligned} \sigma_h(P) &= p_0 \\ \mathrm{sub}_h(P) &= p_1'. \end{aligned} \tag{31}$$

(The main novelty here is the formula for the subprincipal symbol.)

Let $B_s, C, D \in \Psi_{2,h}^{k,l}(X;\mathcal{L})$ have symbols b_s, c, d respectively. We may easily compute in the usual manner (since the weights commute with P to leading order):

$$^2\sigma(i((h^{-1}P)^*A - A(h^{-1}P))) = \rho\mathsf{H}_2(a_1) + 2a\,\mathrm{Im}\,\mathrm{sub}_h(P) = \rho\mathsf{H}_2(a_1)$$

by (24). Thus we have

$$i((h^{-1}P)^*A - A(h^{-1}P)) = \int_0^\delta B_s^*B_s\,ds + C^*C - D^*D + R \tag{32}$$

with $R \in \Psi_{2,h}^{2k-1,2l+1}(X;\mathcal{L})$, and $^2\mathrm{WF}'(R) \subset {}^2\mathrm{WF}'(A)$. Note, then, that a priori R has higher order than C^*C in the second index, as the invariance of $^2\sigma(A)$ along the H_1 flow yields a vanishing of the *principal* symbol of the "commutator" at $SN(\mathcal{L})$, but not necessarily of lower-order terms. However, the use of the special quantization Op gives us a better result:

Lemma 5.3. *We may decompose*

$$R = R_1 + R_2 \in \Psi_{2,h}^{2k-1,2l}(X;\mathcal{L}) + \Psi_{2,h}^{-\infty,2l+1}(X;\mathcal{L}).$$

Proof of Lemma. The subprincipal symbol of P is a real constant on $\mathcal{L}$; let μ denote this constant. Thus we have $P = \mathrm{Op}(p)$, $A = \mathrm{Op}(a)$, with

$$\begin{aligned} p &= p_0 + \mu h + O(Ih) + O(h^2) \\ &= \sum \overline{\omega}_j I_j + \frac{1}{2}\sum \overline{\omega}_{ij} I_i I_j + \mu h + O(\rho_{\mathrm{ff}}^2\rho_{\mathrm{sf}}) \equiv p_0 + p_1, \\ p_0 &= \sum \overline{\omega}_j I_j + \frac{1}{2}\sum \overline{\omega}_{ij} I_i I_j + O(I^3), \end{aligned} \tag{33}$$

where p_0 is independent of h, and a as in (29). Here $O(\rho_{\mathrm{sf}}^{-r}\rho_{\mathrm{ff}}^{-s})$ stands for an element of $S^{r,s}(\mathsf{S})$, and as usual ρ_{ff} denotes a boundary defining function for the front face, and ρ_{sf} a boundary defining function for the side face of the blowup $\mathsf{S} = [T^*X;\mathcal{L}]$.

Now we use (30). Writing $p = p_0 + p_1$ (as in (33)), the terms arising by replacing p by p_0 in (30) with $\alpha = \beta = 0$ cancel, while the terms with $|\alpha + \beta| = 1$ give $\mathrm{Op}(H_{p_0}a) \in \Psi_{2,h}^{2k,2l}(X)$, which differs from $\int_0^\delta B_s^*B_s\,ds + C^*C - D^*D$ by an element $\tilde{R}$

of $\Psi_{2,h}^{2k-1,2l}(X)$ (as they are both in $\Psi_{2,h}^{2k,2l}(X)$ and have the same principal symbol). Thus, R is obtained by taking the terms in (30) arising from p_1, as well as those arising from p_0 with $|\alpha+\beta|>1$, along with the remainder of the formula and $\tilde{R}$.

We now examine (30) in coordinates on $\mathsf{S}=[T^*X;\mathcal{L}]$ given locally by $H=h/I_1$, I_1, $\tilde{I}=I'/I_1=(I_2/I_1,\ldots,I_n/I_1)$, and $\theta_1,\ldots,\theta_n$ (these are of course valid only in one part of the corner of the blowup, but other patches are obtained symmetrically). Thus, H is a defining function for sf and I_1 for ff. By the analogous computation to (4), all terms have, a priori, the same conormal order at $SN(\mathcal{L})$ (the terms all have asymptotics I_1^{-2l-1} in the coordinates employed in (4), with powers of H ascending from H^{2k+1}). However, since p is actually a *smooth* function on $T^*X\times[0,1)$, hence we have $\partial_I^\gamma p=O(1)$ for all γ (with the above notation, so further ∂_θ derivatives have the same property), so if $|\beta|\geq 1$,

$$h^{|\alpha+\beta|-1}(\partial_\theta^\alpha\partial_I^\beta p)(\partial_I^\alpha\partial_\theta^\beta a)\in S^{2k+1-|\alpha|-|\beta|,2l+2-|\beta|}(\mathsf{S}).$$

Moreover by (33),

$$\partial_\theta^\gamma p=O(\rho_{\mathrm{sf}}\rho_{\mathrm{ff}}^2)=O(HI_1^2)\quad\text{ if } |\gamma|>0.$$

Hence for $|\beta|\leq|\alpha|$, $|\alpha|\geq 1$,

$$h^{|\alpha+\beta|-1}(\partial_\theta^\alpha\partial_I^\beta p)(\partial_I^\alpha\partial_\theta^\beta a)\in S^{2k-|\alpha|-|\beta|,2l}(\mathsf{S}).$$

We thus conclude that the terms of the form

$$h^{|\alpha+\beta|-1}(\partial_\theta^\alpha\partial_I^\beta p)(\partial_I^\alpha\partial_\theta^\beta a)$$

with $|\alpha+\beta|>1$ (which thus have either $|\beta|\geq 2$, or $|\alpha|\geq 1$, $|\beta|\leq|\alpha|$) in the sum are in fact $O(H^{-2k+1}I_1^{-2l})$. An analogous calculation holds if we interchange the role of α and β (as well as if we complex conjugate), and we conclude that modulo the terms with $|\alpha+\beta|=1$, we can Borel sum the right hand side of (30) to a symbol in

$$S^{-2k+1,-2l}(\mathsf{S}).$$

This concludes the proof of the Lemma.

The remainder of the proof of the theorem is as follows.

Pairing (32) with u we obtain for all $h>0$,

$$-2\operatorname{Im}\langle Au,h^{-1}Pu\rangle=\int_0^\delta\|B_su\|^2\,ds+\|Cu\|^2-\|Du\|^2+\langle R_1u,u\rangle+\langle R_2u,u\rangle.$$

The assumption of absence of ${}^2\mathrm{WF}^{k,l}(u)$ at ζ (and hence its H_1-orbit) controls $\langle Du,u\rangle$ uniformly as $h\downarrow 0$. The assumption of absence of ${}^2\mathrm{WF}^{k-1/2,l}(u)$ on the whole microsupport of A controls $\langle R_1u,u\rangle$. The assumption $u\in L^2$ controls $\langle R_2u,u\rangle$ since $R_2\in\Psi_{2,h}^{-\infty,2l+1}=\mathcal{R}^{2l+1}\subset\mathcal{R}^0$ since $l\leq -1/2$. Thus, since the left side is $O(h^\infty)$, we obtain absence of ${}^2\mathrm{WF}^{k,l}(u)$ on the elliptic set of C, hence on the time-δ flowout of H_2 (for any small δ). Hence we obtain H_2-invariance of ${}^2\mathrm{WF}^k(u)$. (A corresponding argument works along the backward flowout.) $\square$

6. Consequences for Spreading of Lagrangian Regularity

Recall that an invariant torus in an integrable system (with the notation of §5) is said to be *isoenergetically nondegenerate* if

$$\Omega = \begin{pmatrix} \omega_{11} & \dots & \omega_{1n} & \omega_1 \\ \vdots & \ddots & \vdots & \vdots \\ \omega_{n1} & \dots & \omega_{nn} & \omega_n \\ \omega_1 & \dots & \omega_n & 0 \end{pmatrix} \tag{34}$$

is a nondegenerate matrix. We recall from [21] that a somewhat trivial example of a system in which the invariant tori are nondegenerate is when $P = h^2\Delta - 1$ on $S^1 \times S^1$; we may take $\mathcal{L}$ to be, for instance, $\{\xi_1 = 1,\ \xi_2 = 1\}$. Here Δ is the nonnegative Laplacian, and ξ_i are the fiber variables dual to x_i in $T^*(S^1 \times S^1)$. A considerably less trivial example is the spherical pendulum, where all tori are isoenergetically nondegenerate except for those given by a codimension-one family of exceptional energies and angular momenta—see Horozov [13, 14] for the proof of the nondegeneracy and the description of the exceptional tori.

Definition 6.1. A distribution u is Lagrangian on a closed set $F \subset \mathcal{L}$ if there exists $A \in \tilde{\Psi}_h(X)$, elliptic on F, such that Au is a Lagrangian distribution with respect to $\mathcal{L}$.

We recall that in [21], it was shown that under the hypotheses of Theorem 5.1,[10] and if $\mathcal{L}$ is assumed to be isoenergetically nondegenerate, then local Lagrangian regularity on $\mathcal{L}$ is invariant under the Hamilton flow of P on $\mathcal{L}$, and, additionally, Lagrangian regularity on a small tube of closed bicharacteristics implies regularity along the bicharacteristics inside it.

We now prove the following, generalizing the results of [21].

Corollary 6.2. *Assume that the hypotheses of Theorem 5.1 hold and that, additionally, $\mathcal{L}$ is isoenergetically nondegenerate. If u is Lagrangian microlocally near any point in $\mathcal{L}$ relative to L^2 then u is globally Lagrangian with respect to $\mathcal{L}$ in a microlocal neighborhood of $\mathcal{L}$, relative to $h^\epsilon L^2$ for all $\epsilon > 0$.*

Remark 6.3. If the initial data for the wave equation on $\mathbb{R}^n$ is smooth in an annulus, the solution is smooth near the origin at certain later times. This remark is to Hörmander's propagation of singularities theorem as Corollary 6.2 and the results of [21] are to Theorem 5.1: in both settings the crude statements about singular supports are deducible from a much finer microlocal theorem.

Remark 6.4. An example from [21] shows that the hypothesis of isoenergetic nondegeneracy is necessary: without it, there do exist quasimodes that are Lagrangian only on parts of $\mathcal{L}$.

The reader may wonder if nowhere Lagrangian quasimodes are in fact possible, given the hypotheses of the theorem. An example is as follows: consider

$$P = h^2\Delta - 1$$

on $S^1 \times S^1$ (with Δ the nonnegative Laplacian). Consider the sequence

$$u_k = e^{i(k^2x_1 + kx_2)},\ k \in \mathbb{N}.$$

[10]In [21] the subprincipal symbol assumption was in fact stronger: it was assumed to vanish.

taking the sequence of values $h = h_k = k^{-1}(1+k^2)^{-1/2}$ gives

$$P_{h_k} u_k = 0.$$

Now the u_k's are easily verified (say, by local semiclassical Fourier transform) to have semiclassical wavefront set in the Lagrangian $\mathcal{L} = \{(x_1, x_2, \xi_1 = 1, \xi_2 = 0)\}$. On the other hand, the operator $hD_{x_2} \in \Psi_h(S^1 \times S^1)$ is characteristic on $\mathcal{L}$ and we have, for the sequence $h = h_k$,

$$(hD_{x_2})^m u_k = (hk)^m u_k = (1+k^2)^{-m/2} u_k.$$

Thus, u certainly does not have iterated regularity under the application of $h^{-1}(hD_{x_2})$, hence is not a semiclassical Lagrangian distribution. (We recall, though, from [21] that the hypotheses of the Corollary are satisfied in this case.)

Proof. Let $x \in \mathcal{L}$, and assume that u enjoys Lagrangian regularity at x, relative to L^2, i.e. $(x,\xi) \notin {}^2\mathrm{WF}^{\infty,0}(u)$ for all $\xi \in SN_x(\mathcal{L})$. By Theorem 5.1, ${}^2\mathrm{WF}^{\infty,-1/2}(u)$ is invariant under the flow

$$\sum \overline{\omega}_j \partial_{\theta_j}$$

and

$$\sum \overline{\omega}_{ij} \xi_i \partial_{\theta_j}.$$

and hence under any linear combination of them. In other words, at given $\vec{\xi} = (\xi_1, \dots, \xi_n)^t$, ${}^2\mathrm{WF}(u)$ is invariant under the flow along the vector field

$$V(\xi, s) = (\vec{\xi}^t, s) \cdot \Omega \cdot \begin{pmatrix} \partial_\theta \\ 0 \end{pmatrix}$$

for all $s \in \mathbb{R}$, where Ω is given by (34). By isoenergetic nondegeneracy, the set of ξ such that $V(\xi, s)$ has rationally independent components for some $s \in \mathbb{R}$ has full measure, i.e. in particular such values of ξ are dense in $SN_x(\mathcal{L})$. As the closure of the orbit along such a vector field is all of $\mathcal{L} \times \{\xi\}$, and as ${}^2\mathrm{WF}$ is a closed set, we find that for a dense set of $\xi \in SN_x(\mathcal{L})$, ${}^2\mathrm{WF}(u) \cap (\mathcal{L} \times \{\xi\}) = \emptyset$.[11] Again by closedness of ${}^2\mathrm{WF}(u)$, we now find that ${}^2\mathrm{WF}^{\infty,-1/2}(u) = \emptyset$. Since $u \in L^2$, an interpolation yields

$$ {}^2\mathrm{WF}^{\infty,-\epsilon}(u) = \emptyset. \quad \square$$

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