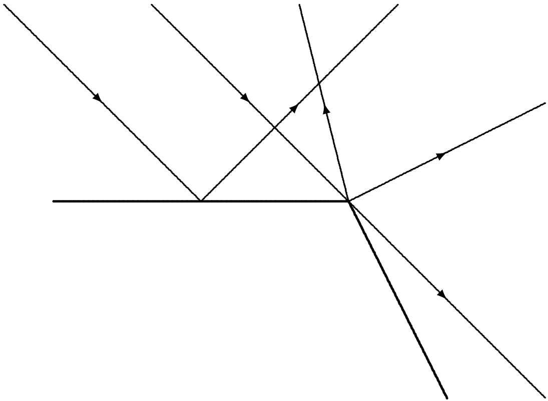
# Diffraction by edges

András Vasy
(with Richard Melrose and Jared Wunsch)

New Orleans
January 2007

According to geometric optics, light propagates in straight lines (in homogeneous media), reflects/refracts from surfaces according to Snell's law: energy and tangential momentum are conserved.



A better description is that light satisfies the wave equation,

$$Pu = 0, \quad Pu = D_t^2 u - \Delta_c u,$$

$\Delta_c$ is the Laplacian, so it is $c^2 \sum_{j=1}^n D_{x_j}^2$ in $\mathbb{R}^n$, where $c$ is the speed of light, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial_{x_j}}$, with suitable boundary conditions, e.g. $u = 0$ on the boundary (Dirichlet BC).

One way of discussing the relationship between these is that singularities (lack of smoothness) of solutions of $Pu = 0$ follow geometric optics rays.

In order to orient ourselves, we start with a simple first order PDE.

Let $V$ be a real $C^\infty$ vector field on a manifold without boundary, $M$. The PDE $Vu = 0$ states that $u$ is constant along integral curves of $V$. In other words, values of $u$ propagate along the integral curves. For instance, the following propagate:

- being 0 in a neighborhood of a point,
- being $C^k$ or $C^\infty$ in a neighborhood of a point,
- being $L^2$, or lying in a Sobolev space $H^s$.

If $s$ is a positive integer, $H^s$ is the space of functions whose derivatives up to $s$th order are in $L^2$, i.e. square integrable.

## Definition

- Let $\operatorname{supp} u$ to be the closure of the set where $u$ is nonzero, i.e. $x \notin \operatorname{supp} u$ if $x$ has a neighborhood on which $u$ is 0.
- Let $\operatorname{sing\,supp} u$ to be the set of points which have no neighborhood on which $u$ is $C^\infty$.

Then:

- if $Vu = 0$ and $x \in \operatorname{supp} u$, then $y \in \operatorname{supp} u$ for every $y$ which is on the integral curve of $V$ through $x$.
- if $Vu = 0$ and $x \in \operatorname{sing\,supp} u$, then $y \in \operatorname{sing\,supp} u$ for every $y$ which is on the integral curve of $V$ through $x$.

A similar result remains true if we instead have a PDE $Pu = 0$, where $P = V + a$, and $a$ is $C^\infty$. Note that $a$ does not influence how the support or singular support propagate.

But what happens if $P$ is replaced by a higher order differential operator? For instance, consider the wave equation on the line:

$$D_t^2 u - c^2 D_x^2 u = 0.$$

Factoring the operator as $(D_t - cD_x)(D_t + cD_x)$ shows that the general solution is

$$u = f(x + ct) + g(x - ct),$$

i.e. the sum of two propagating waves. Note that the two waves do not interact with each other at all. But one goes to the left, the other to the right, so how do we know which one we are to follow?

Associate a direction as well to the singularities of functions!

We want to say not only whether $u$ is singular at a point, but also in which direction it is singular. Introduced by Hörmander, the set describing this information is a refinement of $\operatorname{sing\,supp} u$, and is called the wave front set of $u$.

- On manifolds *without boundary* $X$, this is naturally a subset of the cotangent bundle with the zero section removed, $T^*X \setminus o$.
- If $X = \mathbb{R}^n_z$, then $T^*X = \mathbb{R}^n_z \times \mathbb{R}^n_\zeta$, so $\mathrm{WF}(u)$ consists of points $(z, \zeta)$, $\zeta \neq 0$, and it measures whether $u$ is $C^\infty$ at $z$ in the direction of $\zeta$.

Could also measure singularities with respect to other spaces:

- Sobolev spaces $H^s_{\mathrm{loc}}(X)$: $\mathrm{WF}^s(u)$,
- real analytic functions: $\mathrm{WF}_A(u)$.

Examples:

- $\mathrm{WF}(\delta_0) = \{(0, \zeta) : \zeta \neq 0\} = N^*\{0\} \setminus o$;
- if $\Omega$ has a $C^\infty$ boundary, say $\Omega = \{z : \ f(z) > 0\}$, $f$ is $C^\infty$, $df \neq 0$ when $f = 0$, and $\chi_\Omega$ is the characteristic function of $\Omega$, then

$$\begin{aligned}\mathrm{WF}(\chi_\Omega) = \{(z, \zeta) : z \in \partial\Omega, \ \zeta = \lambda\, df(z), \ \lambda \in \mathbb{R} \setminus \{0\}\} \\ = N^*\partial\Omega \setminus o.\end{aligned}$$

N.B. If $Y$ is a submanifold of $X$, $q \in Y$, $N^*_q Y$ is spanned by differentials of functions vanishing on $Y$.

Basic properties of $\mathrm{WF}(u)$ include:

- $\mathrm{WF}(u)$ is a closed conic subset of $T^*X \setminus o$, i.e. $(z,\zeta) \in \mathrm{WF}(u)$ implies $(z,\lambda\zeta) \in \mathrm{WF}(u)$ for $\lambda > 0$;
- $u$ is $C^\infty$ if and only if $\mathrm{WF}(u) = \emptyset$;
- for $z_0 \in X$, $z_0 \notin \operatorname{sing\,supp} u$ (i.e. $z_0$ has a neigborhood in $X$ on which $u$ is $C^\infty$) if and only if $\mathrm{WF}(u) \cap (T^*_{z_0}X \setminus o) = \emptyset$.

Once one knows $\mathrm{WF}(u)$, one may want to know precisely how $u$ is singular. For solutions of the wave equation this is

- possible in the boundaryless case,
- impossible in general when boundaries are present.

One way to define $\mathrm{WF}(u)$, i.e. to *microlocalize*:

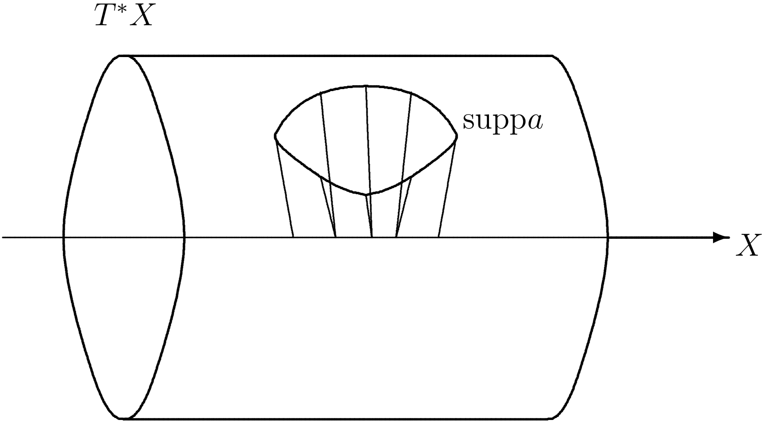
- consider a class of functions, called *symbols*. Symbols are functions $a$ on $T^*X$ which are well-behaved as $\zeta \to \infty$; e.g. they are asymptotically homogeneous, say of degree $m$.
- *quantize*: associate *pseudodifferential operators* $A = \mathrm{Op}(a) \in \Psi(X)$ to $a$. *Not canonical.*
- $A : C^\infty(X) \to C^\infty(X)$,
- $A : H^s(X) \to H^{s-m}(X)$.

- one choice: $a(z,\zeta) = \sum_{|\alpha|\leq m} a_\alpha(z)\zeta^\alpha \Rightarrow \mathrm{Op}(a) = a(z, D_z) = \sum_{|\alpha|\leq m} a_\alpha(z) D_z^\alpha$.
- if $A \in \Psi^m(X)$, there is a canonical *principal symbol* $\sigma_m(A)$ associated to it.
- $\sigma_m(A)$ is a homogeneous degree $m$ function on $T^*X \setminus o$.
- if $a$ is a symbol which near infinity is homogeneous of degree $m$, then $\mathrm{Op}(a) \in \Psi^m(X)$, and $\sigma_m(\mathrm{Op}(a)) = a$.
- $\sigma_m(\sum_{|\alpha|\leq m} a_\alpha(z) D_z^\alpha) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$.

**Definition**

We say $(z, \zeta) \notin \mathrm{WF}(u)$ if for some asymptotically homogeneous degree 0 symbol $a$, with $a$ identically 1 near infinity on the half-line through $(z, \zeta)$, $\mathrm{Op}(a)u \in C^\infty(X)$.

The idea is that $\mathrm{Op}(a)$ microlocalizes at $\mathrm{supp}(a)$, so $\mathrm{Op}(a)$ is a *phase space cutoff*.




Compare with singular support: $z \notin \operatorname{sing\,supp} u$ if there is $f \in C^\infty(X)$ identically 1 near $z$ such that $fu \in C^\infty(X)$.

## Definition

If $p$ is a function on $T^*X$, the Hamilton vector field of $p$ is defined as

$$H_p = \sum_j \frac{\partial p}{\partial \zeta_j} \frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j} \frac{\partial}{\partial \zeta_j}.$$

- if $p$ is homogeneous degree $m$ with respect to the $\mathbb{R}^+$ action, then $H_p$ is homogeneous of degree $m - 1$.
- $H_p$ is invariantly defined, using the *symplectic structure* of $T^*X$.

The geometry associated to an $m$th order (pseudo)differential operator $P$ arises from $p = \sigma_m(P)$ and the structure of $T^*X$.

**Definition**

- $\mathrm{Char}(P) = p^{-1}(\{0\}) \subset T^*X \setminus o$ is the characteristic set of $P$; it is conic (invariant under the $\mathbb{R}^+$-action),
- assuming $p$ is real valued, bicharacteristics $\gamma$ are integral curves of the Hamilton vector field $H_p$ inside $\mathrm{Char}(P)$.

Thus, bicharacteristics $\gamma(s) = (z(s), \zeta(s))$ satisfy $p(z(s), \zeta(s)) = 0$, and

$$\frac{dz_j}{ds} = \frac{\partial p}{\partial \zeta_j}(z(s), \zeta(s)), \ \frac{d\zeta_j}{ds} = -\frac{\partial p}{\partial z_j}(z(s), \zeta(s)),$$

the equations of Hamiltonian mechanics.

If $(M, g)$ Riemannian manifold, $\Delta_g \geq 0$ the Laplacian, $P = D_t^2 - \Delta_g$ is the wave operator on $X = M_z \times \mathbb{R}_t$, then

$$p(z, t, \zeta, \tau) = \sigma(P) = \tau^2 - |\zeta|_z^2,$$

the projection to $M$ of the bicharacteristics are geodesics.

**Theorem (Hörmander; $\partial X = \emptyset$.)**

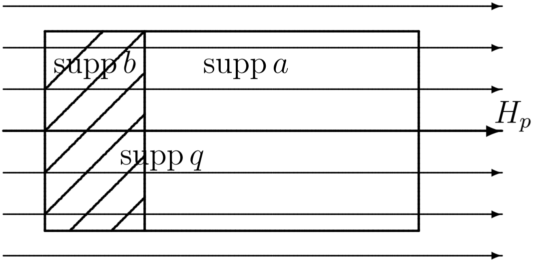
*If $P \in \Psi^m(X)$, $\sigma_m(P)$ real, $Pu = 0$ then $\mathrm{WF}(u) \subset \mathrm{Char}(P)$ and $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics inside $\mathrm{Char}(P)$.*

In other words, singularities of $u$ propagate along bicharacteristics.

Proofs:

- Construct an approximation, or parametrix, for the solution operator. This was the original approach (Hadamard, Lax and others). It was perfected by Duistermaat and Hörmander.
- Prove microlocal energy estimates, i.e. estimate the $L^2$-norm of $\mathrm{Op}(a)u$ in terms of the $L^2$-norm of $\mathrm{Op}(b)u$, here $a$, $b$ are phase space cutoffs.

  Roughly, this says that energy on $\operatorname{supp} a$ is controlled by energy in $\operatorname{supp} b$, i.e. energy could only get to $\operatorname{supp} a$ from $\operatorname{supp} b$.




- construct an operator, say $\mathrm{Op}(q)$, whose commutator with $P$ has form $-i(\mathrm{Op}(b)^2 - \mathrm{Op}(a)^2)$, modulo negligible terms,
- if $Pu = 0$ then

$$\begin{aligned}
&\|\mathrm{Op}(b)u\|^2 - \|\mathrm{Op}(a)u\|^2 \sim \langle i[P, \mathrm{Op}(q)]u, u\rangle \\
&= \langle i(P\mathrm{Op}(q) - \mathrm{Op}(q)P)u, u\rangle \\
&= \langle i\mathrm{Op}(q)u, Pu\rangle - \langle i\mathrm{Op}(q)Pu, u\rangle = 0,
\end{aligned}$$

- $H_p$ enters the picture because the principal symbol of $[P, \mathrm{Op}(q)]$ is $\frac{1}{i} H_p q$,
- so we want

$$H_p q = b^2 - a^2,$$

which can be achieved by taking $q$ to be a bump function along integral curves of $H_p$.

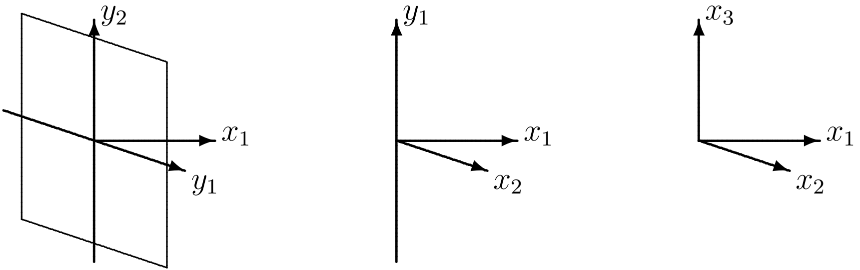
Below we consider the wave equation

$$Pu = 0, \quad Pu = D_t^2 u - \Delta_g u,$$

on manifolds with corners $M$ with a $C^\infty$ Riemannian metric $g$; here $\Delta_g \geq 0$ the Laplacian, $D_t = \frac{1}{i}\partial_t$, and $u$ is a function (or distribution, i.e. generalized function) on $X = M \times \mathbb{R}_t$.

Manifolds with corners $X$ are locally diffeomorphic to quadrants in Euclidean spaces, i.e. each point of $X$ has a neighborhood $U$ with local coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ such that locally $X$ is given by $x_1 \geq 0, \ldots, x_k \geq 0$.

The boundary faces of $X$ intersecting $U$ are thus locally of the form $\{x_j = 0,\ j \in J\}$ where $J$ is a subset of $\{1, \ldots, k\}$, and have codimension $|J|$.

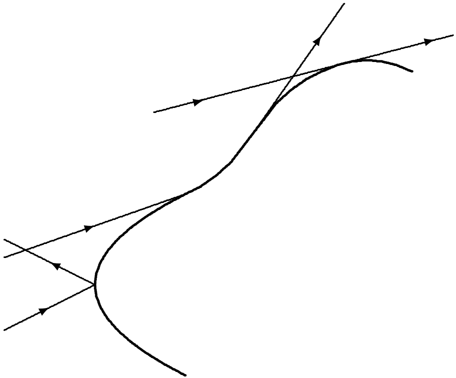


For a manifold with $C^\infty$ boundary, $k \leq 1$ in all local coordinates.

If $M$ has a $C^\infty$ boundary, there is a result very similar to Theorem 1 due to Melrose, Sjöstrand and Taylor. The basic picture is that when a geodesic hits the boundary kinetic energy and tangential momentum are conserved, but the normal momentum may jump. (But its magnitude is conserved!)

**Theorem (Melrose-Sjöstrand-Taylor, $\partial M$ smooth)**

*If $u$ is an* admissible *solution of the wave equation, satisfying Dirichlet or Neumann boundary conditions, then $\mathrm{WF}_b(u) \subset \dot{\Sigma}$ is a union of maximally extended* generalized broken bicharacteristics.



*Generalized broken bicharacteristics* are now best thought of as curves in (i.e. continuous maps from an interval into) a compressed version $\dot{\Sigma}$ of the characteristic set $\Sigma = \mathrm{Char}(P)$ in which points in $\Sigma \subset T^*X$ differing by an element of $N^*\partial X$ are identified.

Special case – *broken bicharacteristic*:

- $\hat{\pi} : \Sigma \to \dot{\Sigma}$ projection,
- $\gamma_- : (a, 0] \to \Sigma$ and $\gamma_+ : [0, b) \to \Sigma$ are bicharacteristics in the usual sense,
- $\hat{\pi}(\gamma_+(0)) = \hat{\pi}(\gamma_-(0))$,
- then the curve

$$\gamma : (a, b) \to \dot{\Sigma}, \;\; \gamma|_{(a,0]} = \hat{\pi}(\gamma_-), \;\; \gamma|_{[0,b)} = \hat{\pi}(\gamma_+),$$

is a generalized broken bicharacteristic.

If $\partial X$ is smooth

- every *normally incident* generalized broken bicharacteristic is a broken bicharacteristic,
- not true for tangentially incident ones.

*Admissibility*: If $X$ is a manifolds with corners equipped with a $C^\infty$ metric, admissible solutions include all $u \in L^2_{\mathrm{loc}}(X)$ with $du \in L^2_{\mathrm{loc}}(X; T^*X)$, i.e. all $u \in H^1_{\mathrm{loc}}(X)$. More generally, time derivatives of such $u$ are admissible.

*Dirichlet boundary conditions*: require $u \in H^1_{0,\mathrm{loc}}(X)$, where $H^1_0(X)$ is the completion of $C^\infty_c(X^\circ)$ in the $H^1$ norm $\|v\|^2_{H^1} = \|v\|^2_{L^2} + \|dv\|^2_{L^2}$. (Again, can take time derivatives.)

If $u \in H^1_{0,\mathrm{loc}}(X)$, $u$ solving the wave equation $Pu = 0$ means that

$$\int_X \left( D_t u \, \overline{D_t v} - \langle du, dv \rangle_g \right) \, dg \, dt = 0$$

for all $v \in H^1_{0,\mathrm{loc}}(X)$. There is a similar formulation for Neumann boundary conditions.

For admissible solutions of the wave equation on manifolds with corners, there is also a wave front set (in ${}^bT^*X \setminus o$; $\dot{\Sigma} \subset {}^bT^*X \setminus o$), due to Melrose, denoted by $\mathrm{WF}_b(u)$:

- away from $\partial X$, this is $\mathrm{WF}(u)$,
- at $\partial X$ it measures if $u$ is microlocally conormal to $\partial X$ relative to $L^2(X)$, i.e. it microlocally tests whether $V_1 \ldots V_m u \in L^2(X)$ for all vector fields $V_1, \ldots, V_m$ tangent to $\partial X$.

In local coordinates, this means that we test whether

$$(x_1\partial_{x_1})^{\alpha_1} \ldots (x_k\partial_{x_k})^{\alpha_k} \partial_{y_1}^{\alpha_{k+1}} \ldots \partial_{y_{n-k}}^{\alpha_n} Au \in L^2_{\mathrm{loc}}(X)$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $A = \mathrm{Op}(a) \in \Psi_b(X)$ is a microlocalizer.

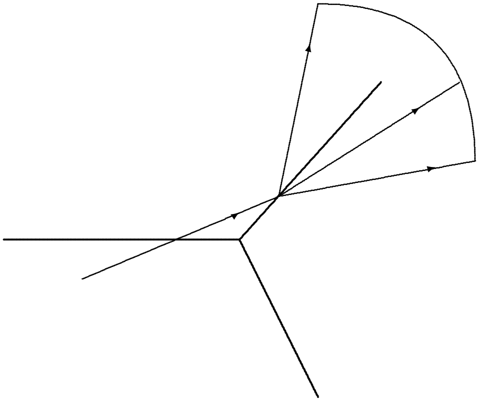
Even for elliptic equations, such as $\Delta u = 0$, on a manifold with corners, one cannot expect that $u$ is $C^\infty$, as is shown by the example of circular sectors of angle $\beta$, in which solutions vanishing on the sides of the angle have the form

$$u \sim r^{n\pi/\beta} \sin(n\pi\theta/\beta)$$

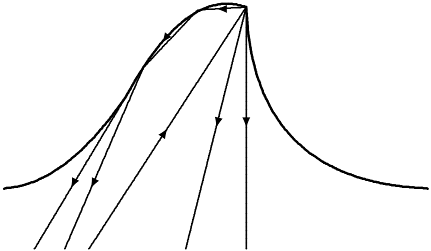
in polar coordinates $(r, \theta)$, in which the sector is $0 \leq \theta \leq \beta$.

If $X$ has corners:

- to define $\dot{\Sigma}$, for each face $F$ of $X$, over $F^\circ$, identify points in $\Sigma = \mathrm{Char}(P)$ differing by a covector in $N^*F$.
- encodes law of reflection: kinetic energy and momentum tangential to $F$ are conserved, but the normal momentum may change.



- Matching bicharacteristic segments $\gamma_\pm$ again give rise to generalized broken bicharacteristics,
- even for normal incidence these are not the only ones as the incoming/outgoing rays can be tangent to another boundary face,
- the continuation of an incoming ray is not unique: one gets a whole cone of reflected rays.

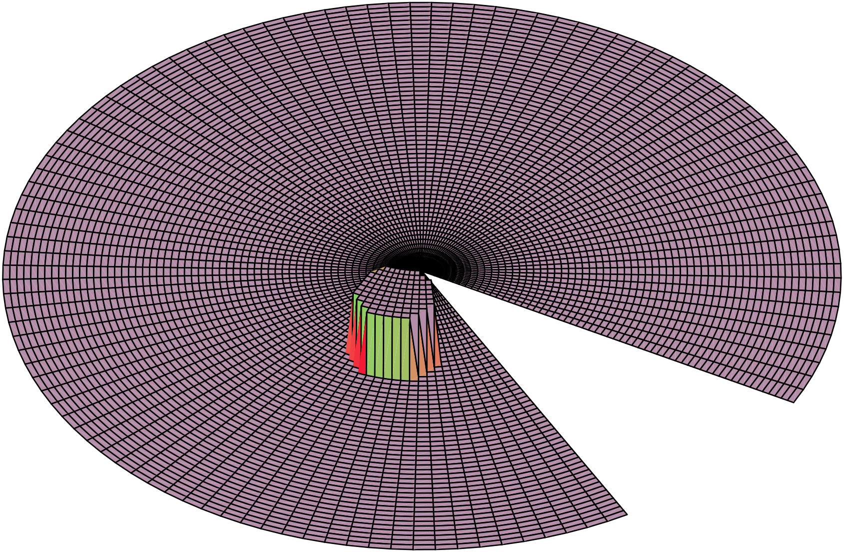


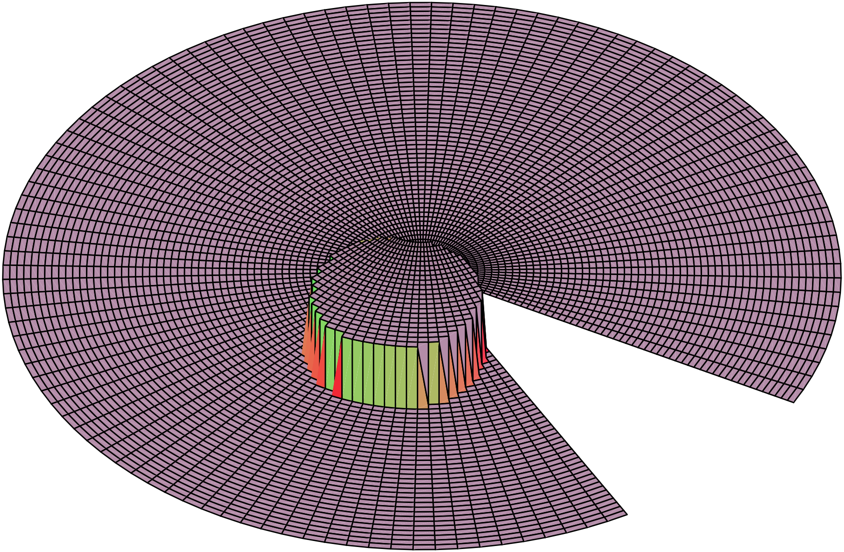
**Theorem (Lebeau in the analytic category, A.V.)**

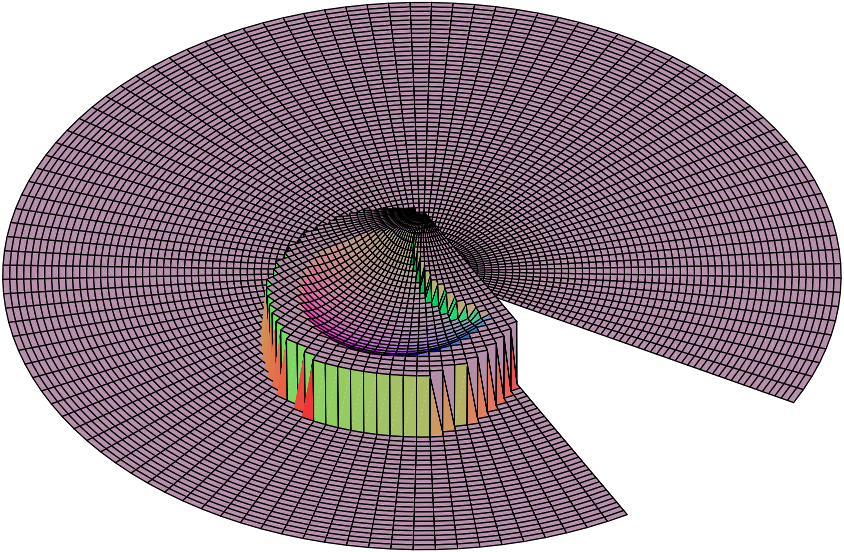
*If $u$ is an admissible solution of the wave equation, satisfying Dirichlet or Neumann boundary conditions at codimension one hypersurfaces, then $\mathrm{WF}_b(u) \subset \dot{\Sigma}$ is a union of maximally extended generalized broken bicharacteristics.*

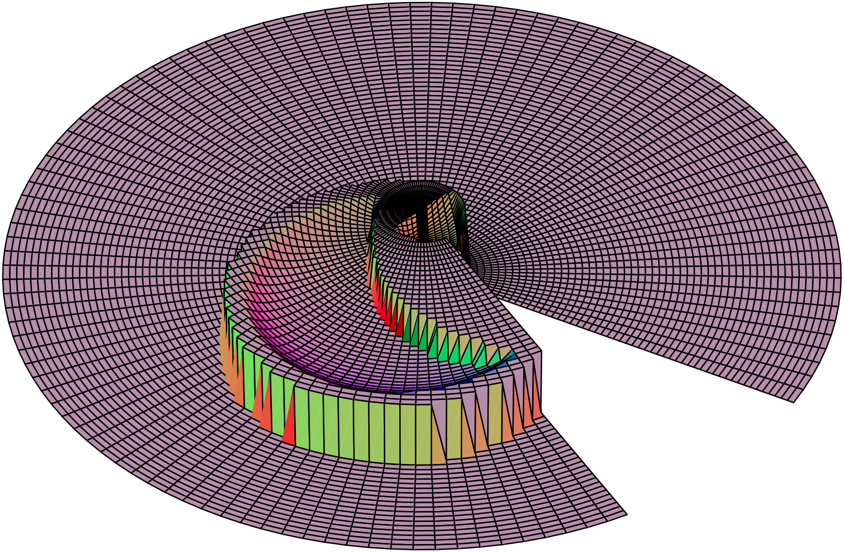
This result is optimal for normally incident bicharacteristics $\gamma_-$ in the sense that in general a solution will have wave front set on *all* generalized broken bicharacteristics extending $\gamma_-$.

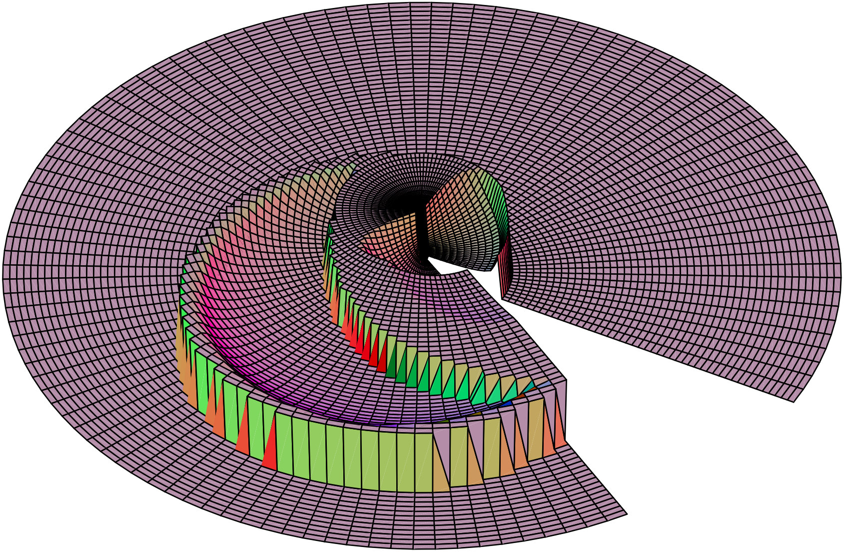
The result also holds for Sobolev wave front sets $\mathrm{WF}_b^m(u)$, which measure in $X^\circ$ whether $u$ is microlocally in $H^m_{\mathrm{loc}}(X^\circ)$. The proof relies on positive commutator estimates (microlocal energy estimates) relative to $H^1(X)$ using Melrose's $\Psi_b(X)$ (totally characteristic operators) as microlocalizers and commutants.

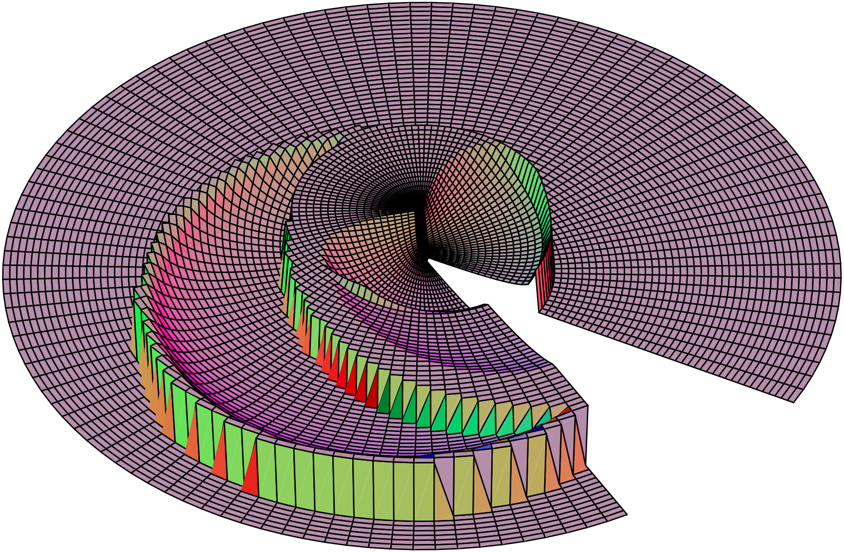


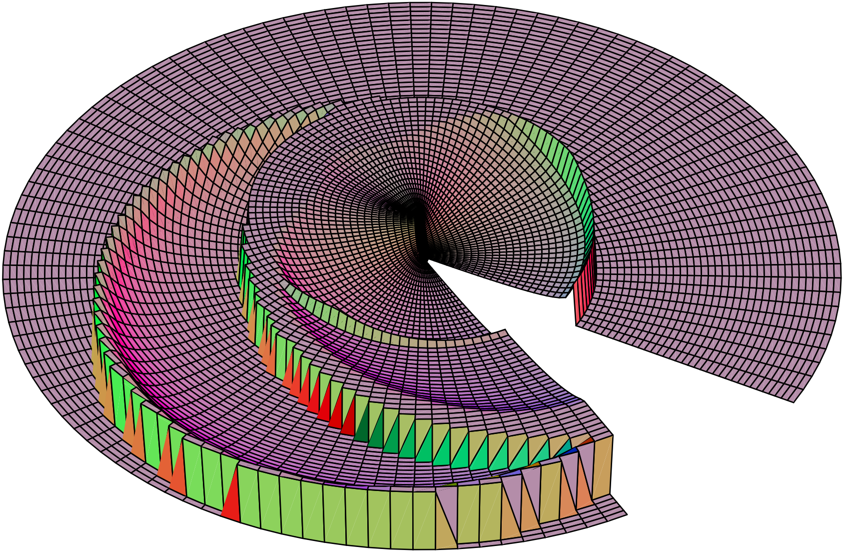


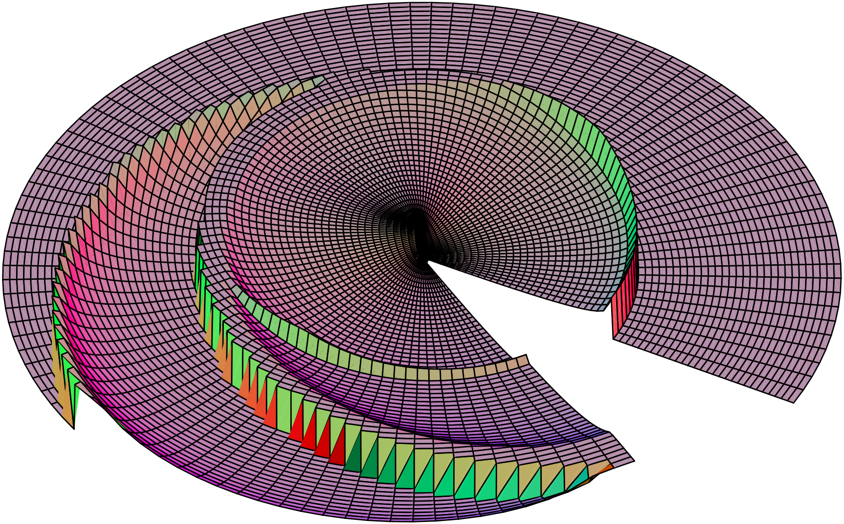










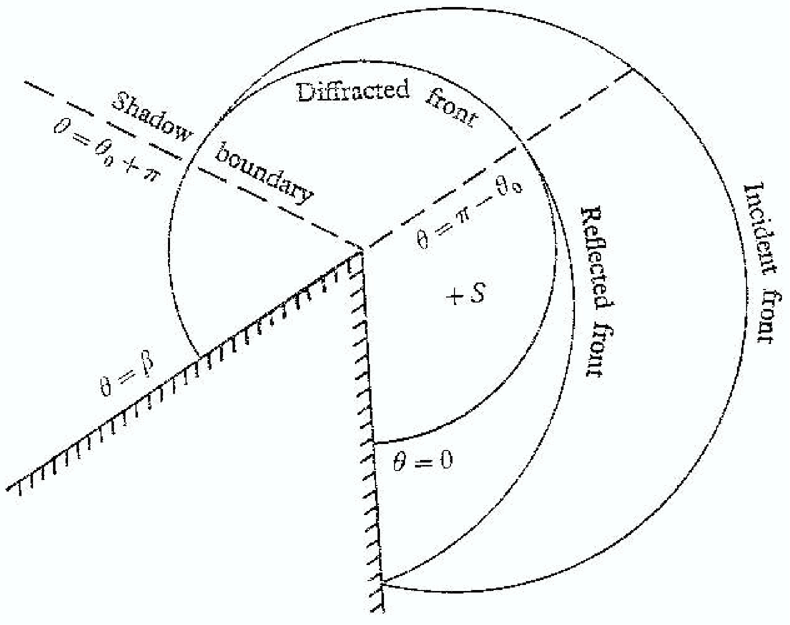


Is the Sobolev result optimal for normally incident bicharacteristics?

- It is,
- but for a rather large class of solutions of the wave equation, namely those 'not focusing' on the corner, it can be improved.

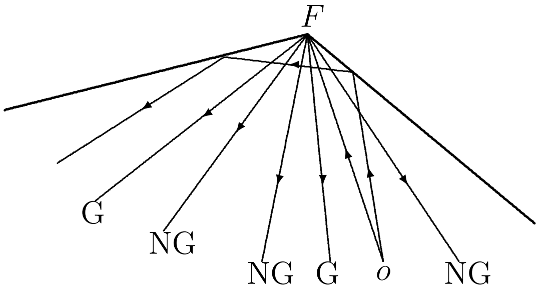
Illustration: spherical waves emanating from a source near the boundary or corner:

- most of the spherical wave misses the corner, i.e. only a lower dimensional part hits it,
- full dimensional part of the spherical wave hits the boundary hypersurfaces (or smooth boundary).


Diffracted front
Shadow boundary
$\theta = \theta_0 + \pi$
$\theta = \pi - \theta_0$
Reflected front
Incident front
$+S$
$\theta = \beta$
$\theta = 0$

**Definition**

Suppose $F$ is a corner. Among generalized broken bicharacteristics hitting the edge $F^\circ$, the geometric bicharacteristics are those which are limits of bicharacteristics in $\dot{\Sigma} \setminus T^*F$.



Expectation: unless one is dealing with a solution that focuses on the corner, on the 'non-geometric' broken bicharacteristics the reflected wave should be less singular than the incident one.

- Full result is too hard with the current state of technology.
- partial results on manifolds with corners (with corners of arbitrary codimension),
- the full result in a model setting (manifolds with so-called edge metrics).

So we consider an edge (or corner) $F$, of codimension $k$, on a manifold with corners. Near an interior point of $F$, one has local coordinates $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ such that locally $X$ is given by $x_1 \geq 0, \ldots, x_k \geq 0$, and $F$ is given by $x = 0$ (i.e. $x_1 = \ldots = x_k = 0$).

### Definition

A generalized broken bicharacteristic segment $\gamma_0$, defined on $(0, s_0)$ or $(-s_0, 0)$, is said to approach $F$ *normally* as $s \to 0$ if for all $j$

$$\lim_{s\to 0\pm} \frac{x_j(\gamma(s))}{s} \neq 0.$$

In particular, the projection of $\gamma_0$ to $M$ is a geodesic for small $s \neq 0$.

## Definition

Suppose $\gamma_0 : (-s_0, 0) \to \dot{\Sigma}$ is a bicharacteristic approaching $F^\circ$ normally.
The *regular part of the diffracted front* emanating from $\gamma_0$ consists of *non-geometric* generalized broken bicharacteristics $\gamma : (-s_0, s_1)$ extending $\gamma_0$, such that $\gamma|_{(0,s_1)}$ approaches $F$ normally.

Example: fundamental solution of wave equation with pole $o$ near the edge.

**Theorem (Melrose-V.-Wunsch)**

*Let $F$ be a codimension $k$ corner of $X$. Let $s$ be such that the fundamental solution of the wave equation with pole in $X^\circ$ lies in $H^{s'}_{\mathrm{loc}}(X^\circ)$ for all $s' < s$.*

*Suppose also that the pole, $o$, is on a bicharacteristic $\gamma_0$ normally incident to $F^\circ$, sufficiently close to $F^\circ$.*

*Then* microlocally *near the regular part of the diffractive front emanating from $\gamma_0$, and the fundamental solution is in $H^{s'+(k-1)/2}_{\mathrm{loc}}(X^\circ)$ for all $s' < s$.*

In 2 dimensions, in the analytic category, there is a corresponding result due to Gérard and Lebeau for conormal incident waves. There is also a long history of the subject in applied mathematics, especially in the work of Keller.

Model: manifolds with edge metrics – manifolds with boundary $\tilde{M}$, whose boundary has a fibration, $\pi_0 : \partial\tilde{M} \to Y$ with compact fibers $Z$ (without boundary), and a Riemannian metric $g$ compatible with this fibration.

More precisely, we assume that on a neighborhood $U$ of $\partial\tilde{M}$, $g$ is of the form

$$g = dx^2 + \pi^* h + x^2 k$$

with

- the boundary defined by $x = 0$,
- $h \in C^\infty([0,\epsilon) \times Y; \mathrm{Sym}^2\, T^*([0,\epsilon) \times Y))$,
- and $k \in C^\infty(U; \mathrm{Sym}^2\, T^*\tilde{M})$;

we further assume that $h|_{x=0}$ is a nondegenerate metric on $Y$ and $k|_{x=0}$ is a nondegenerate fiber metric.

Here we extended the fibration $\pi_0$ to a fibration $\pi : U \to [0, \epsilon) \times Y$ on a neighborhood $U$ of $\partial \tilde{M}$.

Examples: let $\tilde{M}$ be the real blow up of a $C^\infty$ submanifold $Y$ of a manifold without boundary $M$: $\tilde{M} = [M; Y]$, i.e. introduce 'polar coordinates' around $Y$ in $M$.

- the fibers $Z$ are spheres,
- a smooth metric on $M$ would give rise to an edge metric on $\tilde{M}$.
- e.g. $z$ axis in $\mathbb{R}^3$ blown up: cylindrical coordinates $(z, r, \theta) \in \mathbb{R} \times [0, \infty) \times \mathbb{S}^1$. The boundary is $r = 0$ (so $x = r$), the fiber is $\mathbb{S}^1$.
- Euclidean metric becomes $dz^2 + dr^2 + r^2\, d\theta^2$.

More interesting case:

- $M$ manifold with corners,
- $\tilde{M}$ 'total boundary blow up' (blow up all corners)
- fibers $Z$ have boundary: does not quite fit previous framework,
- e.g. $\theta \in [0, \beta]$ rather than $\theta \in \mathbb{S}^1$,
- as long as one stays away from bicharacteristics hitting the face $F^\circ$ in question tangentially to the other faces, the model methods still work.

Back to generalized broken bicharacteristics:

**Definition**

If $\partial X$ is smooth:

$$\dot{T}^*X = T^*\partial X \cup T^*X^\circ \text{ (disjoint union)},$$

with the natural projection $\pi : T^*X \to \dot{T}^*X$. In local coordinates $(x, y_1, \ldots, y_{n-2}, t)$ (where $t$ plays the role of one of the $y$ coordinates), and dual coordinates $(\xi, \eta_1, \ldots, \eta_{n-2}, \tau)$,

$$\pi(x, y, t, \xi, \eta, \tau) = \begin{cases} (x, y, t, \xi, \eta, \tau), & x > 0, \\ (0, y, t, \eta, \tau), & x = 0. \end{cases}$$

If $p|_{x=0} = \tau^2 - \xi^2 - \sum h_{ij}\eta_i\eta_j = \tau^2 - \xi^2 - |\eta|^2$,

$$\dot{\Sigma} \cap T^*\partial X = \{(0, y, t, \eta, \tau) : \ |\eta|^2 \leq \tau^2, \ \tau \neq 0\}.$$

Similar definition for corners.

The projection $\hat{\pi} : \Sigma \to \dot{\Sigma}$ gives the topology of $\dot{\Sigma}$. (It is best to consider $\dot{\Sigma}$ as a subset of the b-cotangent bundle ${}^bT^*X$, which is a $C^\infty$ vector bundle over $X$ introduced by Melrose.)

## Definition

On manifolds with corners, generalized broken bicharacteristics are

- continuous maps $\gamma : I \to \dot{\Sigma}$,
- for $f$ continuous real valued on $\dot{\Sigma}$ such that $\hat{\pi}^* f$ is $C^\infty$ on $\Sigma$,

$$\liminf_{s \to s_0} \frac{f \circ \gamma(s) - f \circ \gamma(s_0)}{s - s_0} \geq \inf\{H_p \hat{\pi}^* f(q) : \ q \in \hat{\pi}^{-1}(\gamma(s_0))\}.$$

In the $C^\infty$ setting one can strengthen this definition to rule out certain bicharacteristics gliding along the boundary, away from corners.

For each $f$ as in the definition, the *one-sided limits*

$$\lim_{s \to s_0 \pm} \frac{f \circ \gamma(s) - f \circ \gamma(s_0)}{s - s_0}$$

exist, and are equal to $H_p \hat{\pi}^* f(q_\pm)$ for some $q_\pm \in \hat{\pi}^{-1}(\gamma(s_0))$.

Possibilities for $\rho \in \dot{\Sigma} \cap T^* F$:

- *glancing*: $\hat{\pi}^{-1}(\{\rho\})$ consists of one point: $\tau^2 = |\eta|^2$ – rays through $\rho$ are tangent to $F$,
- *hyperbolic*: $\hat{\pi}^{-1}(\{\rho\})$ consists of more than one point: $\tau^2 > |\eta|^2$ – rays through $\rho$ are normal to $F$,
- if $F$ has codimension 1, hyperbolic points have exactly two preimages, $\xi = \pm\sqrt{\tau^2 - |\eta|^2}$, corresponding to broken bicharacteristics.