Scattering theory on symmmetric spaces and N-body scattering

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How to solve partial differential equations?

Exact methods, such as Taylor series, an example being Cauchy-Kovalevskaya. This is familiar from ODE's, e.g. for the ODE

$$u'' + u = 0,\ \ u(0) = 1,\ \ u'(0) = 0$$

one obtains

$$u(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} x^{2n},$$

better known as $\cos x$.

- Usually no explicit solution is available.
- Even if it is, its properties may not be obvious. E.g. from the Taylor series one can easily read off that $\cos x$ is smooth, indeed real analytic, but its behavior near infinity is far from clear.

Approximate, or parametrix methods: consider, for example, a PDE $Lu = f$, where L is a linear operator and f is given. The method is:

- Construct G such thar $GL = \mathrm{Id} + K$ and K is nice – e.g. compact on a Hilbert space $\mathcal{H}$ such as L^2.
- Then $\mathrm{Id} + K$ has finite dimensional nullspace, and its range is closed, with finite codimension, so it is invertible as a map between closed subspaces of finite codimension – another way to put it is that K can be made finite rank.
- Sometimes one has additional information so that one can in fact show that L itself is invertible.

Usually one has much more information about G than its mere existence, so for instance one can understand properties of solutions u of $Lu = f$ – they behave like Gf.

A very simple example is perturbation theory:

- If there is an operator L_0 such that $L - L_0$ is compact relative to L_0, $GL_0 = \mathrm{Id}$, then $GL = \mathrm{Id} + G(L - L_0)$.
- For instance, $L_0 = \Delta + 1$ (the positive Laplacian) on a compact manifold M, and $L - L_0$ is a multiplication or first order differential operator, then $G(L - L_0)$ is compact on $L^2(M)$.

A more interesting example: The algebra of *pseudo-differential* operators was invented exactly to provide a machinery for approximate inversion of elliptic differential operators – with compact errors if the underlying manifold is compact. They also show *elliptic regularity* (solutions of elliptic equations are C^∞), provide a convenient background for Atiyah-Singer-type index theorems, etc.

Even these approximate methods often break down. One can still employ *estimates*, such as coercive estimates or positive commutator estimates, to study properties of solutions (including, but not at all limited to existence and uniqueness).

These methods become less explicit, but more generally applicable as we go down the list.

- The traditional way of dealing with analysis on globally symmetric spaces, originating in the work of Harish-Chandra is the first one, the use of explicit solutions, combined with a strong use of the algebraic structure. (The solutions are only *partially* explicit!)
- Today I will describe the second approach in two settings, N-body scattering and globally symmetric spaces to show how the latter is (technically, almost) a special case of the former. Of course, as a special case, it has special features, and it is an interesting question just how big of a role these special features play.

As I emphasize the methods in this talk, I'll first give a flavor of the results one can obtain with this technique:

- the resolvent of the Laplacian, $(\Delta - \sigma)^{-1}$, continues analytically across the continuous spectrum e.g. in the sense that the Green's function continues analytically (this is very easy this way),
- one can describe the asymptotics of the Green's function or of Harish-Chandra's spherical functions near the Weyl chamber walls (less easy, but straightforward; previously done by Trombi-Varadarajan and Anker-Guivarch-Ji-Taylor using Harish-Chandra's approach).

However, as may be expected, certain algebraic miracles are hard to discern from the general framework. For instance, in principle one might expect that the analytic continuation has ramification points in certain regions – but these points don't exist there.

The standard N-body Hamiltonians for d-dimensional particles are operators on functions on $\mathbb{R}^{Nd}$ of the form

$$H = \sum_i \frac{\hbar^2}{2m_i}\Delta_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j)$$

where $\hbar$ is Planck's constant, x_i (in $\mathbb{R}^d$) is the position, m_i the mass of the ith particle,

$$\Delta_{x_i} = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_{i,j}^2},\ x_i = (x_{i,1}, \ldots, x_{i,d}),$$

is the Laplacian and V_{ij} is the interaction between particle i and j which decays at infinity. Some typical examples are Coulomb potentials $V_{ij}(y) = c|y|^{-1}$ and Yukawa potentials $V_{ij}(y) = Ce^{-\kappa|y|}$, $y \in \mathbb{R}^d$.

If $\psi(t, x)$ is the wave function of the system of N particles, u evolves as $-i\hbar \frac{\partial \psi}{\partial t} = H\psi$. Often one considers stationary problems instead, taking a Fourier transform in t: $(H - \sigma)u = 0$.

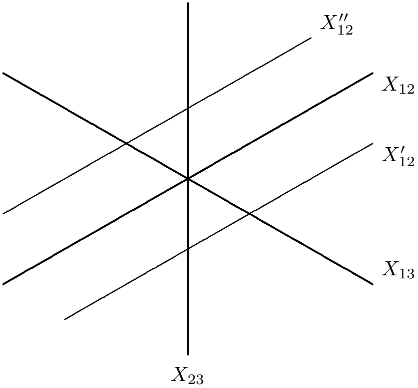
From a classical mechanics point of view, the terms with the Laplacian represent the kinetic energy, while the V_{ij} the potential energy of the system.

Example: 3-body scattering for particles of equal mass.

It is convenient to 'remove the center of mass', i.e. to work in coordinates relative to the center of mass. This corresponds to restricting our attention to the center of mass plane $x_1 + x_2 + x_3 = 0$, and, from the analytical point of view, doing a Fourier transform in the orthogonal variables.

Note that even when restricted to $x_1 + x_2 + x_3 = 0$, $V_{ij}(x_i - x_j)$ does not decay as $x = (x_1, x_2, x_3) \to \infty$, unlike in the 2-body setting, where $V_{12}(x_1 - x_2) \to 0$ if $(x_1, x_2) \to \infty$ and $x_1 + x_2 = 0$.

In the following picture we show these 'collision planes' $X_{ij} = \{x_i = x_j\}$ (inside $x_1 + x_2 + x_3 = 0$); X'_{12} and X''_{12} are translates of X_{12}, so V_{12} is constant (does not decay) along these.



There is a natural 'linear algebraic' generalization of this setup. Let (X, g) be a vector space with a translation invariant Riemannian metric, $\mathcal{X} = \{X_a : a \in I\}$ a finite collection of linear subspaces of X closed under intersections, including $X_0 = X$ and $X_1 = \{0\}$. Let $X^a = X_a^\perp$, V_a real valued functions on X^a decaying at infinity, e.g. $V_a \in S^{-\rho}(X^a)$, $\rho > 0$; for instance $V_a(x^a) \sim c|x^a|^{-1}$ (Coulomb decay). Let

$$H = \Delta_g + \sum_{a \in I} (\pi^a)^* V_a,\ \pi^a : X \to X^a \text{ orth. proj.};$$

assume $V_0 = 0$. Write V_a for $(\pi^a)^* V_a$ as well. Note that V_a is constant on translates of X_a, so it does *not* decay at infinity (except V_1) unlike in 2-body problems.

We may replace V_a by formally self-adjoint first or even second order operators on X^a, with decaying coefficients, as long as H remains elliptic in the usual sense.

I describe $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N,\mathbb{R})$ in some detail, although other higher rank spaces of non-compact type work similarly. The corresponding rank-one example, $\mathbb{H}^2 = \mathrm{SL}(2)/\mathrm{SO}(2)$ is extremely well understood.

If $C \in \mathrm{SL}(N)$, we can write it uniquely as $C = VR$ where $R \in \mathrm{SO}(N)$, V positive definite, $\det V = 1$ – indeed, $V = (CC^t)^{1/2}$. So we may identify $\mathrm{SL}(N)/\mathrm{SO}(N)$ with the real analytic manifold of $N \times N$ positive definite matrices of determinant 1.

Then $\mathrm{SL}(N)$ acts on M by

$$\mathrm{SL}(N) \times M \ni (B, A) \mapsto (BA^2 B^t)^{1/2}.$$

The Killing form of $\mathrm{SL}(N)$ gives a fiber metric on $T_o M$, $o = \mathrm{Id}$, via identifying $T_o M$ with symmetric $N \times N$ matrices of trace 0, and if A is such a matrix, its length is

$$(2N)\,\mathrm{Tr}(AA^t) = (2N) \sum a_{ij}^2.$$

This gives a Riemannian metric g on M via the $\mathrm{SL}(N)$ action.

We now want to study $H = \Delta_g$.

In fact, because of the $\mathrm{SL}(N)$-action, it suffices to study the action of Δ on $K_o = \mathrm{SO}(N)$-invariant functions. Notice that δ_o is K_o-invariant, hence so is $(\Delta_g - \lambda)^{-1}\delta_o$ for λ outside the spectrum of Δ_g.

The main claim is that the radial Laplacian Δ_{rad}, i.e. Δ_g *acting on* $\mathrm{SO}(N)$*-invariant functions*, has a many-body structure ($N-1$-body, to be precise). In particular, the collision planes are (intersections of) the walls of Weyl chambers.

To see this, we diagonalize $A \in M$:

- $A = O\Lambda O^t$, $O \in \mathrm{SO}(N)$, Λ diagonal, $\det\Lambda = 1$.
- Neither Λ nor O is completely determined.
- The diagonal entries λ_i of Λ are the eigenvalues of A, so they are determined up to permutations.
- If all eigenvalues are distinct, there is only a finite indeterminacy of O.
- But if two eigenvalues coincide, the indeterminacy is bigger: the eigenspaces, rather than the eigenvectors, of A are well-defined.

So inside the 'flat' $\exp(\mathfrak{a})$ of diagonal matrices of determinant 1 we have a similar picture to N-body scattering. (Here $\mathfrak{a}$ is the space of diagonal matrices with vanishing trace.)

- With $\lambda_i = \exp(x_i)$, the so-called (restricted) *roots* on $\mathfrak{a}$ are $x_i - x_j$, $i \neq j$; the corresponding expressions $e^{x_i - x_j} = \frac{\lambda_i}{\lambda_j}$ appear prominently in the metric.
- The *walls* in $\mathfrak{a}$ are the zero-sets of the roots; the connected components of their complement $\mathfrak{a}_{\mathrm{reg}}$ are the Weyl chambers.
- The *Weyl group* is generated by reflections across the walls; in this case it is the permutation group $W = S_N$.

- $C^\infty(M)^K$ is naturally identified with $C^\infty(\mathfrak{a})^W$, and so (tautologically) Δ_{rad} extends to this latter space, and then also to W-invariant distributions, etc. It does *not* act on $C^\infty(\mathfrak{a})$.
- We can choose a chamber $\mathfrak{a}^+$ as the *positive* one; in this case this amounts to giving an ordering of the eigenvalues, e.g. $\lambda_1 < \lambda_2 < \lambda_3$. The roots that are positive on this chamber are the positive roots.

Indeed, the metric g reflects this structure. Concretely, for $N = 3$, for $p \in \exp(\mathfrak{a})$, on T_pM, the metric is

$$\begin{aligned} g =& 6\left(\frac{d\lambda_1^2}{\lambda_1^2} + \frac{d\lambda_2^2}{\lambda_2^2} + \frac{d\lambda_3^2}{\lambda_3^2}\right) \\ &+ 3\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right)^2 dc_{12}^2 + 3\left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1}\right)^2 dc_{13}^2 \\ &+ 3\left(\frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2}\right)^2 dc_{23}^2, \end{aligned}$$

where the c_{ij} are coordinates on SO(3) near Id given by the off-diagonal entries in so(3).

Notice the polar coordinate degeneracy at the walls – these arise because our coordinates are not admissible at the walls.

To make the formula truly comparable to the Euclidean setting, we need to write $\Lambda = \exp(D)$, $D \in \mathfrak{a}$ (symmetric, trace 0), so it has diagonal entries $x_i = \log \lambda_i$. The metric on D then is 6 times the Euclidean metric. Since in local coordinates z_j on M,

$$\Delta_g = \sum_{i,j} \frac{1}{\sqrt{g}} D_{z_i} g^{ij} \sqrt{g} D_{z_j},$$

and since we can identify K_o-invariant functions on M with W-invariant functions on $\mathfrak{a}$, Δ_{rad} is a 3-body Hamiltonian, apart from the 'polar coordinate' singularities.

The 3-body structure is due to the factor of $\sqrt{g}$ which involves the components of the metric along K_o-orbits. The interactions between the 'particles' are now given by first order operators.

Let Λ denote the set of restricted roots.

Then

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \frac{1}{2} \sum_{\alpha \in \Lambda} (m_\alpha \coth \alpha)\, H_\alpha,$$

where

- $\Delta_{\mathfrak{a}}$ is the standard Laplacian on the vector space $\mathfrak{a}$,
- m_α is the dimension of the root space ($m_\alpha = 1$ for $\mathrm{SL}(N)/\mathrm{SO}(N)$),
- H_α is the root vector associated to the root α via the metric on $\mathfrak{a}$.

It is helpful to simplify this by considering it on the positive Weyl chamber. Let Λ^+ be the set of positive roots.

Since $m_\alpha = m_{-\alpha}$, $\coth(-\alpha) = -\coth\alpha$ and $H_{-\alpha} = -H_\alpha$, we also have

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \sum_{\alpha\in\Lambda^+} (m_\alpha \coth\alpha)\, H_\alpha.$$

- $\coth \alpha$ is singular at the wall $\alpha^{-1}(0)$,
- and it converges (exponentially fast) to the constant 1 as $\alpha(.) \to +\infty$.

Far away from the walls but inside $\mathfrak{a}^+$, one can consider Δ_{rad} as a perturbation of the translation-invariant operator $\Delta_{\mathfrak{a}} + 2H_\rho$ on $\mathfrak{a}$, where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Lambda^+} m_\alpha \, \alpha \in \mathfrak{a}^*.$$

The resulting analysis is the starting point of Harish-Chandra's work on spherical functions:

- construct local 'eigenfunctions' of Δ_{rad} in $\mathfrak{a}^+$ in perturbation series (in the spirit of Cauchy-Kovalevskaya),
- use group theory to show that these can be combined into genuine global generalized eigenfunctions.

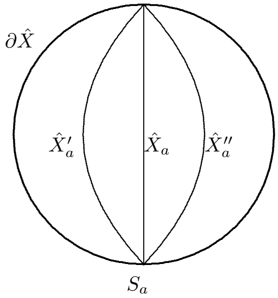
The parametrix point of view: how to construct the resolvent approximately. In N-body scattering, for $\sigma \notin \mathrm{spec}(H)$, one can construct an approximate inverse for $H - \sigma$ by inverting model operators $H_a - \sigma$, $a \in I$, of the various subsystems. Since these are k-body Hamiltonians with $k < N$, these in turn can be inverted by an iterative construction.

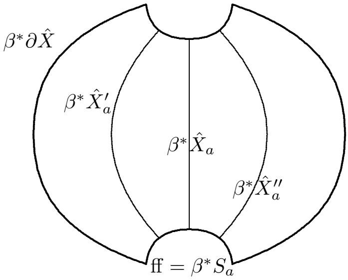
The most elegant version of this construction uses non-commutative (operator valued) symbols. However, in the particular cases we consider, a patching construction suffices. (N.B. The patching construction would not suffice if we had second order interactions, or, worse, considered a truly geometric generalization of N-body Hamiltonians.) To describe this, we need a partition of unity and cutoffs adopted to the geometry.

First, we need a way of succintly stating properties of functions near infinity, to avoid making statements analogous to (but more complicated than) '$\phi \in C^\infty(X)$ homogeneous degree zero, with respect to dilations, outside a compact subset of X'. Thus, we let $\hat{X}$ be the radial (or geodesic) compactification of X to a ball. Namely, we add the sphere $\mathbb{S}^{n-1}$ ($n = \dim X$) to X as its boundary. Near $\partial\hat{X}$, $\hat{X}$ is of the form $[0,1)_\rho \times \mathbb{S}^{n-1}_\omega$; we can take the identification of this region with a subset of X to be given by inverse polar coordinates $(\rho, \omega) \mapsto \rho^{-1}\omega \in X$.

Then $\phi \in C^\infty(\hat{X})$ means, in particular, that ϕ has an asymptotic expansion at infinity, namely the Taylor series at the boundary: $\phi(\rho, \omega) \sim \sum_j \rho^j a_j(\omega)$, i.e. ϕ is a (classical) symbol of order 0.

We also let $S_a = \partial X_a$ be the sphere at infinity for the collision plane X_a. We illustrate the geometry of the compactification below on the left hand side. X_a' and X_a'' are translates of the collision plane X_a; the hat denotes their closure in $\hat{X}$.

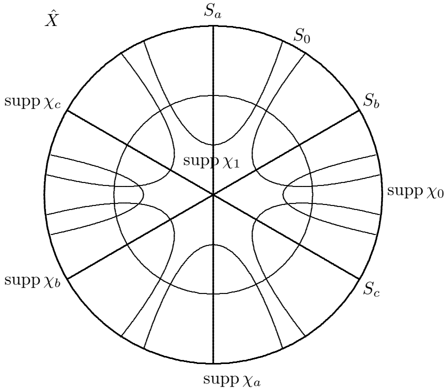




To indicate the relationship of this geometry to N-body potentials, note that V_a is *not* continuous, let alone smooth, on $\hat{X}$; to make it so we need to blow up S_a. The collision planes are really only relevant at infinity, i.e. only the S_a are relevant – the X_a do not play any role in any compact region of X.

Definition: A partition of unity $\{\chi_b : \ b \in I\}$ is $\mathcal{X}$-adapted if

1. each $\chi_b \in C^\infty(\hat{X})$,
2. $\operatorname{supp} \chi_1$ is a compact subset of $X = X_1$,
3. $\operatorname{supp} \chi_b \cap S_c = \emptyset$ unless $S_b \subset S_c$.



There exists such a partition of unity. We can also construct cutoffs ψ_b with the same properties, and in addition with $\psi_b \equiv 1$ on a neighborhood of $\operatorname{supp} \chi_b$.

We also need model operators. Near S_b, $b \neq 1$, the natural model is given by

$$H_b = H^b + \Delta_{X_b},\ H^b = \Delta_{X^b} + \sum_{X^a \subset X^b} V_a;$$

here H^b is the subsystem Hamiltonian for the cluster b, which is an operator on (functions on) X^b. This model 'works' (for at most first order V_a with coefficients in $S^{-\rho}$, i.e. decaying like $|x^a|^{-\rho}$) because on $\operatorname{supp} \psi_b$, $\Delta - H_b$ is a first order differential operator with decaying coefficients (in $S^{-\rho}$).

A parametrix for $H - \sigma$ is then given by

$$P(\sigma) = \sum_{b \in I,\ b \neq 0} \psi_b R_b(\sigma) \chi_b + \psi_0 P_0' \chi_0$$

where P_0' is a parametrix in the usual sense for $H - \sigma$. Note that P_0' is sandwiched between compactly supported functions, so the precise sense in which P_0' is constructed (regarding behavior near infinity) does not make much difference.

Then with

$$E(\sigma) = (H - \sigma)P(\sigma) - \mathrm{Id},\ \ E'(\sigma) = P(\sigma)(H - \sigma) - \mathrm{Id},$$

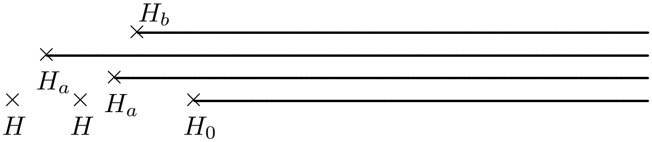
$$P(\sigma) : H^{s,k}(\mathbb{R}^n) \to H^{s+2,k}(\mathbb{R}^n)$$
$$E(\sigma), E'(\sigma) : H^{s,k}(\mathbb{R}^n) \to H^{s+1,k+\rho}(\mathbb{R}^n)$$

for all $s, k \in \mathbb{R}$. Here $H^{s,k}(\mathbb{R}^n)$ is the weighted Sobolev space $\langle z \rangle^{-k} H^s(\mathbb{R}^n_z)$. In particular, the error terms $E(\sigma)$, $E'(\sigma)$ are compact on L^2.

An immediate corollary is the HVZ-theorem:

$$\mathrm{spec}_c(H) = \cup_{b \in I,\ b \neq 1} \mathrm{spec}(H_b),$$
$$\mathrm{spec}(H_b) = \mathrm{spec}(H^b) + [0, +\infty).$$

In general, the spectrum is quite complicated, as only the continuous spectrum is predicted by the theorem, not the eigenvalues. Moreover, the continuous spectrum of H depends on the whole spectrum of its proper subsystems, i.e. L^2-eigenvalues of the subsystems give rise to branches of continuous spectrum.



This analysis remains valid on symmetric spaces G/K as well, as long as one restricts to K-invariant functions. The only additional requirement is that we make the partition of unity, and the cutoffs, W-invariant, and one also needs to work with weighted Sobolev spaces of K-invariant functions. In this case the coefficients decay exponentially, i.e. one can take $\rho > 0$ arbitrarily large.

One slight issue relates to labelling; it is in practice better to work only with the compactified positive Weyl chamber $\hat{\mathfrak{a}}^+$ rather than with $\hat{\mathfrak{a}}$. If X_b^+ is an open face of the closure of $\mathfrak{a}^+$ in $\mathfrak{a}$, there is a unique 'collision plane' (intersection of walls) X_b containing it as an open subset. Let I^+ denote the set of $b \in I$ that arise this way.

Let Λ_b denote the set of roots vanishing at some (hence all) $p \in X_{b,\mathrm{reg}}$,

$$\rho_b = \frac{1}{2} \sum_{\alpha \in \Lambda_b^+} m_\alpha \alpha.$$

Then $H_\rho - H_{\rho_b} \in X_b$, and $H_{\rho_b} \in X^b = \mathfrak{a}^b$.

For such b, the models are now

$$L_b = T_b + \Delta_{b,\mathrm{rad}},$$
$$T_b = \Delta_{X_b} + 2(H_\rho - H_{\rho_b})$$
$$\Delta_{b,\mathrm{rad}} = \Delta_{\mathfrak{a}^b} + 2H_{\rho_b} + \sum_{\alpha\in\Lambda_b^+} m_\alpha(\coth\alpha - 1)H_\alpha.$$

Thus, T_b is a translation-invariant differential operator on X_b, and $\Delta_{b,\mathrm{rad}}$ is the radial part of the Laplacian on a lower rank symmetric space $M_b = G_b/K_b$.

The HVZ-type theorem in this setting is thus

$$\begin{aligned}&\mathrm{spec}_c(\Delta_{\mathrm{rad}})\\&= \cup_{b\in I^+,\ b\neq 1}(\mathrm{spec}(\Delta_{b,\mathrm{rad}}) + [|\rho-\rho_b|^2, +\infty)).\end{aligned}$$

The first sign of a miracle in this setting is that the spectrum is simple: there are no L^2-eigenvalues at all, so in fact

$$\mathrm{spec}(\Delta_{\mathrm{rad}}) = [\rho^2, +\infty).$$

In general, results are similar in N-body scattering and on symmetric spaces, but there are many special coincidences on symmetric spaces. As an example, consider the analytic continuation of the resolvent through the spectrum.

In the N-body setting, under appropriate assumptions for the potentials V_a, $R(\sigma) = (H - \sigma)^{-1}$ continues analytically to a Riemann surface Σ including $\mathbb{C} \setminus \mathrm{spec}(H)$ (e.g. in the sense that its Schwartz kernel, i.e. $R(\sigma)\delta_p$, continues as a distribution).

This Riemann surface is usually very complicated, and one may have only be able to extend only a little beyond the continuous spectrum. (This approach started with the work of Aguilar-Balslev-Combes; the stated result is due to Ch. Gérard.)

(The assumptions: V_a can be compactly supported on X^a, dilation analytic on X^a with respect to the dilations I will discuss, or more generally dilation analytic near infinity on X^a.)

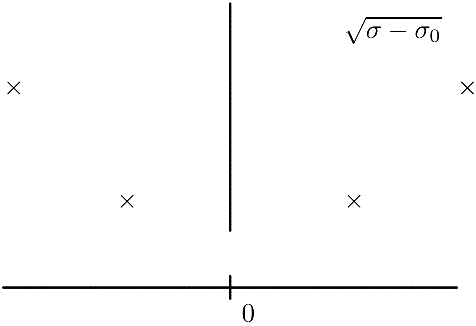
Normalizing $\log z$ on $\mathbb{C} \setminus [0, +\infty)$ to take values in $(-2\pi, 0) + i\mathbb{R}$, and $\sqrt{z}$ to take values in $\operatorname{Im} z < 0$, on symmetric spaces G/K we have

Theorem

For a suitable constant $L > 0$, the Green function $G_o(\sigma)$ continues analytically as a distribution to the logarithmic plane in $\sigma - \sigma_0$ with the half-lines

$$\log(\sigma - \sigma_0) \in i(-\pi + 2k\pi) + [2\log L, +\infty), \ \ k \in \mathbb{Z} \setminus \{0\},$$

removed, if n is even, and to the Riemann surface of $\sqrt{\sigma - \sigma_0}$, with $\sqrt{\sigma - \sigma_0} \in i[L, +\infty)$ removed, if n is odd.


$\sqrt{\sigma - \sigma_0}$
0

A weaker version of this, in which one allows ramifications points arising from poles of the analytic continuation of subsystem resolvents, is actually a rather simple result from the N-body perspective.

One considers the scaling $A \mapsto A^w$, which is dilations along geodesics through $o = \mathrm{Id}$. (The Euclidean analogue is $x \mapsto wx$.) For $w = e^\theta > 0$, this is a diffeomorphism Φ_θ, so

$$(U_\theta f)(A) = (\det D_A \Phi_\theta)^{1/2} (\Phi_\theta^* f)(A)$$

defines a unitary operator on L^2. Then $\Delta_\theta = U_\theta \Delta U_\theta^{-1}$ extends analytically to the strip $|\operatorname{Im}\theta| < \frac{\pi}{2}$.

In the case of $\mathbb{R}^n$, the similar scaling gives $\Delta_\theta = e^{-2\theta}\Delta$; notice that this rotates the continuous spectrum. Same happens for Δ_{rad}. Then

$$\langle f, (\Delta_{\mathrm{rad}} - \sigma)^{-1} g\rangle = \langle U_{\bar{\theta}} f, (\Delta_{\theta,\mathrm{rad}} - \sigma)^{-1} U_\theta g\rangle$$

for a dense set of f, g. One can modify the scaling to actually show that for $g \in \mathcal{D}'_c(M)$, $(\Delta - \sigma)^{-1} g \in \mathcal{D}'(M)$.

The symmetric space miracle is that the ramification points do not occur. Note that an innocent looking improvement, namely that one can improve the statement that the subsystems have only poles (no ramification points; one even knows that these poles cannot go to infinity in certain cones) outside ramification points given by their subsystems' poles, to them having no poles at all, gives a huge improvement due to the inductive construction.

As already hinted at, a parametrix construction contains much more information than invertibility modulo compact operators. For instance, one can read off the asymptotic behavior of the Green's function at infinity very directly.

To make the underlying geometry identical, consider 3-body scattering with the collision planes given by the SL(3)-walls. (This corresponds to three one-dimensional particles with equal masses.) The space on which expansions live in this case is the blow-up of $\hat{X}$ along the S_a:

$$\tilde{X} = [\hat{X}; \{S_a : \ a \in I\}].$$

If we in addition assume that no proper subsystem has bound states, e.g. we have three electrons interacting repulsively, then the Green's function (with pole at p) has the form

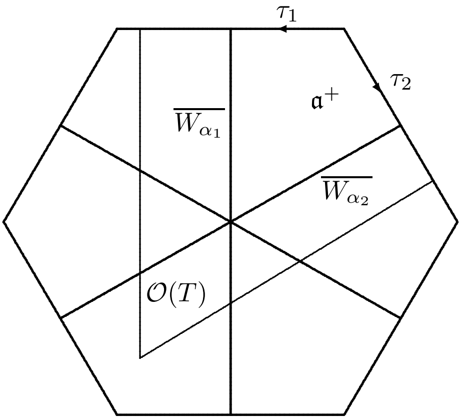
$$R(\sigma)\delta_p = \langle z\rangle^{-1/2} e^{i\sqrt{\sigma}\,\langle z\rangle} a(z),$$

with a smooth on $\tilde{X}$.

(If some subsystems have bound states, we get additional terms decaying rapidly off the lift of S_a to $\tilde{X}$.)

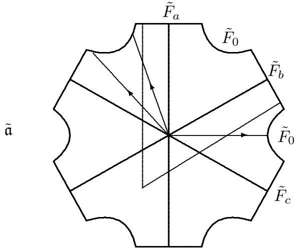
The situation for the Green's function on symmetric spaces is quite similar. The main difference is the appearance of certain additional factors corresponding to the volume form on M. To put these into context, it is useful to consider a smaller space $\bar{\mathfrak{a}}$ than $\tilde{\mathfrak{a}}$.

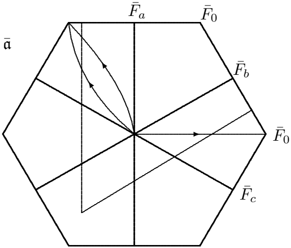
This space $\bar{\mathfrak{a}}$ is the polyhedral or dual-cell compactification. Roughly, it compactifies $\mathfrak{a}^+$ as a cube, using the positive simple roots $\alpha \in \Lambda^+_{\mathrm{ind}}$. Namely, the latter, denoted by α_j, form a basis of $\mathfrak{a}^*$, hence $(\alpha_1, \ldots, \alpha_n)$ (where $\dim \mathfrak{a} = n$) is a coordinate system on $\mathfrak{a}$. We compactify $\mathfrak{a}^+$, the neighborhood $\mathcal{O}(T) = \cap_{j=1}^n \alpha_j^{-1}((T_j, +\infty))$ by identifying it with $\prod_{j=1}^n (T_j, +\infty)$, and compactifying the latter as $\prod_{j=1}^n [0, e^{-T_j})$ via the map $t_j \mapsto e^{-t_j}$. Thus, we effectively make the $\tau_j = e^{-\alpha_j}$ a coordinate system on a neighborhood of $\mathfrak{a}^+$ in $\bar{\mathfrak{a}}$.

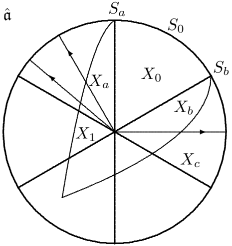

τ_1
τ_2
$\mathfrak{a}^+$
$\overline{W_{\alpha_1}}$
$\overline{W_{\alpha_2}}$
$\mathcal{O}(T)$

It is worth pointing out now what $\tilde{\mathfrak{a}}$ is in terms of $\bar{\mathfrak{a}}$: first, one takes the logarithmic blowup $\bar{\mathfrak{a}}_{\log}$ (i.e. makes α_j^{-1} the coordinates), then one performs a 'total boundary blow-up', starting with boundary faces of the smallest dimension. Thus, the blow-down map $\beta : \tilde{\mathfrak{a}} \to \bar{\mathfrak{a}}$ is smooth, and the smoothness of a function on $\bar{\mathfrak{a}}$ near $\mathfrak{a}^+$ is a much stronger statement than its smoothness on $\tilde{\mathfrak{a}}$ in the same region.

In the next picture, the thin lines without arrows show the boundary of $\mathcal{O}(T)$ for $T_1 < 0$, $T_2 < 0$; the thin lines with arrows are geodesic emanating from 0; in particular they bound conic regions. The labels are $a = (23)$, $b = (12)$, $c = (13)$.


$\tilde{\mathfrak{a}}$
$\tilde{F}_a$
$\tilde{F}_0$
$\tilde{F}_b$
$\tilde{F}_0$
$\tilde{F}_c$


$\bar{\mathfrak{a}}$
$\bar{F}_a$
$\bar{F}_0$
$\bar{F}_b$
$\bar{F}_0$
$\bar{F}_c$


$\hat{\mathfrak{a}}$
S_a
S_0
S_b
X_a
X_0
X_b
X_1
X_c

We then have:

Theorem

On $M = \mathrm{SL}(3)/\mathrm{SO}(3)$, *if* $\sigma \notin \mathrm{spec}(\Delta) = [\sigma_0, +\infty)$, $\sigma_0 = |\rho|^2$, *then*

$$R(\sigma)\delta_o = \rho^\sharp \rho_\sharp x^{1/2} x^\sharp x_\sharp \exp(-i\sqrt{\sigma - \sigma_0}/x) g$$

where $g \in C^\infty(\tilde{M} \setminus \{o\})$. *Here* $\tilde{M}$ *is the compactification of* M *analogous to* $\tilde{\mathfrak{a}}$, *etc.*, $x(.) = d(o, .)$ *is the distance function from* o, $\rho_\sharp$ *and* $\rho^\sharp$ *are defining functions of the two boundary hypersurfaces of* $\bar{M}$, *so* $e^{-\rho} \sim \rho_\sharp \rho^\sharp$ *in* $\mathcal{O}(T)$, *and* $x_\sharp$, $x^\sharp$ *are the defining functions of the lifts of these boundary hypersurfaces to* $\tilde{M}$, *i.e. in the* $\tilde{\mathfrak{a}}$ *picture, of* $\tilde{F}_{12}$ *and* $\tilde{F}_{23}$. *Here the leading term* $g|_{\partial\tilde{M}}$ *is non-zero.*

A weaker version of this theorem, with a continuous rather than smooth statment, is due to Anker, Guivarch, Ji and Taylor.

I went too fast if I get this far! Still, at least for the sake of completeness I describe one more phenomenon, propagation of singularities. This is best stated in terms of a wave front set at infinity, but this would require some background.

Roughly, it is a statement that the (appropriate) wave front set, i.e. the singularities in T^*X measuring microlocal decay at infinity in X, of (tempered) generalized eigenfunctions, is a union of maximally extended generalized broken bicharacteristics. Instead, I state the result for 'perturbed plane waves' in a rank 2 setting. For symmetric spaces, these are Harish-Chandra's spherical functions.

First, what are perturbed plane waves coming in from some direction ξ, of energy $\sigma = \xi \cdot \xi$? In the N-body setting one usually considers ξ real, but this is not necessary.

- An unperturbed plane wave is $e^{-i\xi\cdot z}$; the phase $-\xi \cdot z$ is smallest, resp. largest, where z is parallel, resp. anti-parallel to ξ.
- A perturbed plane wave is supposed to have the same asymptotic behavior 'on the incoming side' (to be precise this would be a microlocal statement); thus,

$$u_\xi(z) = \psi(z)e^{-i\xi\cdot z} - R(\sigma + i0)(H - \sigma)(\psi(z)e^{-i\xi\cdot z})$$

  is a natural candidate; here $\psi \in C^\infty(\hat{X})$ identically 1 near $\frac{\xi}{|\xi|}$.
- Here ψ is not really needed (one can take $\psi \equiv 1$, and the result is independent of ψ anyway), but its presence makes the comparison with symmetric spaces easier.

- The u_ξ can be characterized by the statement that they are the unique (tempered) generalized eigenfunctions of H which microlocally near the 'incoming set' are given by $e^{-i\xi\cdot z}$.
- These u_ξ also show up as the Schwartz kernel of the 'Poisson operator', and they contain (in an accessible manner!) the information about the scattering matrices corresponding to free incoming particles, as well as about part of the spectral projector. (One needs to have plane waves corresponding to bound states in general.)

The complex analogue, when ξ is not real, works similarly.

- One takes ψ identically 1 near $\frac{\operatorname{Im}\xi}{|\operatorname{Im}\xi|}$: the real part of the exponent is $\operatorname{Im}\xi \cdot z$, which is largest where z is parallel to $\operatorname{Im}\xi$.
- Then $(H-\sigma)(\psi(z)e^{-i\xi\cdot z})$ is in a weighted L^2 space corresponding to less growth at infinity than $e^{-i\xi\cdot z}$ itself.
- In certain cases, e.g. if $\xi = c\xi_0$ with ξ_0 real, this implies that $R(\sigma)$ can be applied to it.
- Note that $c \to 1$, $\operatorname{Im} c > 0$, corresponds to the real limit. It is *different* from $c \to 1$, $\operatorname{Im} c < 0$, which would correspond to fixing the 'outgoing' asymptotics of u_ξ, ξ real.

For $\xi = c\xi_0$ with ξ_0 real, $\operatorname{Im}\xi \in \mathfrak{a}^+$, one can define the spherical function U_ξ similarly:

- now $\sigma = \xi \cdot \xi + |\rho|^2$ is the energy,
- choose ψ to vanish near the walls,
- regard $U^0_\xi(z) = \psi(z)e^{-\rho(z)}e^{-i\xi\cdot z}$, supported in $\mathfrak{a}^+$, as a W-invariant function on $\mathfrak{a}$, or a K-invariant function on M;
- let
$$U_\xi(z) = U^0_\xi(z) - R(\sigma)(\Delta_{\mathrm{rad}} - \sigma)(U^0_\xi(z))$$

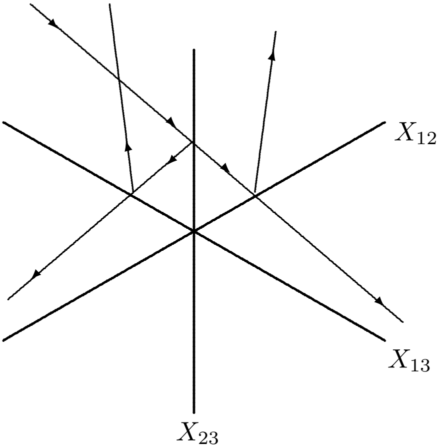
In fact, if one makes a better first approximation than $\psi(z)e^{-\rho(z)}e^{-i\xi\cdot z}$, as I discuss below, it is easy to define U_ξ analogously in general.

If we again assume that no proper subsystem has bound states, e.g. we have three electrons interacting repulsively, then 3-body perturbed plane waves with incoming direction ξ and energy σ (so $\xi \cdot \xi = \sigma$) can be written as a sum

$$u_\xi(z) = \sum_{s \in S_3} c_s(z) e^{-i(s\xi)\cdot z} + \langle z \rangle^{-1/2} e^{i\sqrt{\sigma}\,\langle z \rangle} a(z),$$

with c_s smooth on $\tilde{X}$ away from the direction of $-s\xi$ (where it has a 'conic' singularity), c_s rapidly decreasing in the direction of $s\xi$ for $s \neq 1$, and $c_1 \sim 1$ in the direction of ξ.

The sum over S_3 contains the reflections of the incoming wave from the collision planes, corresponding to particles that have collided. The last term is a spherical wave, much like the Green's function.



The situation in symmetric spaces is quite similar, except that one should consider the coefficients as either living on $\mathfrak{a}^+$, or being W-invariant on $\mathfrak{a}$, so:

- it is no longer the case that c_s is rapidly decreasing in the direction of $s\xi$,
- and there is also an additional exponential weight $e^{-\rho}$.

Denoting the spherical function by U_ξ, we have the following behavior:

$$U_\xi(z) = \sum_{s \in S_3} c_s(z) e^{-\rho(z)} e^{-i(s\xi)\cdot z},$$

valid in a neighborhood of the closure of $\mathfrak{a}^+$ *in* $\mathfrak{a}$. (From a different point of view, based on Harish-Chandra's approach, such a description is due to Trombi and Varadarajan.)

The miracles in this case (apart from the miraculous geometry, which we already assumed for the purposes of illustration in the 3-body setting) are

- $c_s \in C^\infty(\tilde{\mathfrak{a}})^W$ (i.e. there is no 'conic' singularity where waves coming from different directions meet up)
- there is no spherical wave, i.e. $a \equiv 0$.

The absence of the conic singularity actually does follow from the construction; it is a combination of geometric coincidence and that the model operators are Laplacians on symmetric spaces, so their scattering matrices have certain symmetries.

Here, as in the 3-body setting, the coefficients are constructed iteratively, starting with c_1, corresponding to the size of the real part of the exponent in $\mathfrak{a}^+$. (For real ξ, one replaces 'real part of the exponent' by 'phase'.) They are obtained by ensuring that the incident and reflected waves combine to give an approximate generalized eigenfunction at the wall at which the reflection takes place.

(If we do not assume the absence of bound states in subsystems, the spherical wave is replaced by a more complicated expression.)

In fact, there is one more miracle, the coefficients $c_s(z)$ are actually smooth on the smaller space $\bar{\mathfrak{a}}$ rather than merely on $\tilde{\mathfrak{a}}$. As already noted, this miracle *only* holds for the spherical functions, a.k.a. perturbed plane waves, *not* for the asymptotics of the Green's function, a.k.a. spherical waves.

In general, on symmetric spaces of non-compact type G/K, there is a similar expression for the radial Laplacian on $\mathfrak{a}_{\mathrm{reg}}$.

- The Cartan decomposition is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{k}$ the Lie algebra of K, $\mathfrak{p}$ the orthocomplement with respect to the Killing form, so $\mathfrak{p}$ can be identified with $T_o M$.
- Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The symmetric homomorphisms $\operatorname{ad} H$, $H \in \mathfrak{a}$, on $\mathfrak{g}$ commute, hence are simultaneously diagonalizable. A simultaneous eigenvector X satisfies
$$(\operatorname{ad} H)(X) = \alpha(H)X$$
for each $H \in \mathfrak{a}$; here α is a linear functional, so $\alpha \in \mathfrak{a}^*$.
- The set of α's that arise this way is the set Λ of (restricted) roots, and the space of eigenvectors associated to α is the 'root space' $\mathfrak{g}_\alpha$.

Each $\alpha \in \Lambda$ determines the associated Weyl chamber wall $W_\alpha = \alpha^{-1}(0) \subset \mathfrak{a}$;

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \cup_{\alpha \in \Lambda} W_\alpha$$

is the set of regular vectors.

The orthogonal reflections across the walls generate the Weyl group W; this is a finite group. Fixing a connected component $\mathfrak{a}^+$ of $\mathfrak{a}_{\mathrm{reg}}$ to be the positive Weyl chamber, we denote by Λ^+ the set of positive roots (i.e. roots that are positive on $\mathfrak{a}^+$).