Diffraction by Edges

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Consider the wave equation

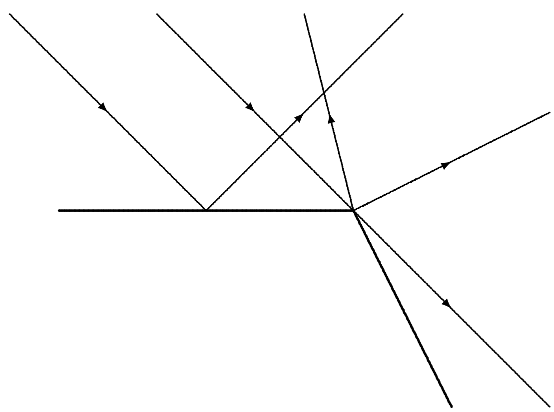
$$Pu = 0, \quad Pu = D_t^2 u - \Delta_g u,$$

on manifolds with corners M; here $\Delta_g \geq 0$ the Laplacian, $D_t = \frac{1}{i}\partial_t$, i.e. u is a distribution on $X = M \times \mathbb{R}_t$.

If $\partial M = \emptyset$, Hörmander's theorem states that singularities of u propagate along bicharacteristics, in the sense that $\mathrm{WF}(u) \subset \mathrm{Char}(P)$ and $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics inside $\mathrm{Char}(P)$. Here recall that

- $p = \sigma_2(P) \in C^\infty(T^*X \setminus o)$ is principal symbol of P,
- $\mathrm{Char}(P) = p^{-1}(\{0\}) \subset T^*X$ is the characteristic set of P,
- bicharacteristics are integral curves of the Hamilton vector field H_p inside $\mathrm{Char}(P)$, and their projections to M are geodesics.

If M has a C^∞ boundary, there is a very similar result due to Melrose, Sjöstrand and Taylor. The basic picture is that when a geodesic hits the boundary kinetic energy and tangential momentum are conserved, but the normal momentum may jump. (But its magnitude is conserved!)



Bicharacteristics are now best thought of as curves in (i.e. continuous maps from an interval into) a compressed version $\dot{\Sigma}$ of the characteristic set $\Sigma = \mathrm{Char}(P)$ in which points in $\Sigma \subset T^*X$ differing by an element of $N^*\partial X$ are identified.

Thus, one has a projection $\widehat{\pi} : \Sigma \to \dot{\Sigma}$; $\dot{\Sigma}$ is given a topology via $\widehat{\pi}$. Using $\dot{\Sigma}$ encodes the law of reflection. (In fact, it is best to consider $\dot{\Sigma}$ as a subset of the b-cotangent bundle ${}^bT^*X$.)

Generalized broken bicharacteristics are

- continuous maps $\gamma : I \to \dot{\Sigma}$,
- for f continuous real valued on $\dot{\Sigma}$ such that $\widehat{\pi}^* f$ is C^∞ on Σ,

$$\liminf_{s \to s_0} \frac{f \circ \gamma(s) - f \circ \gamma(s_0)}{s - s_0}$$
$$\geq \inf\{H_p f(q) : \ q \in \widehat{\pi}^{-1}(\gamma(s_0))\}.$$

In the C^∞ setting one can strengthen this definition to rule out certain bicharacteristics gliding along the boundary, away from corners.

The prototypical example is a broken bicharacteristic: if $\gamma_- : (a, 0] \to \Sigma$ and $\gamma_+ : [0, b) \to \Sigma$ are bicharacteristics in the usual sense, and $\hat{\pi}(\gamma_+(0)) = \hat{\pi}(\gamma_-(0))$, then the curve

$$\gamma : (a, b) \to \dot{\Sigma},\ \gamma|_{(a,0]} = \hat{\pi}(\gamma_-),\ \gamma|_{[0,b)} = \hat{\pi}(\gamma_+),$$

is a (generalized broken) bicharacteristic.

If $q \in \dot{\Sigma}$ is 'hyperbolic', i.e. $\hat{\pi}^{-1}(\{q\})$ consists of more than one (in this case automatically two) points, then every (generalized broken) bicharacteristic through q has this form, and indeed γ is uniquely determined by q. (Not true in general!)

In the presence of boundaries and corners, one needs an admissibility criterion for solutions of the wave equation. If X is a manifolds with corners equipped with a C^∞ metric, we say that u is admissible if there exists $k \in \mathbb{R}$ such that

$$\langle D_t \rangle^{-k} u \in L^2_{\text{loc}}(X) \text{ and } \langle D_t \rangle^{-k} du \in L^2_{\text{loc}}(X; T^*X),$$

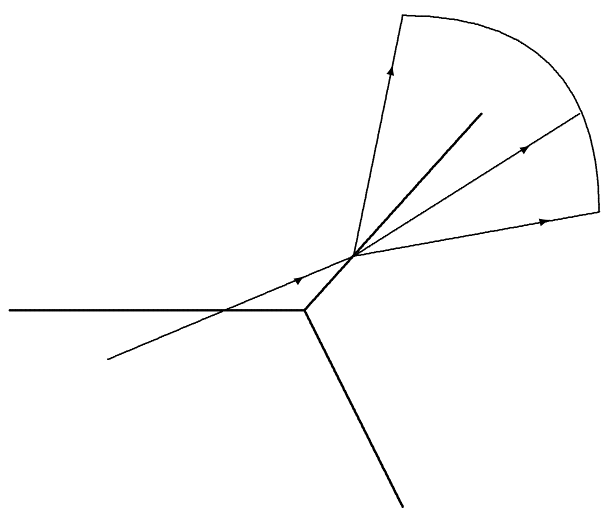
i.e. if $\langle D_t \rangle^{-k} u \in H^1_{\text{loc}}(X)$. Note that imposing Dirichlet boundary conditions on u amounts to requiring $\langle D_t \rangle^{-k} u \in H^1_{0,\text{loc}}(X)$, where $H^1_0(X)$ is the completion of $C^\infty_c(X^\circ)$ in the H^1 norm.

For admissible solutions of the wave equation on manifolds with corners, there is a wave front set also in ${}^bT^*X \setminus o$, due to Melrose, denoted by $\mathrm{WF}_b(u)$. Away from ∂X, this is simply $\mathrm{WF}(u)$, and at ∂X it measures if u is microlocally conormal to ∂X relative to $L^2(X)$, i.e. it microlocally tests whether $V_1 \dots V_m u \in L^2(X)$ for all vector fields $V_1, \dots, V_m$ tangent to ∂X.

The theorem of Melrose, Sjöstrand and Taylor in this formulation is that if $Pu = 0$, u is admissible, and u satisfies Dirichlet or Neumann boundary condition then $\mathrm{WF}_b(u) \subset \dot{\Sigma}$, and it is a union of maximally extended generalized broken bicharacteristics in $\dot{\Sigma}$. (The result also holds in the analytic category, although certain tangential curves that are not bicharacteristics in the C^∞ setting must be allowed there.)

If M, hence X, are manifolds with corners, and we impose Dirichlet or Neumann boundary conditions on the boundary hypersurfaces, then the analogous theorem is still true. In the analytic setting this is due to Lebeau, in the C^∞ setting to A.V.

The definition of $\dot{\Sigma}$ in this setting is that if F° is the interior of a boundary face F of X, then over F°, points in Σ differing by a covector in N^*F are identified. This again is the law of reflection: kinetic energy as well as momentum tangential to F are conserved, but the normal momentum may change.

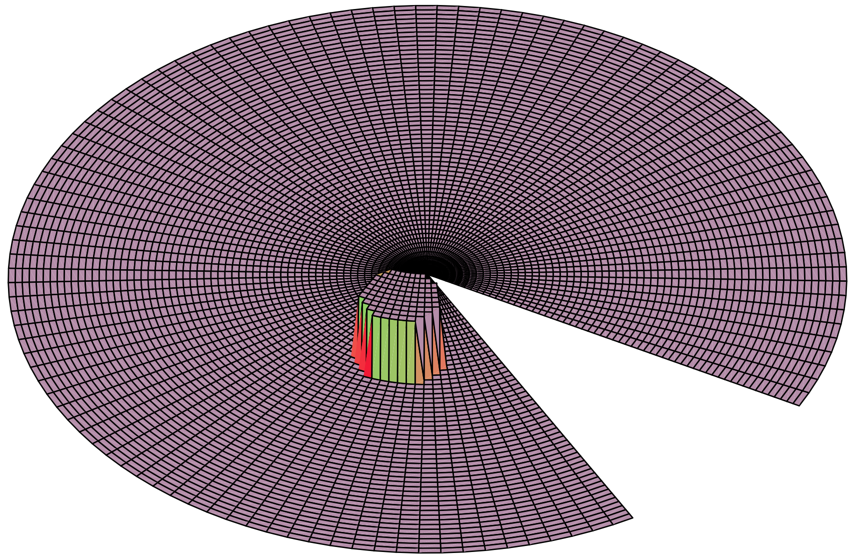


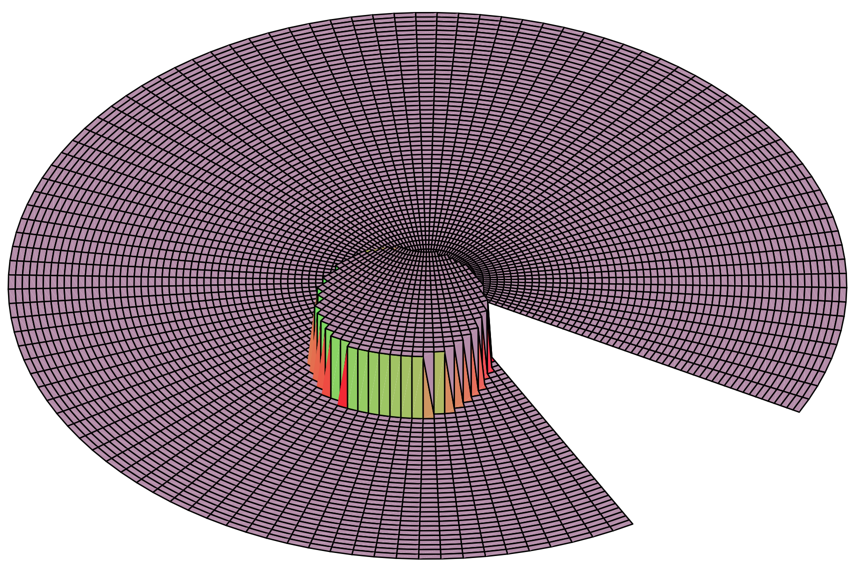
Bicharacteristic segments $\gamma_\pm$ as above again give rise to generalized broken bicharacteristics, but now even for normally incidence these are not the only ones as the incoming/outgoing ones can be tangent to another boundary face. Notice also that the continuation of an incoming ray is not unique: one gets a whole cone of reflected rays.

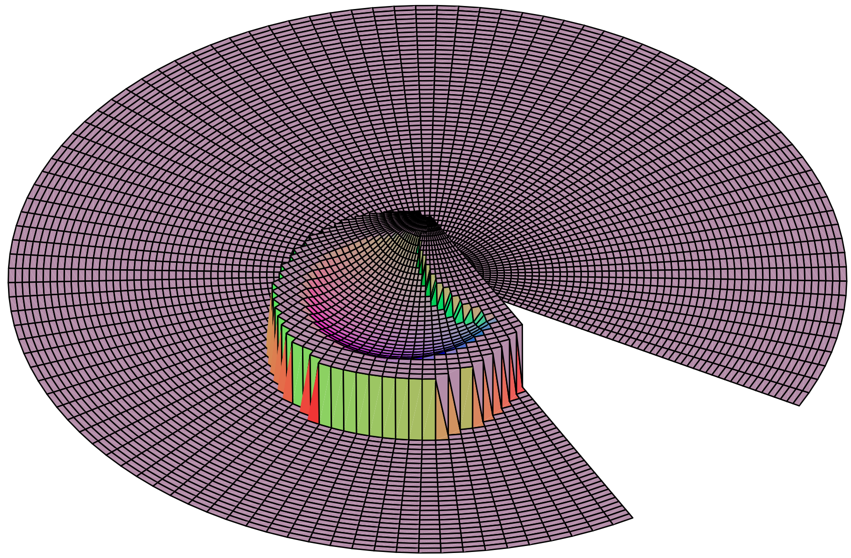
Theorem 1 *If u is an admissible solution of the wave equation, satisfying Dirichlet or Neumann boundary conditions at codimension one hypersurfaces, then $\mathsf{WF}_b(u) \subset \dot{\Sigma}$ is a union of maximally extended generalized broken bicharacteristics.*

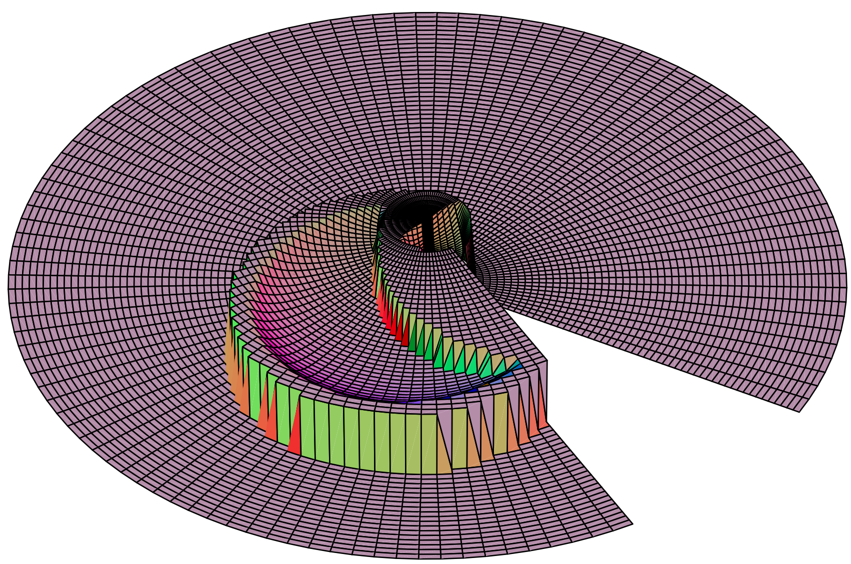
This result is optimal for normally incident bicharacteristics γ_- in the sense that in general a solution will have wave front set on *all* generalized broken bicharacteristics extending γ_-.

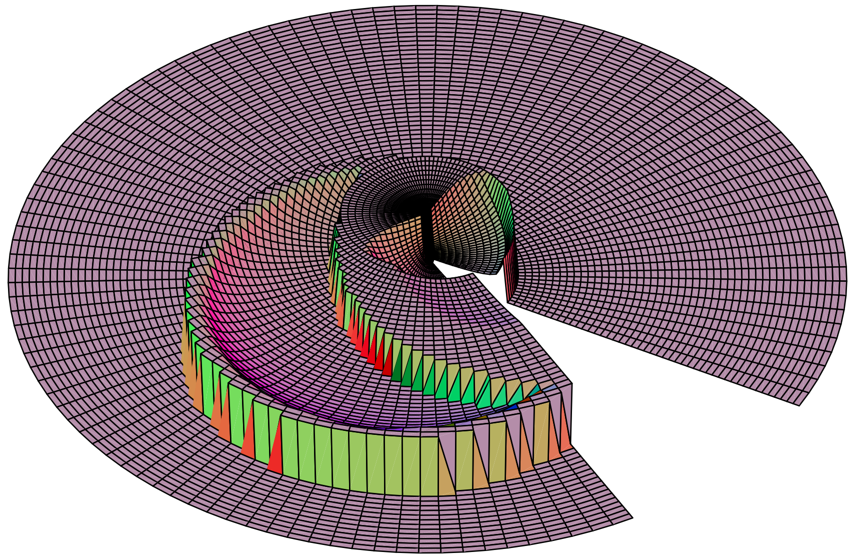
The result also holds for Sobolev wave front sets $\mathrm{WF}_b^m(u)$, which measure in X° whether u is microlocally in $H^m_{\mathrm{loc}}(X^\circ)$. The proof relies on positive commutator estimates (microlocal energy estimates) relative to $H^1(X)$ using Melrose's $\Psi_b(X)$ (totally characteristic operators) as microlocalizers and commutants.

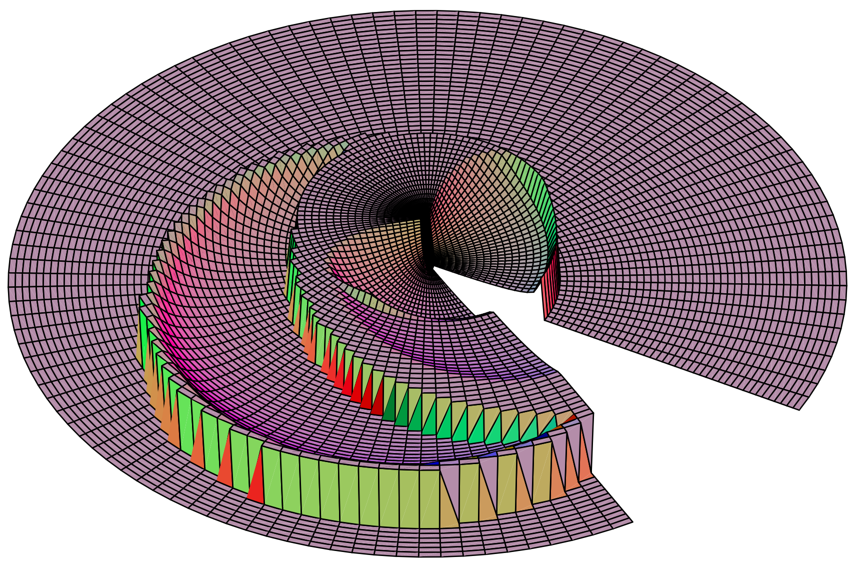


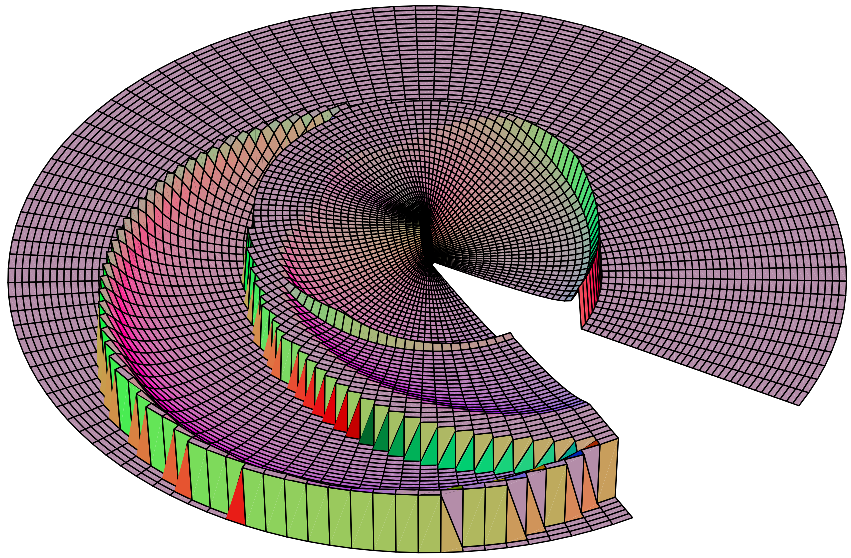


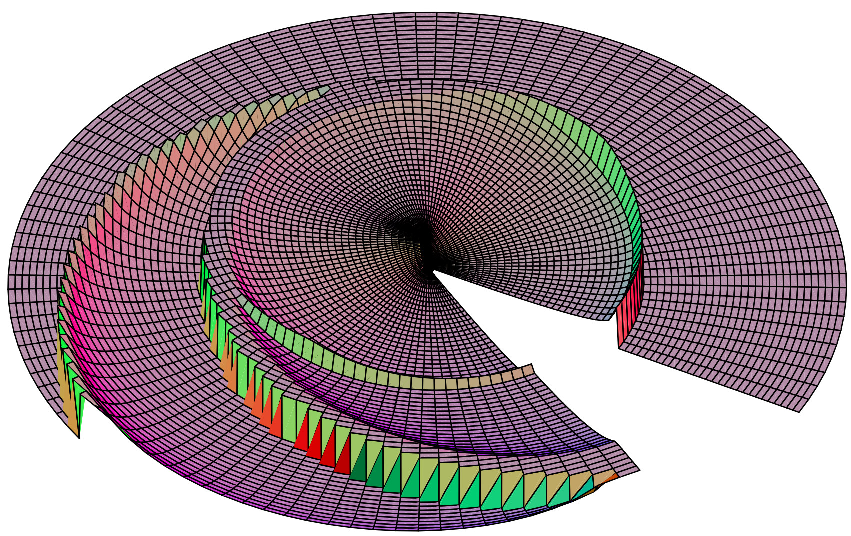












One can then ask whether the Sobolev result is optimal for normally incident bicharacteristics. Indeed it is, but there is a special and rather large class of solutions of the wave equation, namely those ‘not focusing’ on the corner, for which it can be improved.

To see how, remark that not all generalized broken bicharacteristics are ‘geometric’ in the sense that they are limits of bicharacteristics that just miss the corner under consideration. One expects that, unless one is dealing with a solution that focuses on the corner, on the ‘non-geometric’ broken bicharacteristics the reflected wave should be less singular than the incident one.

This is most easily illustrated by spherical waves emanating from a source near the boundary or corner. Then most of the spherical wave misses the corner (i.e. only a lower dimensional part hits it), unlike in the case of a spherical wave hitting a smooth boundary. Thus, one expects that part of the solution comprising the reflected rays from the corner, but away from the reflected rays from the boundary hypersurfaces, is less singular than the spherical wave.

With the current state of technology the geometric improvement at corners is too hard to obtain, although the machinery used in obtaining the 'model' result below can be adapted to get certain partial results on manifolds with corners (with corners of arbitrary codimension).

In the codimension 2 corner setting, in the analytic category, there is a corresponding result due to Gérard and Lebeau for conormal incident waves. There is also a long history of the subject in applied mathematics, especially in the work of Keller.

So we consider a model which is a manifold $(\tilde{M}, g)$ with boundary equipped with a singular Riemannian metric of 'edge-type':

- $\partial\tilde{M}$ has a fibration $\phi_0 : \partial\tilde{M} \to Y$, with *compact* fiber Z,

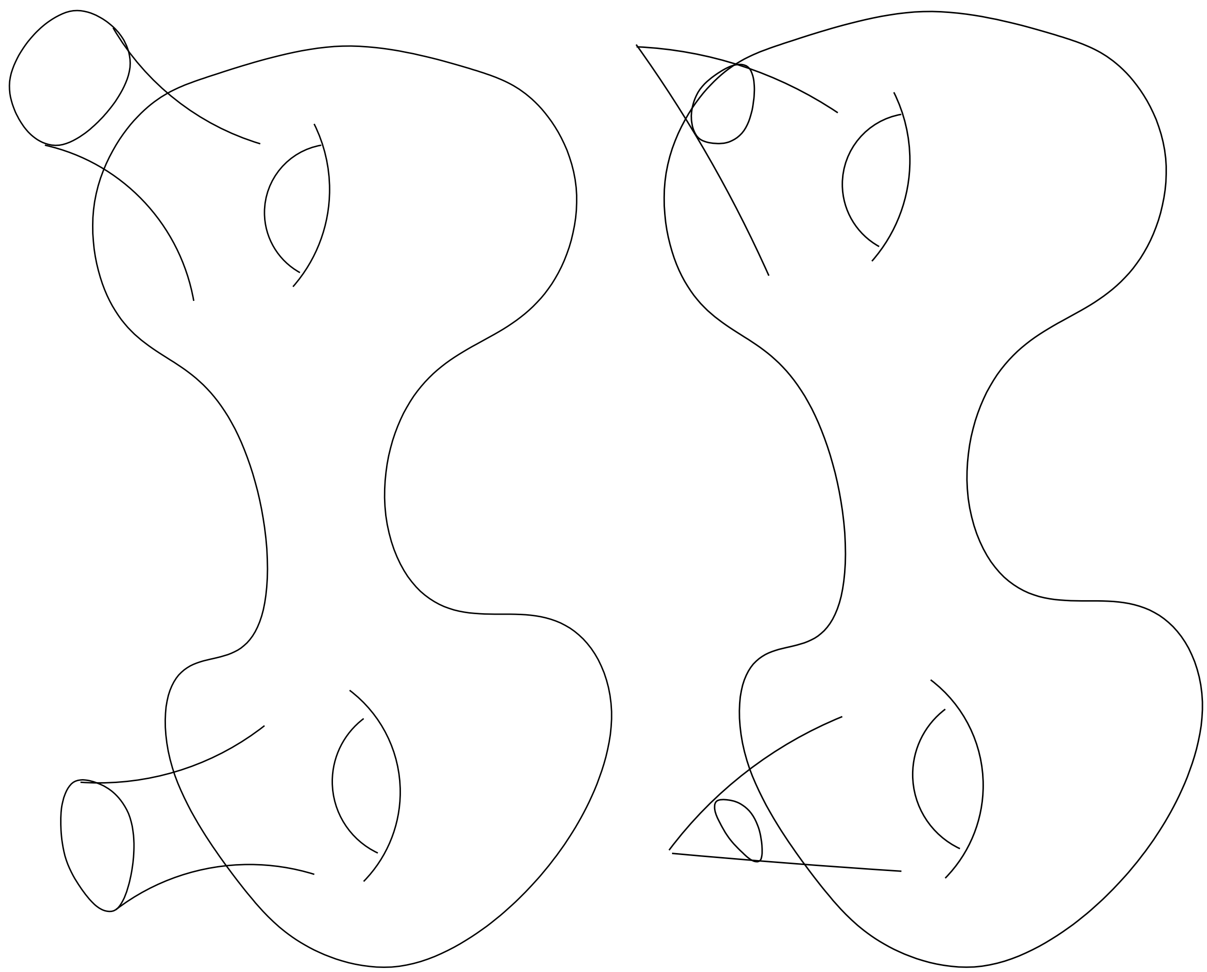
- $\tilde{M}$ has a boundary defining function x,

- near $\partial\tilde{M}$,

  $$g = dx^2 + \phi^* h + x^2 k$$

  with $h \in C^\infty([0,\epsilon) \times Y; \mathrm{Sym}^2 T^*([0,\epsilon) \times Y))$ and $k \in C^\infty(U; \mathrm{Sym}^2 T^*\tilde{M})$; we further assume that $h|_{x=0}$ is a nondegenerate metric on Y and $k|_{x=0}$ is a nondegenerate fiber metric.

Here we extended the fibration ϕ_0 to a fibration $\phi : U \to [0,\epsilon) \times Y$ on a neighborhood U of $\partial\tilde{M}$.



A typical example is if we blow-up a submanifold Y of $\mathbb{R}^n$ (or any Riemannian manifold) and lift the metric to the blown-up space $\tilde{M} = [\mathbb{R}^n; Y]$. In other words, we introduce 'geodesic polar coordinates' around Y, although in this case the propagation of singularities result is trivial, for the metric on $\mathbb{R}^n$ is *not* singular at Y. The fibers Z in this case are spheres, of dimension equal to codim$Y - 1$, while the base is Y. A more interesting example is obtained in this case if the metric is altered, provided it still has the same form.

The truly relevant setting for us arises by taking a domain with corners M in $\mathbb{R}^n$, or taking a manifold with corners M as before, and blow up all corners, in order of increasing dimension (or inclusions) to obtain the total boundary blow-up $\tilde{M}$ of M. In this case, however, $\tilde{M}$ itself is a manifold with corners, and the fibers Z are manifolds with boundary or corners.

As we study the wave equation, we work with $\tilde{X} = \tilde{M} \times \mathbb{R}_t$, which still has a fibration, with base $Y \times \mathbb{R}$ and fiber Z.

Much like before, there are two phase spaces and characteristic sets (analogues of Σ and $\dot{\Sigma}$):

- the identification giving $\dot{\Sigma}$ is not only in the momenta, but also in the base space $\tilde{X}$,

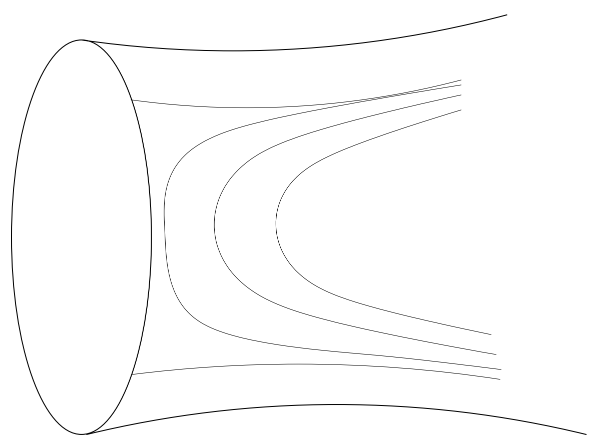
- the compressed (or collapsed) version of $\tilde{X}$ is $\dot{X} = X^\circ \cup (Y \times \mathbb{R}_t)$ (disjoint union), with the projection over $\partial\tilde{X}$ given by the fibration $\phi_0 \times \mathrm{id}_t$,

- if $\tilde{X}$ is the blow-up of a space X, then $\dot{X} = X$,

- over the interior $\tilde{X}^\circ$ of $\tilde{X}$, the characteristic sets are $p^{-1}(\{0\}) \cap T^*\tilde{X}^\circ$,

- at ∂X, $\dot{\Sigma}$ is obtained from Σ by identifying covectors with differing dx and $x\,dz$ components and dropping the fiber coordinate z.

Generalized broken bicharacteristics are defined as curves in $\dot{\Sigma}$, using a Hamilton vector field condition on Σ. Their projection to $\dot{X}$ is continuous – they usually do not lift to continuous curves on $\tilde{X}$.



The theorem on the propagation of singularities for solutions u of the wave equation holds as before. It is again proved by positive commutator estimates, using $\Psi_b(\tilde{X})$ as microlocalizers.

Suppose now that $\gamma_0 : (0, t_0) \to T^*\tilde{X}^\circ$ is a bicharacteristic segment, approaching the boundary normally as $t \to 0$. Then the projection of γ_0 to $\tilde{M}$ is a geodesic; this geodesic extends to a smooth curve c defined on $[0, t_0)$. In particular $c(0) \in \partial\tilde{X}$ is well-defined; we say that γ_0 is outgoing from $c(0)$.

Let Γ denote the set of all generalized broken bicharacteristics extending γ_0 (extending backwards is the interesting part here).

The theorem on the propagation of singularities states that if

$$\Gamma_{-\epsilon} = \bigcup \{\gamma((-\epsilon, 0)) : \ \gamma \in \Gamma\}$$

is disjoint from $\mathrm{WF}_b(u)$, then so is the image of γ_0; similarly for $\mathrm{WF}_b^m(u)$.

Among bicharacteristics hitting the edge normally, the geometric bicharacteristics are those which are limits of bicharacteristics in $T^*\tilde{X}^\circ$. It is straightforward to make this concrete: this means that the incident and outgoing points for the corresponding geodesic lie in the same fiber, distance π away from each other with respect to the fiber metric k.

If $\tilde{M}$ arises from a blow-up $[M;Y]$, the front face of $[M;Y]$ is isomorphic to the spherical normal bundle of Y, i.e. points in the same fiber correspond to approaching Y from different (normal) directions, so, if the metric is just a metric on M lifted by the blow-down map, distance π corresponds exactly to going 'straight' in M, without breaking at Y.

The non-focusing assumption can be stated via the 'backward flow-out' $\mathcal{F}_-$ of the edge microlocally near $\Gamma_{-\epsilon}$. Here $\mathcal{F}_-$ consists of bicharacteristics hitting the edge. Away from the edge, $\mathcal{F}_-$ is a smooth coisotropic submanifold of $T^*\tilde{X}^\circ$, and indeed it extends to a smooth submanifold of the edge cotangent bundle, ${}^eT^*\tilde{X}$, which we discuss later.

Let $\mathcal{M}$ be the set of first order ps.d.o's with symbol vanishing along $\mathcal{F}_-$, and let $\mathcal{M}^j$ be the set of finite sums of products of at most j factors, each of which is in $\mathcal{M}$.

The non-focusing condition of order r' for γ_0 is that, for some $\epsilon > 0$, microlocally near $\Gamma_{-\epsilon}$, and for some N,

$$u = \sum A_j v_j,\ A_j \in \mathcal{M}^N,\ v_j \in H^{r'}.$$

We still need the analogue of boundary conditions, which in this case are obtained by taking the self-adjoint realization of the Laplacian to be the Friedrichs extension of the Laplacian on $C_c^\infty(\tilde{M}^\circ)$:

- the quadratic form domain $\mathcal{D}$ is defined as the completion of $C_c^\infty(\tilde{M}^\circ)$ with respect to the norm $\|u\|^2_{L^2_g} + \|du\|^2_{L^2_g}$,

- the domain of Δ is $\mathcal{D}_2 = \{u \in \mathcal{D} : \ \Delta u \in L^2_g\}$.

In general, $\mathcal{D}_s$ will denote the domain of $\Delta^{s/2}$.

An admissible solution of the wave equation $Pu = 0$, $P = D_t^2 - \Delta$, is then one satisfying

$$u \in L^2(\mathbb{R}; \mathcal{D}_s),\ D_t u \in L^2(\mathbb{R}; \mathcal{D}_{s-1}),$$

for some $s \in \mathbb{R}$. For $s = 1$, this states that $u \in L^2(\mathbb{R}; \mathcal{D})$, $D_t u \in L^2(\mathbb{R}; L^2_g)$.

Theorem 2 *Suppose that* $(\tilde{M}, g)$ *is a manifold with an edge metric,* $\tilde{X} = \tilde{M} \times \mathbb{R}$*, and* u *is an admissible solution of* $Pu = 0$*,* $P = D_t^2 - \Delta$*. Let* γ_0 *be a bicharacteristic segment as above, and suppose that* u *satisfies the non-focusing assumption of order* r' *for* γ_0*.*

Then for $R < r'$*,* $\gamma_0 \cap \mathrm{WF}^R(u) = \emptyset$ *provided that, for some* $\epsilon > 0$*, all* geometric *generalized broken bicharacteristics* $\gamma \in \Gamma$ *extending* γ_0 *satisfy* $\gamma((-\epsilon, 0)) \cap \mathrm{WF}^R(u) = \emptyset$*.*

That is, singularities of order $R < r'$ can only propagate into γ_0 from geometric generalized broken bicharacteristics extending it.

The proof of this theorem relies on

- the propagation of singularities, in the sense of a compressed phase space, and

- the microlocal propagation of coisotropy.

Thus, along bicharacteristics which are not geometrically related to incoming bicharacteristics carrying singularities, one shows that u is coisotropic of an order given by the background regularity of u – this does *not* require the non-focusing hypothesis. (The 'background regularity' is the regularity along all related incoming bicharacteristics, not only the geometric ones.)

The combination of non-focusing (which is dual to coisotropy) and coisotropy gives the improved result of the theorem via interpolation.

This statement is quite natural: the non-focusing condition, in this form, states that while $u \in H^{r'-N}$ only, it is in a better space, $H^{r'}$, 'to finite order along Γ' (rather than in any neighborhood of Γ), as reflected by the presence of $\mathcal{M}^N$ in the condition. (This 'finite order' corresponds to saying that an operator in $\mathcal{M}$, while first order, is in fact zeroth order to 'first order along $\mathcal{F}_-$'.) Thus, modulo $H^{r'}$, one can expect singularities to follow limits of integral curves of H_p, i.e. geometrically related broken bicharacteristics.

In the conic setting, where Y is a point, the metric can be brought to the form $g = dx^2 + x^2 k$ *near* $x = 0$, $\Delta_Z \in \mathcal{M}^2$. Thus, the non-focusing assumption is equivalent to the non-focusing assumption used in previous work of Melrose and Wunsch for conic points: $u = (\Delta_Z + 1)^N v$, with v microlocally in $H^{r'}$.

Lagrangian distributions, such as the fundamental solution with initial condition a delta distribution near, but not at, the edge, often satisfy the non-focusing condition simply by virtue of the Lagrangian Λ intersecting the coisotropic manifold $\mathcal{F}_-$ transversally inside the characteristic set.

Inside Λ, the codimension of this intersection is the dimension f of the fibers (i.e. in the corner setting this would be the codimension of the corner before the blow up, minus 1), which implies that u satisfies the non-focusing condition with an improvement of $f/2$.

Roughly speaking, a Lagrangian distribution u associated to Λ is smooth along Λ, so one can divide u by some first order factors vanishing at $\mathcal{F}_- \cap \Lambda$ (symbols of ps.d.o.'s) and still improve Sobolev regularity.

One can associate to a boundary fibration $\phi_0 \times \mathrm{id}_t$ an edge tangent bundle, ${}^e T\tilde{X}$, whose smooth sections are the vector fields on $\tilde{X}$ which are tangent to $\phi_0 \times \mathrm{id}_t$.

In local coordinates (x, y, z) as above, so z_j are the fiber variables, y_j are coordinates on $Y \times \mathbb{R}_t$, these vector fields have the form

$$ax\partial_x + \sum b_j x\partial_{y_j} + \sum c_j \partial_{z_j},$$

with $a, b_j, c_j \in C^\infty(\tilde{X})$ arbitrary. Correspondingly, ${}^e T\tilde{X}$ is locally spanned by $x\partial_x, x\partial_{y_j}, \partial_{z_j}$. The dual bundle is the edge cotangent bundle, ${}^e T^*\tilde{X}$; it is spanned by $\frac{dx}{x}, \frac{dy_j}{x}, dz_j$, with corresponding dual coordinates ξ, η_j, ζ_j.

The analogue of the phase space T^*X from beforehand (M a manifold with corners with a non-degenerate Riemannian metric) is $x\,{}^e T^*\tilde{X}$: this consists of covectors of finite length.

- The principal symbol $p = \sigma_2(P)$ has the property that $x^2 p \in C^\infty({}^eT^*\tilde{X} \setminus o)$,

- the characteristic set $\Sigma = p^{-1}(\{0\})$ is a C^∞ submanifold of ${}^eT^*\tilde{X} \setminus o$,

- the Hamilton vector field H_p is such that $W = x^2 H_p$ is a C^∞ vector field on ${}^eT^*\tilde{X} \setminus o$, tangent to its boundary and to Σ,

- in Σ, W is radial only at $\partial\tilde{X}$, and there precisely at the set R of points $(0, y, z, \xi, \eta, \zeta)$ with $\zeta = 0$,

- working on ${}^eS^*\tilde{X} = ({}^eT^*\tilde{X} \setminus o)/\mathbb{R}^+$, the non-tangential flow-out $\mathcal{F}$ of the edge is the stable/unstable submanifold of $R' = R/\mathbb{R}^+$ outside $x = 0$ (depending on the sign of ξ; $\xi \neq 0$),

- maximally extended integral curves of W over $\partial\tilde{X}$ conserve the 'slow variables' (y, η), the projections to $\partial\tilde{X}$ are (reparameterized) geodesics in Z of length π; they tend to the radial set R' as $s \to \pm\infty$.

The result on the propagation of coisotropy is thus a result on *propagation through radial points*, and is thus related to earlier work of Hassell-Melrose-V. It roughly states that if a solution of the wave equation is coisotropic along bicharacteristics flowing towards a radial point, then it has no edge wave front set on bicharacteristics flowing out of this radial point inside the boundary, and conversely. (This statement needs to be made a little more precise.)

The propagation of singularities theorem on the compressed phase space only keeps track of the 'slow variables' at the edge.

One can also describe $\mathcal{D}_s$ rather explicitly. For $0 \leq s < \frac{f+1}{2}$, where $f = \dim Z$,

$$\mathcal{D}_s = x^s H_e^s(\tilde{M}),$$

where $H_e^s(\tilde{M})$ is the Sobolev space associated to $\mathcal{V}_e(\tilde{M})$, consisting of smooth sections of ${}^e T\tilde{M}$, relative to $L_g^2(\tilde{M})$. For general s, a similar description is possible, and one can also describe the admissibility criterion for solutions of the wave equation similarly.