

SEMILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY DE SITTER, KERR-DE SITTER AND MINKOWSKI SPACETIMES

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ABSTRACT. In this paper we show the small data solvability of suitable semilinear wave and Klein-Gordon equations on geometric classes of spaces, which include so-called asymptotically de Sitter and Kerr-de Sitter spaces, as well as asymptotically Minkowski spaces. These spaces allow general infinities, called conformal infinity in the asymptotically de Sitter setting; the Minkowski type setting is that of non-trapping Lorentzian scattering metrics introduced by Baskin, Vasy and Wunsch. Our results are obtained by showing the *global* Fredholm property, and indeed invertibility, of the underlying linear operator on suitable L^2 -based function spaces, which also possess appropriate algebra or more complicated multiplicative properties. The linear framework is based on the b-analysis, in the sense of Melrose, introduced in this context by Vasy to describe the asymptotic behavior of solutions of linear equations. An interesting feature of the analysis is that *resonances*, namely poles of the inverse of the Mellin transformed b-normal operator, which are ‘quantum’ (not purely symbolic) objects, play an important role.

1. INTRODUCTION

In this paper we consider semilinear wave equations in contexts such as asymptotically de Sitter and Kerr-de Sitter spaces, as well as asymptotically Minkowski spaces in which recent progress [41] and [2] allows one to set up the analysis of the associated linear problem *globally* as a Fredholm problem. This allows one to use the contraction mapping theorem to solve semilinear equations with small data in many cases since typically the semilinear terms can be considered perturbations of the linear problem. That is, as opposed to solving an evolution equation on time intervals of some length, possibly controlling this length in some manner, and iterating the solution using (almost) conservation laws, we solve the equation globally in one step. There is a catch, however, namely just as in the context of linear problems, one needs to consider a symbol or normal operator at infinity, conveniently encoded by a compactification of the space, with respect to which lower order terms in the differential order sense can play an equal role to the principal symbol terms. One way this can show up is that a zero resonance for the Mellin transformed normal operator significantly affects the analysis; resonances in the ‘physical half plane’, $\text{Im } \sigma > 0$, would have an even more pronounced effect, while resonances in the ‘unphysical half plane’, $\text{Im } \sigma < 0$, affect the expansion of the solutions of the non-linear equation, just as they affect that for linear equations, see

Date: June 19, 2013.

The authors were supported in part by A.V.’s National Science Foundation grants DMS-0801226 and DMS-1068742 and P. H. was supported in part by a Gerhard Casper Stanford Graduate Fellowship and the German National Academic Foundation.

[41]. Another way is that trapping in Kerr-de Sitter spaces elevates first order nonlinearities to effectively order $1 + \epsilon$, $\epsilon > 0$, (or at least logarithmically worse), which is too big of a loss for these to be considered perturbations directly. Yet another way is that when fitting asymptotically Minkowski spaces, in the weak sense of Baskin, Vasy and Wunsch (i.e. really non-trapping scattering Lorentzian metrics), into the framework, weight factors are introduced in the non-linearities, and these strongly influence which problems can be considered perturbations. In addition to providing solvability of semilinear problems, our results can also be used to obtain the asymptotic expansion of the solution. We emphasize that while the way these spaces fit into it differs somewhat, the underlying linear framework is that of L^2 -based b-analysis, on manifolds with boundary, as introduced for these equations in this form in [41] (previously, Wang [45] in her PhD thesis used the b-structure on perturbations of Minkowski space to analyze the precise form of solutions of Einstein's equation; this was however a non-microlocal approach), except that in the global view of asymptotically de Sitter spaces one can eliminate the boundary altogether.¹

We thus consider semilinear wave equations of the form

$$(\square_g - \lambda)u = f + q(u, du)$$

on a manifold M where q is (typically; more general functions are also considered) a polynomial vanishing at least quadratically at $(0, 0)$ (so contains no constant or linear terms, which should be included either in f or in the operator on the left hand side). As weights make a difference, we make du have coefficients which are well-behaved relative to the metric structure, as specified below. Here g and λ and the function spaces used fit in one of the following scenarios, which we state slightly informally, discussing the terminology afterwards, but the reader unfamiliar with the terms could drop the word 'asymptotically' and 'even' to obtain specific examples:

- (1) A neighborhood of the backward light cone from future infinity in an asymptotically de Sitter space. (This may be called a static region/patch of an asymptotically de Sitter space, even when there is no time like Killing vector field.) Due to a zero resonance for the linear problem when $\lambda = 0$, which moves to the lower half plane for $\lambda > 0$, in this setting $\lambda > 0$ works in general; $\lambda = 0$ works if q depends on du but not on u . The relevant function spaces are L^2 -based b-Sobolev spaces on the bordification (partial compactification) of the space, or analogous spaces plus a finite expansion. Further, the semilinear terms involving du have coefficients corresponding to the b-structure, i.e. b-objects are used to create functions from the differential forms, or equivalently b-derivatives of u are used.
- (2) Kerr-de Sitter space, including a neighborhood of the event horizon. This is an extension of (1), i.e. the framework is essentially the same, with the difference being that there is now trapping corresponding to the 'photon sphere'. This makes first order terms in the non-linearity non-perturbative,

¹The b-analysis itself originates in Melrose's work on the propagation of singularities for the wave equation on manifolds with smooth boundary, and Melrose described a systematic framework for elliptic b-equations in [32]. Here 'b' refers to analysis based on vector fields tangent to the boundary of the space; we give some details later in the introduction and further details in Section 2.1, where we recall the setting of [41].

unless they are well-adapted to the trapping. Thus, we assume $\lambda > 0$. The relevant function spaces are as in the asymptotically de Sitter setting.

- (3) Global *even* asymptotically de Sitter spaces. These are in some sense the easiest examples as they correspond, via extension across the conformal boundary, to working on a manifold without boundary. However, in order to get reasonable results, one needs to measure regularity relatively finely, using the module of vector fields tangent to what used to be the conformal boundary in the extension. The relevant function spaces are thus Sobolev spaces with additional (finite) conormal regularity. Further, du has coefficients corresponding to the 0-structure of Mazzeo and Melrose, in the same sense the b-structure was used in (1). The range of λ here is limited by the process of extension across the boundary; for non-linearities involving u only, the restriction amounts to (at least very slowly) decaying solutions for the linear problem (without extension across the conformal boundary).

Another possibility is to view global de Sitter space as a union of static patches. Here, the b-Sobolev spaces on the static parts translate into 0-Sobolev spaces on the global space, which have weights that are shifted by a dimension-dependent amount relative to the weights of the b-spaces. This approach allows many of the non-linearities that we can deal with on static parts; however, the resulting decay estimates on u are quite lossy relative to the decay of the forcing term f .

- (4) Non-trapping Lorentzian scattering spaces (generalized asymptotically Minkowski spaces), $\lambda = 0$. Note that if $\lambda > 0$, the type of the equation changes drastically; it naturally fits into Melrose's scattering algebra² rather than the b-algebra which can be used for $\lambda = 0$. While the results here are quite robust and there are no issues with trapping, they are more involved as one needs to keep track of regularity relative to the module of vector fields tangent to the light cone at infinity. The relevant function spaces are b-Sobolev spaces with additional b-conormal regularity corresponding to the aforementioned module. Further, du has coefficients corresponding to Melrose's scattering structure. These spaces, in the special case of Minkowski space, are related to the spaces used by Klainerman [24], using the infinitesimal generators of the Lorentz group, but while Klainerman works in an $L^\infty L^2$ setting, we remain purely in a (weighted) L^2 based setting, as the latter is more amenable to the tools of microlocal analysis.

The results in these cases are, roughly stated, with references to the precise theorems:

- (1) 'Static' asymptotically de Sitter. If $\lambda > 0$, one can allow q an arbitrary polynomial with quadratic vanishing at the origin, or indeed a more general function. If $\lambda = 0$ and q depends on du only, the same conclusion holds. Further, in either case, one obtains an expansion of the solution at infinity. See Theorems 2.24 and 2.35, and Corollary 2.27.
- (2) Kerr-de Sitter. In the main part of the section we assume $\lambda > 0$, and allow $q = q(u)$ with quadratic vanishing at the origin. We also obtain an expansion at infinity. See Theorems 3.5 and 3.9, and Corollary 3.8. However, in

²In many ways the scattering algebra is actually much better behaved than the b-algebra, in particular it is symbolic in the sense of weights/decay. Thus, with numerical modifications, our methods should extend directly.

Section 3.3 we briefly discuss non-linearities involving derivatives which are appropriately behaved at the trapped set.

- (3) ‘Global’ asymptotically de Sitter. Here $\lambda = (n-1)^2/4 + \sigma^2$. If³ $\text{Im } \sigma < 0$ is sufficiently small and the dimension satisfies $n \geq 6$, quadratic vanishing of q suffices; if $n \geq 4$ then cubic vanishing is sufficient. If q does not involve derivatives, $\text{Im } \sigma \geq 0$ small also works, and if $\text{Im } \sigma > 0, n \geq 5$, or $\text{Im } \sigma = 0, n \geq 6$, then quadratic vanishing of q is sufficient. See Theorems 4.10, 4.12 and 4.15. Using the results from ‘static’ asymptotically de Sitter spaces, quadratic vanishing of q in fact suffices for all $\lambda > 0$, and indeed $\lambda \geq 0$ if $q = q(du)$, but the decay estimates for solutions are lossy relative to the decay of the forcing. See Theorem 4.17.
- (4) Non-trapping Lorentzian scattering (generalized asymptotically Minkowski). If $q = q(du)$, we allow q with quadratic vanishing at 0 if $n \geq 5$; cubic if $n \geq 4$. If $q = q(u)$, we allow q with quadratic vanishing if $n \geq 6$; cubic if $n \geq 4$. Further, for $q = q(du)$ quadratic satisfying a null-condition, $n = 4$ also works. See Theorems 5.10, 5.12 and 5.18.

In order to underline the generality of the method, we emphasize that, corresponding to cases (1) and (2), in b-settings in which one can work on standard b-Sobolev spaces, which means that one propagates singularities away from radial sets, the restrictions on the solvability of the semilinear equations are simply given by the presence of resonances for the Mellin-transformed normal operator in $\text{Im } \sigma \geq 0$, which would allow growing solutions⁴ to the equation, making the non-linearity non-perturbative, and the losses at high energy estimates for this Mellin-transformed operator and the closely related b-principal symbol estimates when one has trapping.⁵ In particular, the results are necessarily optimal in the non-trapping setting of (1), as shown even by an ODE, see Remark 2.29. In the trapping setting it is not clear precisely what improvements are possible for non-linearities with derivatives, though when there are no derivatives in the non-linearity, we already have no restrictions on the non-linearity and to this extent the result is optimal.

On Lorentzian scattering spaces more general function spaces are used⁶, and it is not in principle clear whether the results are optimal, but at least comparison with the work of Klainerman and Christodoulou for perturbations of Minkowski space [7,

³The equation is unchanged if one replaces σ by $-\sigma$. The process of extending across the boundary, however, breaks this symmetry, and in Section 4 we mostly consider $\text{Im } \sigma \leq 0$.

⁴With an exception if $\text{Im } \sigma = 0$, in which case the non-linear iterative arguments produce growth unless the non-linearity has a special structure.

⁵These losses cause the difference in the trapping setting for non-linearities with or without derivatives.

⁶There are two basic properties of spaces of functions on manifolds with boundaries: differentiability and decay. Whether one can have both at the same time for the linear analysis depends on the (Hamiltonian) dynamical nature of radial points: when defining functions of the corresponding boundaries of the compactified cotangent bundle have opposite character (stable vs. unstable) one can have both at the same time, otherwise not; see Propositions 2.1 and 5.1 for details. For non-linear purposes, the most convenient setting, in which we are in (1), is if one can work with spaces of arbitrarily high regularity and fast decay, and corresponds to saddle points of the flow in the above sense. In (4) however, working in higher regularity spaces, which is necessary in order to be able to make sense of the non-linearity, requires using faster growing (or at least less decaying) weights, which is problematic when dealing with non-linearities (e.g., polynomials) since multiplication gives even worse growth properties then. Thus, to make the non-linear analysis work, the function spaces we use need to have more structure; it is a module regularity that is used to capture some weaker regularity in order to enable work in spaces with acceptable weights.

[24, 23] gives consistent results; see the comments below. On global asymptotically de Sitter spaces, the framework of [41] and [39] is very convenient for the linear analysis, but it is not clear to what extent it gives optimal results in the non-linear setting, though see the comments below.

While all results are stated for the scalar equation, analogous results hold in many cases for operators on natural vector bundles, such as the d'Alembertian (or Klein-Gordon operator) on differential forms, since the linear arguments work in general for operators with scalar principal symbol whose subprincipal symbol satisfies appropriate estimates at radial sets, see [41, Remark 2.1], though of course for semilinear applications the presence of resonances in the closed upper half plane has to be checked. This already suffices to obtain the well-posedness of the semilinear equations on asymptotically de Sitter spaces that we consider in this paper.⁷ On asymptotically Minkowski spaces, the absence of poles of an asymptotically hyperbolic resolvent has to be checked in addition,⁸ see Theorem 5.2, and the numerology depends crucially on the delicate balance of weights and regularity, see Footnote 6.

The degree to which these non-linear problems have been studied differ, with the Minkowski problem (on perturbations of Minkowski space, as opposed to our more general setting) being the most studied. There semilinear and indeed even quasilinear equations are well understood due to the work of Christodoulou [7] and Klainerman [24, 23], with their book on the global stability of Einstein's equation [8] being one of the main achievements. (We also refer to the work of Lindblad and Rodnianski [26, 27] simplifying some of the arguments, of Bieri [4, 5] relaxing some of the decay conditions, of Wang [45] obtaining asymptotic expansions, and of Lindblad [25] for results on a class of quasilinear equations. Hörmander's book [22] provides further references in the general area. There are numerous works on the *linear* problem, and estimates this yields for the non-linear problems, such as Strichartz estimates; here we refer to the recent work of Metcalfe and Tataru [33] for a parametrix construction in low regularity, and references therein.) Here we obtain results comparable to these (when restricted to the semilinear setting), on a larger class of manifolds, see Remark 5.15. For non-linearities which do not involve derivatives, slightly stronger results have been obtained, in a slightly different setting, in [9]; see Remark 5.16. On the other hand, there is little work on the asymptotically de Sitter and Kerr-de Sitter settings; indeed the only paper the authors are aware of is that of Baskin [1] in roughly comparable generality in terms of the setting,⁹ though in *exact* de Sitter space Yagdjian [49, 48] has studied a large class of semilinear equations with no derivatives. Baskin's result is for a semilinear equation with no derivatives and a single exponent, using his parametrix

⁷Here one needs to know the poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space only.

⁸On *perturbations* of Minkowski space, this follows from the appropriate behavior of poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space.

⁹There is more work on the linear problem in de Sitter, de Sitter-Schwarzschild and Kerr-de Sitter spaces. We refer to [41] for more detail; some references are Polarski [35], Yagdjian and Galstian [50], Sá Barreto and Zworski [36], Bony and Häfner [6], Vasy [43], Baskin [3], Dafermos and Rodnianski in [10], Melrose, Sá Barreto and Vasy [30], Dyatlov [16, 15]. Also, while it received more attention, the linear problem on Kerr space does not fit directly into our setting; see the introduction of [41] for an explanation and for further references, [11] for more background and additional references.

construction [3], namely u^p with¹⁰ $p = 1 + \frac{4}{n-2}$, and for $\lambda > (n-1)^2/4$. In the same setting, $p > 1 + \frac{4}{n-1}$ works for us, and thus Baskin's setting is in particular included. Yagdjian works with the explicit solution operator (derived using special functions) in exact de Sitter space, again with no derivatives in the non-linearity. While there are some exponents that his results cover (for $\lambda > (n-1)^2/4$, all $p > 1$ work for him) that ours do not directly (but indirectly, via the static model, we in fact obtain such results), the range $(\frac{(n-1)^2}{4} - \frac{1}{4}, \frac{(n-1)^2}{4})$ is excluded by him while covered by our work for sufficiently large p . In the (asymptotically) Kerr-de Sitter setting, to our knowledge, there has been no similar semilinear work.

We next recall from [43] that an asymptotically de Sitter space is an appropriate generalization of the Riemannian conformally compact spaces of Mazzeo and Melrose [28], namely a smooth manifold with boundary, \tilde{M} , with interior \tilde{M}° equipped with a Lorentzian metric \tilde{g} , which we take to be of signature $(1, n-1)$ for the sake of definiteness, and with a boundary defining function ρ , such that $\hat{g} = \rho^2 \tilde{g}$ is a smooth symmetric 2-cotensor of signature $(1, n-1)$ up to the boundary of \tilde{M} and $\hat{g}(d\rho, d\rho) = 1$ (thus, the boundary defining function is timelike, and thus the boundary is spacelike; the $= 1$ statement makes the curvature asymptotically constant), and in addition $\partial\tilde{M}$ has two components (each of which may be a union of connected components) \tilde{X}_\pm , with all null-geodesics $c = c(s)$ of \tilde{g} tending to \tilde{X}_+ as $s \rightarrow +\infty$ and to \tilde{X}_- as $s \rightarrow -\infty$, or vice versa.¹¹ Analogously to asymptotically hyperbolic spaces, where this was shown by Graham and Lee [18], on such a space one can always introduce a product decomposition $(\partial\tilde{M})_z \times [0, \delta)_\rho$ near $\partial\tilde{M}$ (possibly changing ρ) such that the metric has a warped product structure $\hat{g} = d\rho^2 - h(\rho, z, dz)$, $\tilde{g} = \rho^{-2} \hat{g}$; the metric is called even if h can be taken even in ρ , i.e. a smooth function of ρ^2 . We refer to Guillarmou [19] for the introduction of even metrics in the asymptotically hyperbolic context, and to [43], [41] and [40] for further discussion.

Blowing up a point p at \tilde{X}_+ , which essentially means introducing spherical coordinates around it, we obtain a manifold with corners $[\tilde{M}; p]$, with a blow-down map $\beta : [\tilde{M}; p] \rightarrow \tilde{M}$, which is a diffeomorphism away from the *front face*, which gets mapped to p by β . Just like blowing up the origin in Minkowski space desingularizes the future (or past) light cone, this blow-up desingularizes the backward light cone from p on \tilde{M} , which lifts to a smooth submanifold transversal to the front face on $[\tilde{M}; p]$ which intersects the front face in a sphere Y . The interior of this lifted backward light cone, at least near the front face, is a generalization of the static patch in de Sitter space, and we refer to a neighborhood M_δ , $\delta > 0$, of the closure of the interior M_\pm of the lifted backward lightcone in $[\tilde{M}; p]$ which only intersects the boundary of $[\tilde{M}; p]$ in the interior of the front face (so M_δ is a non-compact manifold with boundary, with boundary X_δ , and with say boundary defining function τ) as the 'static' asymptotically de Sitter problem. See Figure 1. Via a doubling process, X_δ can be replaced by a compact manifold without boundary, X , and M_δ by $M = X \times [0, \tau_0)_\tau$, an approach taken in [41] where complex absorption was used,

¹⁰The dimension of the spacetime in Baskin's paper is $n+1$; we continue using our notation above.

¹¹Notice that in the interior of \tilde{M} , the conformal factor ρ^{-2} simply reparameterizes the null-geodesics, so equivalently one can require that null-geodesics of \hat{g} reach \tilde{X}_\pm at finite parameter values.

or indeed one can instead work in a compact region $\Omega \subset M_\delta$ by adding artificial, spacelike, boundaries, as we do here in Section 2.1. With such an Ω , the distinction between M and M_δ is irrelevant, and we simply write M below.

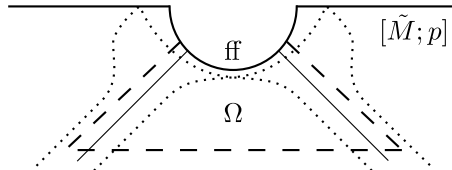


FIGURE 1. Setup of the ‘static’ asymptotically de Sitter problem. Indicated are the blow-up of \tilde{M} at p and the front face, the lift of the backward light cone to $[\tilde{M}; p]$ (solid), and lifts of backward light cones from points nearby p (dotted); moreover, $\Omega \subset M$ (dashed boundary) is a submanifold with corners within M (which is not drawn here; see [41] for a description of M using a doubling procedure in a similar context). The role of Ω is explained in Section 2.1.

See [43, 41] for relating the ‘global’ and the ‘static’ problems. We note that the lift of \tilde{g} to M in the static region is a Lorentzian b-metric, i.e. is a smooth symmetric section of signature $(1, n-1)$ of the second tensor power of the b-cotangent bundle, ${}^bT^*M$. The latter is the dual of bTM , whose smooth sections are smooth vector fields on M tangent to ∂M ; sections of ${}^bT^*M$ are smooth combinations of $\frac{d\tau}{\tau}$ and smooth one forms on X , relative to a product decomposition $X \times [0, \delta)_\tau$ near $X = \partial M$.

As mentioned earlier, the methods of [41] work in a rather general b-setting, including generalizations of ‘static’ asymptotically de Sitter spaces. Kerr-de Sitter space, described from this perspective in [41, Section 6], can be thought of as such a generalization. In particular, it still carries a Lorentzian b-metric, but with a somewhat more complicated structure, of which the only important part for us is that has trapped rays. More concretely, it is best to consider the bicharacteristic flow in the b-cosphere bundle (projections of null-bicharacteristics being just the null-geodesics), ${}^bS^*M$, quotienting out by the \mathbb{R}^+ -action on the fibers of ${}^bT^*M \setminus o$. On the ‘static’ asymptotically de Sitter space each half of the spherical b-conormal bundle¹² ${}^bSN^*Y$ consists of (a family of) saddle points of the null-bicharacteristic flow (these are called *radial sets*, the stable/unstable directions are normal to ${}^bSN^*Y$ itself), with one of the stable and unstable manifolds being the conormal bundle of the lifted light cone (which plays the role of the *event horizon* in black hole settings), and the other being the characteristic set within the boundary

¹²The b-conormal bundle ${}^bN^*Y$ of a boundary submanifold Y of a manifold with boundary M is the subbundle of ${}^bT^*_Y M$ whose fiber over $p \in Y \subset \partial M = X$ is the annihilator of vector fields on M tangent to Y and X . In local coordinates (τ, y, z) , where Y is defined by $z = 0$ in X , these vector fields are smooth linear combinations of $\tau\partial_\tau, \partial_{y_j}, z_j\partial_{z_k}, \tau\partial_{z_k}$, whose span in ${}^bT_p M$ is that of $\tau\partial_\tau$ and ∂_{y_j} , and thus the fiber of the b-conormal bundle is spanned by the dz_j , i.e. has the same dimension as the codimension of Y in X (and *not* that in M , corresponding to $\frac{d\tau}{\tau}$ not annihilating $\tau\partial_\tau$). The spherical b-conormal bundle is the quotient of ${}^bN^*Y \setminus o$ by the \mathbb{R}^+ -action; it is a submanifold of ${}^bS^*M$.

X (so within the boundary, the radial sets ${}^bSN^*Y$, are actually sources or sinks). Then on asymptotically de Sitter spaces all null-bicharacteristics over $\overline{M_+} \setminus X$ either leave Ω in finite time or (if they lie on the conormal bundle of the event horizon) tend to ${}^bSN^*Y$ as the parameter goes to $\pm\infty$, with each bicharacteristic tending to ${}^bSN^*Y$ in at most one direction. The main difference for Kerr-de Sitter space is that there are null-bicharacteristics which do not leave $\overline{M_+} \setminus X$ and do not tend to ${}^bSN^*Y$. On de Sitter-Schwarzschild space (non-rotating black holes) these future trapped rays project to a sphere, called the photon sphere, times $[0, \delta)_\tau$; on general Kerr-de Sitter space the trapped set deforms, but is still *normally hyperbolic*, a setting studied by Wunsch and Zworski in [47] and by Dyatlov in [17].

We refer to [2, Section 3] and to Section 5.1 here for a definition of asymptotically Minkowski spaces, but roughly they are manifolds with boundary M with Lorentzian metrics g on the interior M° conformal to a b-metric \hat{g} as $g = \tau^{-2}\hat{g}$, with τ a boundary defining function¹³ (so these are Lorentzian *scattering* metrics in the sense of Melrose [29], i.e. symmetric cotensors in the second power of the scattering cotangent bundle, and of signature $(1, n-1)$), with a real C^∞ function v defined on M with $dv, d\tau$ linearly independent at $S = \{v=0, \tau=0\}$, and with a specific behavior of the metric at S which reflects that of Minkowski space on its radial compactification near the boundary of the light cone at infinity (so S is the light cone at infinity in this greater generality). Concretely, the specific form is¹⁴

$$\tau^2 g = \hat{g} = v \frac{d\tau^2}{\tau^2} - \left(\frac{d\tau}{\tau} \otimes \alpha + \alpha \otimes \frac{d\tau}{\tau} \right) - \tilde{h},$$

where α is a smooth one form on M , equal to $\frac{1}{2} dv$ at S , \tilde{h} is a smooth 2-cotensor on M , which is positive definite on the annihilator of $d\tau$ and dv (which is a codimension 2 space). The difference between the de Sitter-type and Minkowski settings is in part this conformal factor, τ^{-2} , but more importantly, as this conformal factor again does not affect the behavior of the null-bicharacteristics so one can consider those of \hat{g} on ${}^bS^*M$, at the spherical conormal bundle ${}^bSN^*S$ of S the nature of the radial points is source/sink rather than a saddle point of the flow. (One also makes a non-trapping assumption in the asymptotically Minkowski setting.)

While the basic ingredients of the necessary linear b-analysis were analyzed in [41], the solvability framework was only discussed in the dilation invariant setting, and in general the asymptotic expansion results were slightly lossy in terms of derivatives in the non-dilation invariant case. We remedy these issues in this paper, providing a full Fredholm framework. The key technical tools are the propagation of b-singularities at b-radial points which are saddle points of the flow in ${}^bS^*M$, see Proposition 2.1, as well as the b-normally hyperbolic versions, proved in the Appendix, of the semiclassical normally hyperbolic trapping estimates of Wunsch and Zworski [47]; the rest of the Fredholm setup is discussed in Section 2.1 in the non-trapping and Section 3.1 in the normally hyperbolic trapping setting. The analogue of Proposition 2.1 for sources/sinks was already proved in [2, Section 4];

¹³In Section 5 we switch to ρ as the boundary defining function for consistency with [2].

¹⁴More general, ‘long-range’ scattering metrics also work for the purposes of this paper without any significant changes; the analysis of these is currently being completed by Baskin, Vasy and Wunsch. The difference is the presence of smooth multiples of $\tau \frac{d\tau^2}{\tau^2}$ in the metric near $\tau=0, v=0$. These do not affect the normal operator, but slightly change the dynamics in ${}^bS^*M$. This, however, does not affect the function spaces to be used for our semilinear problem.

our Lorentzian scattering metric Fredholm discussion, which relies on this, is in Section 5.1.

We emphasize that our analysis would be significantly less cumbersome in terms of technicalities if we were not including Cauchy hypersurfaces and solved a globally well-behaved problem by imposing sufficiently rapid decay at past infinity instead (it is standard to convert a Cauchy problem into a forward solution problem). Cauchy hypersurfaces are only necessary for us if we deal with a problem ill-behaved in the past because complex absorption does not force appropriate forward supports even though it does so at the level of singularities; otherwise we can work with appropriate (weighted) Sobolev spaces. The latter is the case with Lorentzian scattering spaces, which thus provide an ideal example for our setting. It can also be done on the global setting of de Sitter spaces, as in setting (3) above, essentially by realizing these as the boundary of the appropriate compactification of a Lorentzian scattering space, see [40]. In the case of Kerr-de Sitter black holes, in the presence of dilation invariance, one has access to a similar luxury; complex absorption does the job as in [41]; the key aspect is that it needs to be imposed *outside* the static region we consider. For a general Lorentzian b-metric with a normally hyperbolic trapped set, this may not be easy to arrange, and we do work by adding Cauchy hypersurfaces, even at the cost of the resulting, rather artificial in terms of PDE theory, technical complications. For perturbations of Kerr-de Sitter space, however, it is possible to forego the latter for well-posedness by an appropriate gluing to complete the space with actual Kerr-de Sitter space in the past for the purposes of functional analysis. We remark that Cauchy hypersurfaces are somewhat ill-behaved for L^2 based estimates, which we use, but match $L^\infty L^2$ estimates quite well, which explains the large role they play in existing hyperbolic theory, such as [24] or [21, Chapter 23.2]. We hope that adopting this more commonly used form of ‘truncation’ of hyperbolic problems will aid the readability of the paper.

We also explain the role that the energy estimates (as opposed to microlocal energy estimates) play. These mostly enter to deal with the artificially introduced boundaries; if other methods were used to truncate the flow, their role reduces to checking that in certain cases, when the microlocal machinery only guarantees Fredholm properties of the underlying linear operators, the potential finite dimensional kernel and cokernel are indeed trivial. Asymptotically Minkowski spaces illustrate this best, as the Hamilton flow is globally well-behaved there; see Section 5.1.

The other key technical tool is the algebra property of b-Sobolev spaces and other spaces with additional conormal regularity. These are stated in the respective sections; the case of the standard b-Sobolev spaces reduces to the algebra property of the standard Sobolev spaces on \mathbb{R}^n . Given the algebra properties, the results are proved by applying the contraction mapping theorem to the linear operator.

In summary, the plan of this paper is the following. In each of the sections below we consider one of these settings, and first describe the Sobolev spaces on which one has invertibility for the linear problems of interest, then analyze the algebra properties of these Sobolev spaces, finally proving the solvability of the semilinear equations by checking that the hypotheses of the contraction mapping theorem are satisfied.

The authors are grateful to Dean Baskin, Rafe Mazzeo, Richard Melrose, Gunther Uhlmann, Jared Wunsch and Maciej Zworski for their interest and support. In

particular, the overall strategy reflects Melrose's vision for solving non-linear PDE globally.

2. STATIC MODEL OF DE SITTER SPACE

2.1. The linear Fredholm framework. While the results of [41] on linear estimates for wave equations for b-metrics are optimally stated when the metrics and thus the corresponding operators are dilation-invariant, i.e. when near $\tau = 0$ the normal operator can be identified with the operator itself, see [41, Lemma 3.1], the estimates for Sobolev derivatives are lossy for general b-metrics in [41, Proposition 3.5], essentially because one should not treat the difference between the normal operator and the actual operator purely as a perturbation. While perhaps the main interest here is in the dilation-invariant case, we first strengthen the linear results in [41] in the non-dilation invariant setting by analyzing b-radial points which are saddle points of the Hamilton flow. This is similar to [2, Section 4], where the analogous result was proved when the b-radial points are sources/sinks. This is then used to set up a Fredholm framework for the linear problem. If one is mainly interested in the dilation invariant case, one can use [41, Lemma 3.1] in place of Theorems 2.18-2.20 below, either adding the boundary corresponding to H_2 below, or still using complex absorption as was done in [41].

So suppose $\mathcal{P} \in \Psi_b^m(M)$, M a manifold with boundary. We refer to Section 3 of [41] for a discussion of the b-geometry relevant for the present context, and [41, Section 2] for the results for the Mellin transformed normal operator. Let p be the principal symbol of \mathcal{P} , which is thus a homogeneous degree m function on ${}^bT^*M \setminus o$, which we assume to be *real-valued*, and let \mathbf{H}_p be the Hamilton vector field of p (more precisely, the extension of the Hamilton vector field from $T^*M^\circ \setminus o$ to ${}^bT^*M \setminus o$), which is a homogeneous degree $m - 1$ vector field on ${}^bT^*M \setminus o$ tangent to the boundary ${}^bT_X^*M$. In local coordinates (τ, z) on M near X , with b-dual coordinates (σ, ζ) , i.e. writing b-covectors as

$$\sigma \frac{d\tau}{\tau} + \sum_j \zeta_j dz_j,$$

this has the form

$$\mathbf{H}_p = (\partial_\sigma p)(\tau \partial_\tau) - (\tau \partial_\tau p) \partial_\sigma + \sum_j ((\partial_{\zeta_j} p) \partial_{z_j} - (\partial_{z_j} p) \partial_{\zeta_j}), \quad (2.1)$$

see [2, Equation (3.20)], where a somewhat different notation is used, given by [2, Equation (3.19)]. It is then convenient to consider the fiber compactification $\overline{{}^bT^*M}$ of ${}^bT^*M$, where the fibers are replaced by their radial compactification. The new boundary of $\overline{{}^bT^*M}$ at fiber infinity is ${}^bS^*M$; it still possesses the compactification of the 'old' boundary $\overline{{}^bT_X^*M}$, see Figure 2. Recall here that ${}^bS^*M$ is naturally the quotient of ${}^bT^*M \setminus o$ by the \mathbb{R}^+ -action of dilation in the fibers of the cotangent bundle.

Let $\tilde{\rho}$ denote a homogeneous degree -1 defining function of ${}^bS^*M$. Then the rescaled Hamilton vector field

$$V = \tilde{\rho}^{m-1} \mathbf{H}_p$$

is a C^∞ vector field on $\overline{{}^bT^*M}$ away from the 0-section, and it is tangent to all boundary faces. The characteristic set Σ is the zero-set of the smooth function $\tilde{\rho}^m p$ in ${}^bS^*M$. We refer to the flow of V in $\Sigma \subset {}^bS^*M$ as the Hamilton, or

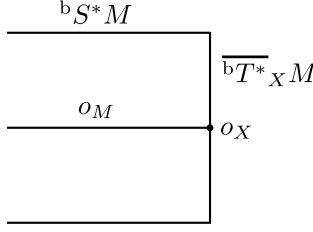


FIGURE 2. The radially compactified cotangent bundle $\overline{bT^*M}$ near $\overline{bT^*}_X M$; the cosphere bundle ${}^bS^*M$, viewed as the boundary at fiber infinity of ${}^bT^*M$, is also shown, as well as the zero section $o_M \subset {}^bT^*M$ and the zero section over the boundary $o_X \subset \overline{bT^*}_X M$.

(null)bicharacteristic flow; its integral curves, the (null)bicharacteristics, are reparameterizations of those of the Hamilton vector field H_p , projected by the quotient map ${}^bT^*M \setminus o \rightarrow {}^bS^*M$.

2.1.1. *Generalized b-radial sets.* The standard propagation of singularities theorem in the characteristic set Σ in the b-setting is that for $u \in H_b^{-\infty, r}(M)$, within Σ , $\text{WF}_b^{s, r}(u) \setminus \text{WF}_b^{s-m+1, r}(\mathcal{P}u)$ is a union of maximally extended integral curves (i.e. null-bicharacteristics) of \mathcal{P} . This is vacuous at points where V vanishes (as a smooth vector field); these points are called *radial points*, since at such a point H_p itself (on ${}^bT^*M \setminus o$) is radial, i.e. is a multiple of the generator of the dilations of the fiber of the b-cotangent bundle. At a radial point α , V acts on the ideal \mathcal{I} of C^∞ functions vanishing at α , and thus on $T_\alpha^* \overline{bT^*M}$, which can be identified with $\mathcal{I}/\mathcal{I}^2$. Since V is tangent to both boundary hypersurfaces, given by $\tau = 0$ and $\tilde{\rho} = 0$, $d\tau$ and $d\tilde{\rho}$ are automatically eigenvectors of the linearization of V . We are interested in a generalization of the situation in which we have a smooth submanifold L of ${}^bS_X^*M$ consisting of radial points which is a source/sink for V within ${}^bT_X^*M$ but if it is a source, so in particular $d\tilde{\rho}$ is in an unstable eigenspace, then $d\tau$ is in the (necessarily one-dimensional) stable eigenspace, and vice versa. Thus, L is a saddle point of the Hamilton flow.

In view of the bicharacteristic flow on Kerr-de Sitter space (which, unlike the non-rotating de Sitter-Schwarzschild black holes, does not have this precise radial point structure), it is important to be slightly more general, as in [41, Section 2.2]. Thus, we assume that dp does not vanish where p does, i.e. at Σ , and is linearly independent of $d\tau$ at $\{\tau = 0, p = 0\} = \Sigma \cap {}^bS_X^*M$, so Σ is a smooth submanifold of ${}^bS^*M$ transversal to ${}^bS_X^*M$, and for L assume simply that $L = L_+ \cup L_-$, L_\pm smooth disjoint submanifolds of ${}^bS_X^*M$, given by $\mathcal{L}_\pm \cap {}^bS_X^*M$ where \mathcal{L}_\pm are smooth disjoint submanifolds¹⁵ of Σ transversal to ${}^bS_X^*M$, defined locally near ${}^bS_X^*M$, with $\tilde{\rho}^{m-1}H_p$ tangent to \mathcal{L}_\pm , with a homogeneous degree zero quadratic defining function¹⁶ ρ_0 of

¹⁵These play the role of the two halves of the conormal bundles of event horizons.

¹⁶That is, ρ_0 vanishes quadratically at \mathcal{L} (and vanishes only at \mathcal{L}), with the vanishing non-degenerate, in the sense that the Hessian is positive definite, corresponding to ρ_0 being a sum of squares of linear defining functions whose differentials span the conormal bundle of \mathcal{L} within Σ .

\mathcal{L} within Σ such that

$$\begin{aligned} \tilde{\rho}^{m-2} \mathbf{H}_p \tilde{\rho}|_{L_\pm} &= \mp \beta_0, & -\tilde{\rho}^{m-1} \tau^{-1} \mathbf{H}_p \tau|_{L_\pm} &= \mp \tilde{\beta} \beta_0, \\ \beta_0, \tilde{\beta} &\in C^\infty(L_\pm), & \beta_0, \tilde{\beta} &> 0, \end{aligned} \quad (2.2)$$

and, with $\beta_1 > 0$,

$$\mp \tilde{\rho}^{m-1} \mathbf{H}_p \rho_0 - \beta_1 \rho_0 \quad (2.3)$$

is ≥ 0 modulo cubic vanishing terms at L_\pm . Then L_- is a source and L_+ is a sink within ${}^b S_X^* M$ in the sense of nearby bicharacteristics *within* ${}^b S_X^* M$ all tend to L_\pm as the parameter along them goes to $\pm\infty$, but at L_- there is also a stable, and at L_+ an unstable, manifold, namely \mathcal{L}_- , resp. \mathcal{L}_+ .¹⁷ In order to simplify the statements, we assume that

$$\tilde{\beta} \text{ is constant on } L_\pm; \quad \tilde{\beta} = \beta > 0;$$

we refer the reader to [41, Equation (2.5)-(2.6)], and the discussion throughout that paper, where a general $\tilde{\beta}$ is allowed, at the cost of either $\sup \tilde{\beta}$ or $\inf \tilde{\beta}$ playing a role in various statements depending on signs. Finally, we assume that $\mathcal{P} - \mathcal{P}^* \in \Psi_b^{m-2}(M)$ for convenience (with respect to some b-metric), as this is the case for the Klein-Gordon equation.¹⁸

Proposition 2.1. *Suppose \mathcal{P} is as above.*

If $s \geq s'$, $s' - (m-1)/2 > \beta r$, and if $u \in H_b^{-\infty, r}(M)$ then L_\pm (and thus a neighborhood of L_\pm) is disjoint from $\text{WF}_b^{s, r}(u)$ provided $L_\pm \cap \text{WF}_b^{s-m+1, r}(\mathcal{P}u) = \emptyset$, $L_\pm \cap \text{WF}_b^{s', r}(u) = \emptyset$, and in a neighborhood of L_\pm , $\mathcal{L}_\pm \cap \{\tau > 0\}$ are disjoint from $\text{WF}_b^{s, r}(u)$.

On the other hand, if $s - (m-1)/2 < \beta r$, and if $u \in H_b^{-\infty, r}(M)$ then L_\pm (and thus a neighborhood of L_\pm) is disjoint from $\text{WF}_b^{s, r}(u)$ provided $L_\pm \cap \text{WF}_b^{s-m+1, r}(\mathcal{P}u) = \emptyset$ and a punctured neighborhood of L_\pm , with L_\pm removed, in $\Sigma \cap {}^b S_X^ M$ is disjoint from $\text{WF}_b^{s, r}(u)$.*

Remark 2.2. The decay order r plays the role of $-\text{Im} \sigma$ in [41] in view of the Mellin transform in the dilation invariant setting identifying weighted b-Sobolev spaces with weight r with semiclassical Sobolev spaces on the boundary on the line $\text{Im} \sigma = -r$, see [41, Equation (3.8)-(3.9)]. Thus, the numerology in this proposition is a direct translation of that in [41, Propositions 2.3-2.4].

Proof. We remark first that $\tilde{\rho}^{m-1} \mathbf{H}_p \rho_0$ vanishes quadratically on \mathcal{L}_\pm since $\tilde{\rho}^{m-1} \mathbf{H}_p$ is tangent to \mathcal{L}_\pm and ρ_0 itself vanishes there quadratically. Further, this quadratic expression is positive definite near $\tau = 0$ since it is such at $\tau = 0$. Correspondingly, we can strengthen (2.3) to

$$\mp \tilde{\rho}^{m-1} \mathbf{H}_p \rho_0 - \frac{\beta_1}{2} \rho_0 \quad (2.4)$$

¹⁷Bicharacteristics in \mathcal{L}_\pm remain there by the tangency of $\tilde{\rho}^{m-1} \mathbf{H}_p$ to \mathcal{L}_\pm ; further $\tau \rightarrow 0$ along them as the parameter goes to $\mp\infty$ by (2.2), at least sufficiently close to $\tau = 0$, since L_\pm are defined in \mathcal{L}_\pm by $\tau = 0$.

¹⁸The natural assumption is that the principal symbol of $\frac{1}{2i}(\mathcal{P} - \mathcal{P}^*) \in \Psi_b^{m-1}(M)$ at L_\pm is

$$\pm \hat{\beta} \beta_0 \tilde{\rho}^{-m+1}, \quad \hat{\beta} \in C^\infty(L_\pm).$$

If $\hat{\beta}$ vanishes, Proposition 2.1 is valid without a change; otherwise it shifts the threshold quantity $s - (m-1)/2 - r$ below in Proposition 2.1 to $s - (m-1)/2 - \beta r - \hat{\beta}$ if $\hat{\beta}$ is constant, with modifications as in [41, Proof of Propositions 2.3-2.4] otherwise.

being non-negative modulo cubic terms vanishing at \mathcal{L}_\pm in a neighborhood of $\tau = 0$.

Notice next that, using (2.4) in the first case and (2.2) in the second, and that L_\pm is defined in Σ by $\tau = 0$, $\rho_0 = 0$, there exist $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\alpha \in \Sigma, \rho_0(\alpha) < \delta_0, \tau(\alpha) < \delta_1, \rho_0(\alpha) \neq 0 \Rightarrow (\mp \tilde{\rho}^{m-1} \mathbf{H}_p \rho_0)(\alpha) > 0$$

and

$$\alpha \in \Sigma, \rho_0(\alpha) < \delta_0, \tau(\alpha) < \delta_1 \Rightarrow (\pm \tilde{\rho}^{m-1} \tau^{-1} \mathbf{H}_p \tau)(\alpha) > 0.$$

Similarly to [41, Proof of Propositions 2.3-2.4], which is not in the b-setting, and [2, Proof of Proposition 4.4], which is but concerns only sources/sinks (corresponding to Minkowski type spaces), we consider commutants

$$C \in \tau^{-r} \Psi_{\mathbf{b}}^{s-(m-1)/2}(M) = \Psi_{\mathbf{b}}^{s-(m-1)/2, -r}(M)$$

with principal symbol

$$c = \phi(\rho_0) \phi_0(p_0) \phi_1(\tau) \tilde{\rho}^{-s+(m-1)/2} \tau^{-r}, \quad p_0 = \tilde{\rho}^m p,$$

where $\phi_0 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi' \leq 0$ in $[0, \infty)$ and ϕ supported in $(-\delta_0, \delta_0)$, while $\phi_1 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi_1' \leq 0$ in $[0, \infty)$ and ϕ_1 supported in $(-\delta_1, \delta_1)$, so that

$$\alpha \in \text{supp } d(\phi \circ \rho_0) \cap \text{supp}(\phi_1 \circ \tau) \cap \Sigma \Rightarrow \mp(\tilde{\rho}^{m-1} \mathbf{H}_p \rho_0)(\alpha) > 0,$$

and

$$\pm \tilde{\rho}^{m-1} \tau^{-1} \mathbf{H}_p \tau$$

remains positive on $\text{supp}(\phi_1 \circ \tau) \cap \text{supp}(\phi \circ \rho_0)$.

The main contribution then comes from the weights, which give

$$\tilde{\rho}^{m-1} \mathbf{H}_p(\tilde{\rho}^{-s+(m-1)/2} \tau^{-r}) = \mp(-s + (m-1)/2 + \beta r) \beta_0 \tilde{\rho}^{-s+(m-1)/2} \tau^{-r},$$

where the sign of the factor in parentheses on the right hand side being negative, resp. positive, gives the first, resp. the second, case of the statement of the proposition. Further, the sign of the term in which $\phi_1(\tau)$, resp. $\phi(\rho_0)$, gets differentiated, yielding $\pm \tau \beta \beta_0 \phi_1'(\tau)$, resp. $\phi'(\rho_0) \tilde{\rho}^{m-1} \mathbf{H}_p \rho_0$, is, when $s - (m-1)/2 - \beta r > 0$, the opposite, resp. the same, of these terms, while when $s - (m-1)/2 - \beta r < 0$, it is the same, resp. the opposite, of these terms. Correspondingly,

$$\begin{aligned} \sigma_{2s}(i[\mathcal{P}, C^*C]) &= \mp 2 \left(-\beta_0 \left(s - \frac{m-1}{2} - \beta r \right) \phi \phi_0 \phi_1 - \beta_0 \tilde{\beta} \tau \phi \phi_0 \phi_1' \right. \\ &\quad \left. \mp (\tilde{\rho}^{m-1} \mathbf{H}_p \rho_0) \phi' \phi_0 \phi_1 + m \beta_0 p_0 \phi \phi_0' \phi_1 \right) \phi \phi_0 \phi_1 \tilde{\rho}^{-2s} \tau^{-2r}. \end{aligned}$$

We can regularize using using $S_\epsilon \in \Psi_{\mathbf{b}}^{-\delta}(M)$ for $\epsilon > 0$, uniformly bounded in $\Psi_{\mathbf{b}}^0(M)$, converging to Id in $\Psi_{\mathbf{b}}^{\delta'}(M)$ for $\delta' > 0$, with principal symbol $(1 + \epsilon \tilde{\rho}^{-1})^{-\delta}$, as in [41, Proof of Propositions 2.3-2.4], where the only difference was that the calculation was on $X = \partial M$, and thus the pseudodifferential operators were standard ones, rather than b-pseudodifferential operators. The a priori regularity assumption on $\text{WF}_{\mathbf{b}}^{s', r}(u)$ arises as the regularizer has the opposite sign as compared to the contribution of the weights, thus the amount of regularization one can do is limited. The positive commutator argument then proceeds completely analogously to [41, Proof of Propositions 2.3-2.4], except that, as in [41], one has to assume a priori bounds on the term with the sign opposite to that of $s - (m-1)/2 - \beta r$, of which there is exactly one for either sign (unlike in [41], in which only $s - (m-1)/2 + \beta \text{Im } \sigma < 0$ has such a term), thus on $\Sigma \cap \text{supp}(\phi_1' \circ \tau) \cap \text{supp}(\phi \circ \rho_0)$ when $s - (m-1)/2 - \beta r > 0$ and on $\Sigma \cap \text{supp}(\phi_1 \circ \tau) \cap \text{supp}(\phi' \circ \rho_0)$ when $s - (m-1)/2 - \beta r < 0$.

Using the openness of the complement of the wave front set we can finally choose ϕ and ϕ_1 (satisfying the support conditions, among others) so that the a priori assumptions are satisfied, choosing ϕ_1 first and then shrinking the support of ϕ in the first case, with the choice being made in the opposite order in the second case, completing the proof of the proposition. \square

2.1.2. Complex absorption. In order to have good Fredholm properties we either need a complete Hamilton flow, or need to ‘stop it’ in a manner that gives suitable estimates; one may want to do the latter to avoid global assumptions on the flow on the ambient space. The microlocally best behaved version is given by complex absorption; it is microlocal, works easily with Sobolev spaces of arbitrary order, and makes the operator elliptic in the absorbing region, giving rise to very convenient analysis. The main downside of complex absorption is that it does not automatically give forward mapping properties for the support of solutions in wave equation-like settings, even though at the level of singularities, it does have the desired forward property. It was used extensively in [41] – in the dilation invariant setting, the bicharacteristics on $X \times (0, \infty)_\tau$ are controlled (by the invariance) as $\tau \rightarrow \infty$ as well as when $\tau \rightarrow 0$, and thus one need not use complex absorption there, instead decay as $\tau \rightarrow \infty$ (corresponding to growth as $\tau \rightarrow 0$ on these dilation invariant spaces) gives the desired forward property; complex absorption was only used to cut off the flow within X . Here we want to localize in τ as well, and while complex absorption can achieve this, it loses the forward *support* character of the problem. However, as it is conceptually much cleaner, we discuss Fredholm properties using it first before turning to adding artificial (spacelike) boundary hypersurfaces instead.

Thus, we now consider $\mathcal{P} - i\mathcal{Q} \in \Psi_b^m(M)$, $\mathcal{Q} \in \Psi_b^m(M)$, with real principal symbol q , being the complex absorption similarly to [41, Sections 2.2 and 2.8]; we assume that $\text{WF}'_b(\mathcal{Q}) \cap L = \emptyset$. Here the semiclassical version, discussed in [41] with further references there, is a close parallel to our b-setting; it is essentially equivalent to the b-setting in the special case that \mathcal{P} , \mathcal{Q} are dilation-invariant, for then the Mellin transform gives rise exactly to the semiclassical problem as the Mellin-dual parameter goes to infinity. Thus, we assume that the characteristic set Σ of \mathcal{P} has the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,$$

with each of Σ_\pm being a union of connected components, and

$$\mp q \geq 0 \text{ near } \Sigma_\pm.$$

Recall from [41, Section 2.5], which in turn is a simple modification of the semiclassical results of Nonnenmacher and Zworski [34], and Datchev and Vasy [12], that under these sign conditions on q , estimates can be propagated in the backward direction along the Hamilton flow on Σ_+ and in the forward direction for Σ_- , or, phrased as a wave front set statement (the property of being singular propagates in the opposite direction as the property of being regular!), $\text{WF}^s(u)$ is invariant in¹⁹ $(\Sigma_+ \setminus {}^bS_X^*M) \setminus \text{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow, and in $(\Sigma_- \setminus {}^bS_X^*M) \setminus \text{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow. Since this is a principal symbol argument, given in [41, Section 2.5] and [12, Lemma 5.1], its extension to the b-setting only requires minimal changes. Namely, assuming one is away from radial points as one may (since at these the statement is vacuous), one

¹⁹That is, the invariance is away from the boundary X ; we address the behavior at the boundary in the rest of the paragraph.

constructs the principal symbol c of the commutant on ${}^bT^*M \setminus o$ as a C^∞ function c_0 on ${}^bS^*M$ with derivative of a fixed sign along the Hamilton flow in the region where one wants to obtain the estimate (exactly the same way as for real principal type proofs) multiplied by weights in τ and $\tilde{\rho}$, making the Hamilton derivative of c_0 large relative to c_0 to control the error terms from the weights, and computes $\langle u, -i[C^*C, \tilde{\mathcal{P}}]u \rangle$, where $\tilde{\mathcal{P}}$ is the symmetric part of $\mathcal{P} - i\mathcal{Q}$ (so has principal symbol p) and $\tilde{\mathcal{Q}}$ is the antisymmetric part. This gives

$$-2\operatorname{Re}\langle u, iC^*C(\mathcal{P} - i\mathcal{Q})u \rangle - 2\operatorname{Re}\langle u, C^*C\tilde{\mathcal{Q}}u \rangle.$$

The issue here is that the second term on the right hand side involves $C^*C\tilde{\mathcal{Q}}$, which is one order higher than $[C^*C, \tilde{\mathcal{P}}]$, so while it itself has a desirable sign, one needs to be concerned about subprincipal terms.²⁰ However, one rewrites

$$2\operatorname{Re}\langle u, C^*C\tilde{\mathcal{Q}}u \rangle = 2\operatorname{Re}\langle u, C^*\tilde{\mathcal{Q}}Cu \rangle + 2\operatorname{Re}\langle u, C^*[C, \tilde{\mathcal{Q}}]u \rangle.$$

Now the first term is positive modulo a controllable error²¹ by the sharp Gårding inequality, or if one arranges that q is the square of a symbol, while the second can be rewritten in terms of $[C, [C, \tilde{\mathcal{Q}}]]$, $(C^* - C)[C, \tilde{\mathcal{Q}}]$, etc, which are all controllable as they drop *two* orders relative to the product $C^*C\tilde{\mathcal{Q}}$, giving rise to the result, namely that for $u \in H_b^{-\infty, r}$, $\operatorname{WF}_b^{s, r}(u)$ is invariant in $\Sigma_+ \setminus \operatorname{WF}^{s-m+1, r}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow, and in $\Sigma_- \setminus \operatorname{WF}^{s-m+1, r}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow.

In analogy with [41, Definition 2.12], we say that $\mathcal{P} - i\mathcal{Q}$ is *non-trapping* if all bicharacteristics in Σ from any point in $\Sigma \setminus (L_+ \cup L_-)$ flow to $\operatorname{Ell}(q) \cup L_+ \cup L_-$ in both the forward and backward directions (i.e. either enter $\operatorname{Ell}(q)$ in finite time or tend to $L_+ \cup L_-$). Notice that as Σ_\pm are closed under the Hamilton flow, bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of \mathcal{Q} in the forward (in Σ_+), resp. backward (in Σ_-) direction. Indeed, by the non-trapping hypothesis, these bicharacteristics have to reach the elliptic set of \mathcal{Q} as they cannot tend to L_+ , resp. L_- : for \mathcal{L}_+ and \mathcal{L}_- are unstable, resp. stable manifolds, and these bicharacteristics cannot enter the boundary (which is preserved by the flow), so cannot lie in the stable, resp. unstable, manifolds of $L_+ \cup L_-$, which are within ${}^bS_X^*M$. Similarly, bicharacteristics in $(\Sigma \cap {}^bS_X^*M) \setminus (L_+ \cup L_-)$ necessarily reach the elliptic set of \mathcal{Q} in the backward (in Σ_+), resp. forward (in Σ_-) direction. Then for s, r satisfying

$$s - (m - 1)/2 > \beta r$$

one has an estimate

$$\|u\|_{H_b^{s, r}} \leq C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s-m+1, r}} + C\|u\|_{H_b^{s', r}}, \quad (2.5)$$

provided one assumes $s' < s$,

$$s' - (m - 1)/2 > \beta r, \quad u \in H_b^{s', r}.$$

²⁰In fact, as the principal symbol of $C^*C\tilde{\mathcal{Q}}$ is real, the real part of its subprincipal symbol is well-defined, and is the real part of c^2q where c and q include the real parts of their subprincipal terms, and is all that matters for this argument, so one could proceed symbolically. See also Footnote 70.

²¹This uses the derivative of c , arising in the symbol of the commutator with $\tilde{\mathcal{P}}$, to provide the control: since \mathcal{Q} is positive modulo an operator one order lower, and in the term involving this operator, the principal symbol c of C is not differentiated, taking the derivative of c_0 large relative to c_0 , as is already used to control weights, etc., controls this error term as well.

Indeed, this is a simple consequence of $u \in H_b^{s',r}$, $(\mathcal{P} - i\mathcal{Q})u \in H_b^{s-m+1,r}$ implying $u \in H_b^{s,r}$ via the closed graph theorem, see [21, Proof of Theorem 26.1.7] and [39, Section 4.3]. This implication in turn holds as on the elliptic set of \mathcal{Q} one has the stronger statement $u \in H_b^{s+1,r}$ under these conditions, and then using real-principal type propagation of regularity in the *backward* direction on Σ_+ and the *forward* direction on Σ_- , one can propagate the microlocal membership of $H_b^{s,r}$ (i.e. the absence of the corresponding wave front set) in the backward, resp. forward, direction on Σ_+ , resp. Σ_- . Since bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of \mathcal{Q} in the forward, resp. backward direction, and thus one has $H_b^{s,r}$ membership along them by what we have shown, Proposition 2.1 extends this membership to L_\pm , and hence to a neighborhood of these, and by our non-trapping assumption every bicharacteristic enters either this neighborhood of L_\pm or the elliptic set of \mathcal{Q} in finite time in the backward, resp. forward, direction, so by the real principal type propagation of singularities we have the claimed microlocal membership everywhere.

Reversing the direction in which one propagates estimates, one also has a similar estimate for the adjoint, $\mathcal{P}^* + i\mathcal{Q}^*$ except now one needs to have

$$s - (m - 1)/2 < \beta r$$

in order to propagate through the saddle points in the opposite direction, i.e. from within ${}^bS_X^*M$ to \mathcal{L}_\pm . Then for $s' < s$,

$$\|u\|_{H_b^{s,r}} \leq C\|(\mathcal{P}^* + i\mathcal{Q}^*)u\|_{H_b^{s-m+1,r}} + C\|u\|_{H_b^{s',r}}. \quad (2.6)$$

The issue with these estimates is that $H_b^{s',r}$ does not include compactly into the error term $H_b^{s',r}$ on the right hand side due to the lack of additional decay. Thus, these estimates are insufficient to show Fredholm properties, which in fact do not hold in general.

We thus further assume that there are no poles of the inverse of the Mellin conjugate $(\widehat{\mathcal{P} - i\mathcal{Q}})(\sigma)$ of the normal operator, $N(\mathcal{P} - i\mathcal{Q})$, on the line $\text{Im } \sigma = -r$. Here we refer to [41, Section 3.1] for a brief discussion of the normal operator and the Mellin transform; this cited section also contains more detailed references to [32]. Then using the Mellin transform, which is an isomorphism between weighted b-Sobolev spaces and semiclassical Sobolev spaces (see Equations (3.8)-(3.9) in [41]), and the estimates for $(\widehat{\mathcal{P} - i\mathcal{Q}})(\sigma)$ (including the high energy, i.e. semiclassical, estimates²², all of which is discussed in detail in [41, Section 2] — the high energy assumptions of [41, Section 2] hold by our assumptions on the b-flow at ${}^bS_X^*M$, and which imply that for all but a discrete set of r the aforementioned lines do not contain such poles), we obtain that on $\mathbb{R}_\rho^+ \times \partial M$

$$\|v\|_{H_b^{s,r}} \leq C\|N(\mathcal{P} - i\mathcal{Q})v\|_{H_b^{s-m+1,r}} \quad (2.7)$$

when

$$s - (m - 1)/2 > \beta r.$$

Again, we have an analogous estimate for $N(\mathcal{P}^* + i\mathcal{Q}^*)$:

$$\|v\|_{H_b^{s,r}} \leq C\|N(\mathcal{P}^* + i\mathcal{Q}^*)v\|_{H_b^{s-m+1,r}}, \quad (2.8)$$

²²The high energy estimates are actually implied by b-principal symbol based estimates on the normal operator space, $M_\infty = X \times \mathbb{R}^+$, $X = \partial M$, on spaces $\tau^r H_b^s(M_\infty)$ corresponding to $\text{Im } \sigma = -r$, but we do not explicitly discuss this here.

provided $-r$ is not the imaginary part of a pole of the inverse of $\widehat{\mathcal{P}^* + i\mathcal{Q}^*}$, and provided

$$s - (m - 1)/2 < \beta r.$$

As $\widehat{\mathcal{P}^* + i\mathcal{Q}^*}(\sigma) = (\widehat{\mathcal{P}} - i\widehat{\mathcal{Q}})^*(\bar{\sigma})$, see the discussion in [41] preceding Equation (3.25), the requirement on $-r$ is the same as r not being the imaginary part of a pole of the inverse of $\widehat{\mathcal{P}} - i\widehat{\mathcal{Q}}$.

We apply these results by first letting $\chi \in C_c^\infty(M)$ be identically 1 near ∂M supported in a collar neighborhood of ∂M , which we identify with $(0, \epsilon)_\tau \times \partial M$ of the normal operator space. Then, *assuming* $s' - (m - 1)/2 > \beta r$,

$$\|u\|_{H_b^{s',r}} \leq \|\chi u\|_{H_b^{s',r}} + \|(1-\chi)u\|_{H_b^{s',r}} \leq C\|N(\mathcal{P}-i\mathcal{Q})\chi u\|_{H_b^{s'-m+1,r}} + \|(1-\chi)u\|_{H_b^{s',r}}. \quad (2.9)$$

Now, if $K = \text{supp}(1 - \chi)$, then

$$\|(1 - \chi)u\|_{H_b^{s',r}} \leq C\|u\|_{H^{s'(K)}} \leq C'\|u\|_{H_b^{s',\tilde{r}}} \leq C''\|u\|_{H_b^{s'+1,\tilde{r}}}$$

for any \tilde{r} . On the other hand, $N(\mathcal{P} - i\mathcal{Q}) - (\mathcal{P} - i\mathcal{Q}) \in \tau\Psi_b^m([0, \epsilon) \times \partial M)$, so

$$\begin{aligned} N(\mathcal{P} - i\mathcal{Q})\chi u &= (\mathcal{P} - i\mathcal{Q})\chi u + (N(\mathcal{P} - i\mathcal{Q}) - (\mathcal{P} - i\mathcal{Q}))\chi u \\ &= \chi(\mathcal{P} - i\mathcal{Q})u + [\mathcal{P} - i\mathcal{Q}, \chi]u + (N(\mathcal{P} - i\mathcal{Q}) - (\mathcal{P} - i\mathcal{Q}))\chi u \end{aligned}$$

plus the fact that $[\mathcal{P} - i\mathcal{Q}, \chi]$ is supported in K and $\|\chi(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s'-m+1,r}} \leq \|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s'-m+1,r}}$ show that for all \tilde{r}

$$\|N(\mathcal{P} - i\mathcal{Q})\chi u\|_{H_b^{s'-m+1,r}} \leq \|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s'-m+1,r}} + C\|u\|_{H_b^{s'+1,\tilde{r}}} + C\|u\|_{H_b^{s'+1,r-1}}. \quad (2.10)$$

Combining (2.5), (2.9) and (2.10) we deduce that (with new constants, and taking s' sufficiently small and $\tilde{r} = r - 1$)

$$\|u\|_{H_b^{s,r}} \leq C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s-m+1,r}} + C\|u\|_{H_b^{s'+1,r-1}}, \quad (2.11)$$

where now the inclusion $H_b^{s,r} \rightarrow H_b^{s'+1,r-1}$ is compact when we choose, as we may, $s' < s - 1$, requiring, however, $s' - (m - 1)/2 > \beta r$. Recall that this argument required that s, r, s' satisfied the requirements preceding (2.5), and that $-r$ is not the imaginary part of any pole of $\widehat{\mathcal{P} - i\mathcal{Q}}$.

Analogous estimates hold for $(\mathcal{P} - i\mathcal{Q})^*$ where now we write \tilde{s}, \tilde{r} and \tilde{s}' for the Sobolev orders for the eventual application:

$$\|u\|_{H_b^{\tilde{s},\tilde{r}}} \leq C\|(\mathcal{P} - i\mathcal{Q})^*u\|_{H_b^{\tilde{s}-m+1,\tilde{r}}} + C\|u\|_{H_b^{\tilde{s}'+1,\tilde{r}-1}}, \quad (2.12)$$

provided \tilde{s}, \tilde{r} in place of s and r satisfy the requirements stated before (2.6), and provided $-\tilde{r}$ is not the imaginary part of a pole of $\widehat{\mathcal{P}^* + i\mathcal{Q}^*}$ (i.e. \tilde{r} of $\widehat{\mathcal{P}} - i\widehat{\mathcal{Q}}$). Note that we *do not* have a stronger requirement for \tilde{s}' , unlike for s' above, since upper bounds for s imply those for $s' \leq s$.

Via a standard functional analytic argument, see [21, Proof of Theorem 26.1.7] and also [41, Section 2.6] in the present context, we thus obtain Fredholm properties of $\mathcal{P} - i\mathcal{Q}$, in particular solvability, modulo a (possible) finite dimensional obstruction, in $H_b^{s,r}$ if

$$s - (m - 1)/2 - 1 > \beta r. \quad (2.13)$$

Concretely, we take $\tilde{s} = m - 1 - s$, $\tilde{r} = -r$, $s' < s - 1$ sufficiently close to $s - 1$ so that $s' - (m - 1)/2 > \beta r$ (which is possible by (2.13)). Thus, $s - (m - 1)/2 > \beta r$ means $\tilde{s} - (m - 1)/2 = (m - 1)/2 - s < -\beta r = \beta\tilde{r}$, so the space on the left hand side

of (2.11) is dual to that in the first term on the right hand side of (2.12), and the same for the equations interchanged, and notice that the condition on the poles of the inverse of the Mellin transformed normal operators is the same for both $\mathcal{P} - i\mathcal{Q}$ and $\mathcal{P}^* + i\mathcal{Q}^*$: $-r$ is not the imaginary part of a pole of $\widehat{\mathcal{P} - i\mathcal{Q}}$. This yields:

Proposition 2.3. *Suppose that \mathcal{P} is non-trapping. Suppose $s, r \in \mathbb{R}$, $s - (m - 1)/2 - 1 > \beta r$, $-r$ is not the imaginary part of a pole of $\widehat{\mathcal{P} - i\mathcal{Q}}$ and let²³*

$$\mathcal{Y}^{s,r} = H_b^{s,r}(M), \quad \mathcal{X}^{s,r} = \{u \in H_b^{s,r}(M) : (\mathcal{P} - i\mathcal{Q})u \in H_b^{s-1,r}(M)\},$$

where $\mathcal{P} - i\mathcal{Q}$ is a priori a map

$$\mathcal{P} - i\mathcal{Q} : H_b^{s,r}(M) \rightarrow H_b^{s-2,r}(M).$$

Then

$$\mathcal{P} - i\mathcal{Q} : \mathcal{X}^{s,r} \rightarrow \mathcal{Y}^{s-1,r}$$

is Fredholm.

2.1.3. *Initial value problems.* As already mentioned, the main issue with this argument using complex absorption that it does not guarantee the forward nature (in terms of supports) of the solution for a wave-like equation, although it does guarantee the correct microlocal structure. So now we assume that $\mathcal{P} \in \text{Diff}_b^2(M)$ and that there is a Lorentzian b-metric g such that

$$\mathcal{P} - \square_g \in \text{Diff}_b^1(M), \quad \mathcal{P} - \mathcal{P}^* \in \text{Diff}_b^0(M). \quad (2.14)$$

Then one can run a completely analogous argument using energy type estimates by restricting the domain we consider to be a manifold with corners, where the new boundary hypersurfaces are spacelike with respect to g , i.e. given by level sets of timelike functions. Such a possibility was mentioned in [41, Remark 2.6], though it was not described in detail as it was not needed there, essentially because the existence/uniqueness argument for forward solutions was given only for dilation invariant operators. The main difference between using complex absorption and adding boundary hypersurfaces is that the latter limit the Sobolev regularity one can use, with the most natural choice coming from energy estimates. However, a posteriori one can improve the result to better Sobolev spaces using propagation of singularities type results.

So assume now that $U \subset M$ is open, and we have two functions \mathfrak{t}_1 and \mathfrak{t}_2 in $C^\infty(M)$, both of which, restricted to U , are timelike (in particular have non-zero differential) near their respective 0-level sets H_1 and H_2 , and

$$\Omega = \mathfrak{t}_1^{-1}([0, \infty)) \cap \mathfrak{t}_2^{-1}([0, \infty)) \subset U.$$

Notice that the timelike assumption forces $d\mathfrak{t}_j$ to not lie in $N^*X = N^*\partial M$ (for its image in the b-cosphere bundle would be zero), and thus if the H_j intersect X , they do so transversally. We assume that the H_j intersect only away from X , and that they do so transversally, i.e. the differentials of \mathfrak{t}_j are independent at the intersection. Then Ω is a manifold with corners with boundary hypersurfaces H_1 , H_2 and X (all intersected with Ω). We however keep thinking of Ω as a domain in M . The role of the elliptic set of \mathcal{Q} is now played by ${}^bS_{H_j}^*M$, $j = 1, 2$. The *non-trapping* assumption becomes that

²³Note that $\mathcal{Y}^{s,r}$, $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being $\|u\|_{\mathcal{X}^{s,r}}^2 = \|u\|_{H_b^{s,r}(M)}^2 + \|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s-1,r}(M)}^2$. See Footnote 27.

- (1) all bicharacteristics in $\Sigma_\Omega = \Sigma \cap {}^bS_\Omega^*M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ flow (within Σ_Ω) to ${}^bS_{H_1}^*M \cup L_+$ in the forward direction (i.e. either enter ${}^bS_{H_1}^*M$ in finite time or tend to L_+) and to ${}^bS_{H_2}^*M \cup L_+$ in the backward direction,
- (2) and from any point in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ the bicharacteristics flow to ${}^bS_{H_2}^*M \cup L_-$ in the forward direction and to ${}^bS_{H_1}^*M \cup L_-$ in the backward direction;

see Figure 3. In particular, orienting the characteristic set by letting Σ_- be the *future oriented* and Σ_+ the *past oriented* part, dt_1 is future oriented, while dt_2 is past oriented.

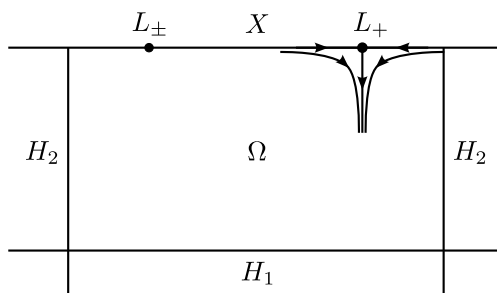


FIGURE 3. Setup for the discussion of the forward problem. Near the spacelike hypersurfaces H_1 and H_2 , which are the replacement for the complex absorbing operator \mathcal{Q} , we use standard (non-microlocal) energy estimates, and away from them, we use b-microlocal propagation results, including at the radial sets L_\pm . The bicharacteristic flow, in fact its projection to the base, is only indicated near L_+ ; near L_- , the directions of the flowlines are reversed.

On a manifold with corners, such as Ω , one can consider supported and extendible distributions; see [21, Appendix B.2] for the smooth boundary setting, with simple changes needed only for the corners setting, which is discussed e.g. in [42, Section 3]. Here we consider Ω as a domain in M , and thus its boundary face $X \cap \Omega$ is regarded as having a different character from the $H_j \cap \Omega$, i.e. the support/extendibility considerations do not arise at X – all distributions are regarded as acting on a subspace of C^∞ functions on Ω vanishing at X to infinite order, i.e. they are automatically extendible distributions at X . On the other hand, at H_j we consider both extendible distributions, acting on C^∞ functions vanishing to infinite order at H_j , and supported distributions, which act on all C^∞ functions (as far as conditions at H_j are concerned). For example, the space of supported distributions at H_1 extendible at H_2 (and at X , as we always tacitly assume) is the dual space of the subspace of $C^\infty(\Omega)$ consisting of functions vanishing to infinite order at H_2 and X (but not necessarily at H_1). An equivalent way of characterizing this space of distributions is that they are restrictions of elements of the dual of $\hat{C}^\infty(M)$ (consisting of C^∞ functions on M vanishing to infinite order at X) with support in $\mathfrak{t}_1 \geq 0$ to C^∞ functions on Ω which vanish to infinite order at X and H_2 , i.e. in the terminology of [21], restriction to $\Omega \setminus (H_2 \cup X)$. The main interest

is in spaces induced by the Sobolev spaces $H_b^{s,r}(M)$. For instance,²⁴

$$H_b^{s,r}(M)^{\bullet,-},$$

with the first superscript on the right denoting whether supported (\bullet) or extendible ($-$) distributions are discussed at H_1 , and the second the analogous property at H_2 , consists of restrictions of elements of $H_b^{s,r}(M)$ with support in $t_1 \geq 0$ to $\Omega \setminus (H_2 \cup X)$. Then elements of $C^\infty(\Omega)$ with the analogous vanishing conditions, so in the example vanishing to infinite order at H_2 and X , are dense in $H_b^{s,r}(M)^{\bullet,-}$; further the dual of $H_b^{s,r}(M)^{\bullet,-}$ is $H_b^{-s,-r}(M)^{-,\bullet}$ with respect to the L^2 (sesquilinear) pairing.

First we work locally. For this purpose it is convenient to introduce another timelike function \tilde{t}_j , of the same timelike nature as t_j (both future or both past oriented) and consider

$$\Omega_{[t_0, t_1]} = \mathfrak{t}_j^{-1}([t_0, \infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty, t_1]), \quad \Omega_{(t_0, t_1)} = \mathfrak{t}_j^{-1}((t_0, \infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty, t_1)),$$

and similarly on half-open, half-closed intervals. Thus, $\Omega_{[t_0, t_1]}$ becomes smaller as t_0 becomes larger or t_1 becomes smaller.

We then consider energy estimates on $\Omega_{[T_0, T_1]}$. In order to set up the following arguments, choose

$$T_- < T'_- < T_0, \quad T_1 < T'_+ < T_+,$$

and assume that $\Omega_{[T_-, T_+]}$ is compact, $\Omega_{[T_0, T_1]}$ is non-empty, t_j, \tilde{t}_j are timelike on $\Omega_{[T_-, T_+]}$, with the same timelike orientation. The energy estimates propagate estimates in the direction of either increasing or decreasing t_j . With the extendible/supported character of distributions at $t_j = T_+$ being irrelevant for this matter in the case being considered and thus dropped from the notation (so $-$ refers to extendibility at $t_j = T_0$), consider

$$\mathcal{P} : H_b^{s,r}(\Omega_{[T_0, T_+]})^- \rightarrow H_b^{s-2,r}(\Omega_{[T_0, T_+]})^-, \quad s, r \in \mathbb{R}.$$

The energy estimate, with backward propagation in t_j , from $\tilde{\mathfrak{t}}_j^{-1}([T'_+, T_+])$, in this setting takes the form:

Lemma 2.4. *Let $r \in \mathbb{R}$. There is $C > 0$ such that for $u \in H_b^{2,r}(\Omega_{[T_0, T_+]})^-$,*

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0, T_1]})^-} \leq C \left(\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0, T_+]})^-} + \|u\|_{H_b^{1,r}(\Omega_{[T_0, T_+]}) \cap \tilde{\mathfrak{t}}_j^{-1}([T'_+, T_+])^-} \right) \quad (2.15)$$

This also holds with \mathcal{P} replaced by \mathcal{P}^ , acting on the same spaces.*

Remark 2.5. The lemma is also valid if one replaces $\mathfrak{t}_j^{-1}([t_0, \infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0}, \infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0}, \infty))$, and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty, t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty, t_{j,1}]) \cap \tilde{\mathfrak{t}}_k^{-1}((-\infty, t_{k,1}])$ in the definition of $\Omega_{[t_0, t_1]}$, i.e. regarding t_j and/or \tilde{t}_j as vector valued. (One can also have more than two such functions.) To see this, simply replace $\chi(t_j)$ by $\chi_j(t_j)\chi_k(t_k)$, and analogously with $\tilde{\chi}$ in the definition of V in (2.16). Then $\chi' \tilde{\chi} \tau^\alpha A^\sharp$ is replaced by $\chi'_j \chi_k \tilde{\chi}_j \tilde{\chi}_k \tau^\alpha A^\sharp + \chi_j \chi'_k \tilde{\chi}_j \tilde{\chi}_k \tau^\alpha A^\sharp$, etc. Then χ'_j can still be used to dominate χ_j , while the terms in which $\tilde{\chi}_j$ is differentiated have support where \tilde{t}_j is in $(T'_{+,j}, T_{+,j})$, so the control region on the right hand side of (2.15) is the union of these sets.

In our application this situation arises as we need the estimates on $\mathfrak{t}_1^{-1}([T_0, T_1]) \cap \mathfrak{t}_2^{-1}([0, \infty))$ and $\mathfrak{t}_1^{-1}([0, \infty)) \cap \mathfrak{t}_2^{-1}([T_0, T_1])$, with $T_0 = 0$, $T_1 > 0$ small. For instance,

²⁴Notice that the Sobolev norm is of completely different nature at X than at the H_j , namely the derivatives are based on complete, rather than incomplete, vector fields: $\mathcal{V}_b(M)$ is being restricted to Ω , so one obtains vector fields tangent to X but not to the H_j .

in the latter case t_2 plays the role of t_j above, while $-t_1$ and t_2 play the role of \tilde{t}_j and \tilde{t}_k ; see Figure 4.

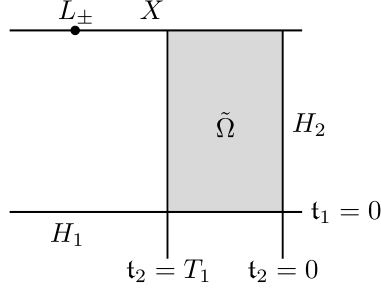


FIGURE 4. A domain $\tilde{\Omega} = t_2^{-1}([0, \infty)) \cap ((-t_1)^{-1}((-\infty, 0]) \cap t_2^{-1}((-\infty, T_1]))$ on which we will apply the energy estimate (2.15).

Proof. To see (2.15), one proceeds as in [41, Section 3.3] and considers

$$V = -i\chi(t_j)\tilde{\chi}(\tilde{t}_j)\tau^\alpha W \quad (2.16)$$

with $W = G(t_j, \cdot)$ a timelike vector field and with $\chi, \tilde{\chi} \in C^\infty(\mathbb{R})$, both non-negative, to be specified. Then choosing a Riemannian b-metric \tilde{g} ,

$$-i(V^*\square_g - \square_g^*V) = {}^b d_{\tilde{g}}^* C^b {}^b d,$$

with the subscript on the adjoint on the right hand side denoting the metric with respect to which it is taken, ${}^b d : C^\infty(M) \rightarrow C^\infty(M; {}^b T^*M)$ being the b-differential, and with

$$C^b = \chi' \tilde{\chi} \tau^\alpha A^\sharp + \chi \tilde{\chi}' \tau^\alpha \tilde{A}^\sharp + \chi \tilde{\chi} \tau^\alpha R^b$$

where $A^\sharp, \tilde{A}^\sharp$ and R^b are bundle endomorphisms of ${}^{\mathbb{C}b} T^*M$ and $A^\sharp, \tilde{A}^\sharp$ are positive definite. Proceeding further, replacing \square_g by \mathcal{P} , one has

$$\begin{aligned} -i(V^*\mathcal{P} - \mathcal{P}^*V) &= {}^b d_{\tilde{g}}^* C^\sharp {}^b d + (\tilde{E}_1)_{\tilde{g}}^* \tau^\alpha \chi \tilde{\chi} {}^b d + {}^b d_{\tilde{g}}^* \tau^\alpha \chi \tilde{\chi} \tilde{E}_2, \\ C^\sharp &= \chi' \tilde{\chi} \tau^\alpha A^\sharp + \chi \tilde{\chi}' \tau^\alpha \tilde{A}^\sharp + \chi \tilde{\chi} \tau^\alpha \tilde{R}^\sharp \end{aligned} \quad (2.17)$$

with \tilde{E}_j bundle maps from the trivial bundle over M to ${}^{\mathbb{C}b} T^*M$, $A^\sharp, \tilde{A}^\sharp$ as before, and \tilde{R}^\sharp a bundle endomorphism of ${}^{\mathbb{C}b} T^*M$, as follows by expanding

$$-i(V^*(\mathcal{P} - \square_g) - (\mathcal{P}^* - \square_g^*)V)$$

using that $\mathcal{P} - \square_g \in \text{Diff}_b^1(M)$. We regard the second term on the right hand side of (2.17) as the one requiring a priori control by $\|u\|_{H_b^{1,r}(\Omega_{[T_0, T_+]} \cap \tilde{t}_j^{-1}([T'_+, T_+]))^-}$; we achieve this by making $\tilde{\chi}$ supported in $(-\infty, T_+)$, identically 1 near $(-\infty, T'_+]$, so $d\tilde{\chi}$ is supported in (T'_+, T_+) . Now making $\chi' \geq 0$ large relative to χ on $\text{supp}(\chi\tilde{\chi})$, as in²⁵ [41, Equation (3.27)], allows one to dominate all terms without derivatives of χ , proving the energy estimate (2.15) when one takes $\alpha = -2r$. \square

²⁵In [41, Equation (3.27)] the sign of χ' is opposite, as the estimate is propagated in the opposite direction.

Propagating in the forward direction, from $\mathfrak{t}_j^{-1}([T_-, T'_-])$, where now $-$ denotes the character of the space at T_1 (so $-$ refers to extendibility at $\mathfrak{t}_j = T_1$)

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0, T_1]}^-)} \leq C \left(\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_-, T_1]}^-)} + \|u\|_{H_b^{1,r}(\Omega_{[T_-, T_1]} \cap \mathfrak{t}_j^{-1}([T_-, T'_-])^-) \right) \quad (2.18)$$

In particular, for u supported in $\mathfrak{t}_j \geq T_0$, the last estimate becomes, with the first superscript on the right denoting whether supported (\bullet) or extendible ($-$) distributions are discussed at $\mathfrak{t} = T_0$, the second superscript the same at $\mathfrak{t} = T_1$,

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0, T_1]}^{\bullet,-})} \leq C \|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0, T_1]}^{\bullet,-})}, \quad (2.19)$$

when

$$\mathcal{P} : H_b^{s,r}(\Omega_{[T_0, T_1]}^{\bullet,-}) \rightarrow H_b^{s-2,r}(\Omega_{[T_0, T_1]}^{\bullet,-})$$

and $u \in H_b^{2,r}(\Omega_{[T_0, T_1]}^{\bullet,-})$. To summarize, we state both this and (2.15) in terms of these supported spaces:

Corollary 2.6. *Let $r, \tilde{r} \in \mathbb{R}$. For $u \in H_b^{2,r}(\Omega_{[T_0, T_1]}^{\bullet,-})$, one has*

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0, T_1]}^{\bullet,-})} \leq C \|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0, T_1]}^{\bullet,-})}, \quad (2.20)$$

while for $v \in H_b^{2,\tilde{r}}(\Omega_{[T_0, T_1]}^{-,\bullet})$, the estimate

$$\|v\|_{H_b^{1,\tilde{r}}(\Omega_{[T_0, T_1]}^{-,\bullet})} \leq C \|\mathcal{P}^*v\|_{H_b^{0,\tilde{r}}(\Omega_{[T_0, T_1]}^{-,\bullet})} \quad (2.21)$$

holds.

A duality argument, combined with propagation of singularities, thus gives:

Lemma 2.7. *Let $s \geq 0, r \in \mathbb{R}$. Then there is $C > 0$ with the following property.*

If $f \in H_b^{s-1,r}(\Omega_{[T_0, T_1]}^{\bullet,-})$, then there exists $u \in H_b^{s,r}(\Omega_{[T_0, T_1]}^{\bullet,-})$ such that $\mathcal{P}u = f$ and

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0, T_1]}^{\bullet,-})} \leq C \|f\|_{H_b^{s-1,r}(\Omega_{[T_0, T_1]}^{\bullet,-})}.$$

Remark 2.8. As in Remark 2.5, the lemma also remains valid if one replaces $\mathfrak{t}_j^{-1}([t_0, \infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0}, \infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0}, \infty))$, and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty, t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty, t_{j,1}]) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty, t_{k,1}])$ in the definition of $\Omega_{[t_0, t_1]}$, provided that the \mathfrak{t}_j have linearly independent differentials on their joint zero set, and similarly for the $\tilde{\mathfrak{t}}_j$. The place where this linear independence is used (the energy estimate above does not need this) is for the continuous Sobolev extension map, valid on manifolds with corners, see [42, Section 3].

Proof. We work on the slightly bigger region $\tilde{\Omega}' = \Omega_{[T'_-, T'_+]}$, applying the energy estimates with T_0 replaced by T'_- , T_1 replaced by T'_+ . First, by the supported property at $\mathfrak{t}_j = T_0$, one can regard f as an element of $H_b^{s-1,r}(\Omega_{[T'_-, T_1]}^{\bullet,-})$ with support in $[T_0, T_1]$. Let

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T'_-, T'_+]}^{\bullet,-}) \subset H_b^{-1,r}(\Omega_{[T'_-, T'_+]}^{\bullet,-})$$

be an extension of f , so \tilde{f} is supported in $\Omega_{[T_0, T'_+]}$, and restricts to f ; by the definition spaces of extendible distributions as quotients of spaces of distributions on a larger space, see [21, Appendix B.2], we can assume

$$\|\tilde{f}\|_{H_b^{s-1,r}(\Omega_{[T'_-, T'_+]}^{\bullet,-})} \leq 2 \|f\|_{H_b^{s-1,r}(\Omega_{[T'_-, T_1]}^{\bullet,-})}. \quad (2.22)$$

By (2.15) applied with \mathcal{P} replaced by \mathcal{P}^* , $\tilde{r} = -r$,

$$\|\phi\|_{H_b^{1,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \leq C \|\mathcal{P}^* \phi\|_{H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}}, \quad (2.23)$$

for $\phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$. Correspondingly, by the Hahn-Banach theorem, there exists

$$\tilde{u} \in (H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet})^* = H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

such that

$$\langle \mathcal{P}\tilde{u}, \phi \rangle = \langle \tilde{u}, \mathcal{P}^* \phi \rangle = \langle \tilde{f}, \phi \rangle, \quad \phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet},$$

and

$$\|\tilde{u}\|_{H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \leq C \|\tilde{f}\|_{H_b^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}}. \quad (2.24)$$

One can regard \tilde{u} as an element of $H_b^{0,r}(\Omega_{[T_-,T'_+]})^{\bullet,-}$ with support in $[T'_-,T'_+]$, with \tilde{f} similarly extended; then $\langle \mathcal{P}\tilde{u}, \phi \rangle = \langle \tilde{f}, \phi \rangle$ for $\phi \in \dot{C}_c^\infty(\Omega_{(T_-,T'_+)})$ (here the dot over C^∞ refers to infinite order vanishing at $X = \partial M!$), so $\mathcal{P}\tilde{u} = \tilde{f}$ in distributions. Since \tilde{u} vanishes on (T_-,T'_-) , and

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T_-,T'_+]})^{\bullet,-},$$

propagation of singularities applied on $\Omega_{(T_-,T'_+)}$ (which has only the boundary $\partial M = X$) gives that $\tilde{u} \in H_{b,\text{loc}}^{s,r}(\Omega_{(T_-,T'_+)})$ (i.e. here we are ignoring the two boundaries, $\mathfrak{t}_j = T_-, T'_+$, not making a uniform statement there, but we are not ignoring $\partial M = X$). In addition, for $\chi, \tilde{\chi} \in C_c^\infty(\Omega_{(T_-,T'_+)})$, $\tilde{\chi} \equiv 1$ on $\text{supp } \chi$, we have the estimate

$$\|\chi \tilde{u}\|_{H_b^{s,r}(\Omega_{[T_-,T'_+]})} \leq C \left(\|\tilde{\chi} \mathcal{P}\tilde{u}\|_{H_b^{s-1,r}(\Omega_{[T_-,T'_+]})} + \|\tilde{\chi} \tilde{u}\|_{H_b^{0,r}(\Omega_{[T_-,T'_+]})} \right). \quad (2.25)$$

In view of the support property of \tilde{u} , this gives that restricting to $(T_-,T_1]$, we obtain an element of $H_b^{s,r}(\Omega_{(T_-,T_1]})^-$, with support in $[T_0,T_1]$, i.e. an element of $H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. The desired estimate follows from (2.24), controlling the second term of the right hand side of (2.25), and (2.22) as well as using $\mathcal{P}\tilde{u} = \tilde{f}$. \square

At this point, u given by Lemma 2.7 is not necessarily unique. However:

Lemma 2.9. *Let $s, r \in \mathbb{R}$. If $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ is such that $\mathcal{P}u = 0$, then $u = 0$.*

Proof. Propagation of singularities, as in the proof of Lemma 2.7, regarding u as a distribution on (T_-,T_1) with support in $[T_0,T_1]$ gives that $u \in H_{b,\text{loc}}^{\infty,r}(\Omega_{(T_-,T_1)})$. Taking $T_0 < T'_1 < T_1$, letting $u' = u|_{[T_0,T'_1]}$, (2.20) shows that $u' = 0$. Since T'_1 is arbitrary, this shows $u = 0$. \square

Corollary 2.10. *Let $s \geq 0$, $r \in \mathbb{R}$. Then there is $C > 0$ with the following property.*

If $f \in H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists a unique $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$.

Further, this unique u satisfies

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C \|f\|_{H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Proof. Existence is Lemma 2.7, uniqueness is linearity plus Lemma 2.9, which together with the estimate in Lemma 2.7 prove the corollary. \square

Corollary 2.11. *Let $s \geq 0$, $r, \tilde{r} \in \mathbb{R}$.*

For $u \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$,

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}} \leq C \|\mathcal{P}u\|_{H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}}, \quad (2.26)$$

*while for $v \in H_b^{s,\tilde{r}}(\Omega_{[T_0, T_1]})^{-,\bullet}$ with $\mathcal{P}^*v \in H_b^{s-1,\tilde{r}}(\Omega_{[T_0, T_1]})^{-,\bullet}$,*

$$\|v\|_{H_b^{s,\tilde{r}}(\Omega_{[T_0, T_1]})^{-,\bullet}} \leq C \|\mathcal{P}^*v\|_{H_b^{s-1,\tilde{r}}(\Omega_{[T_0, T_1]})^{-,\bullet}}. \quad (2.27)$$

Remark 2.12. Again, this estimate remains valid for vector valued \mathfrak{t}_j and $\tilde{\mathfrak{t}}_j$, see Remarks 2.5 and 2.8, under the linear independence condition of the latter.

Proof. It suffices to consider (2.26). Let $f = \mathcal{P}u \in H_b^{-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$, and let $u' \in H_b^{0,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$ be given by Corollary 2.10. In view of the uniqueness statement of Corollary 2.10, $u = u'$. Then the estimate of Corollary 2.10 proves the claim. \square

This yields the following propagation of singularities type result:

Proposition 2.13. *Let $s \geq 0$, $r \in \mathbb{R}$. If $u \in H_b^{-\infty,-\infty}(\Omega_{[T_0, T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$, then $u \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$.*

If instead $u \in H_b^{-\infty,-\infty}(\Omega_{[T_0, T_1]})^{-,-}$ with $\mathcal{P}u \in H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{-,-}$ and for some $\tilde{T}_0 > T_0$, $u \in H_b^{s,r}(\Omega_{[T_0, T_1]} \setminus \Omega_{(\tilde{T}_0, T_1)})^{-,-}$, then $u \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{-,-}$.

Remark 2.14. One can ‘mix and match’ the two parts of the lemma in the setting of Remark 2.5, with say a supportedness condition at $\tilde{\mathfrak{t}}_j$, and only an extendibility assumption at \mathfrak{t}_k , but with $H_b^{s,r}$ membership assumption on u in $\Omega_{[T_0, T_1]} \setminus \tilde{\mathfrak{t}}_k^{-1}((-\infty, \tilde{T}_1))$, $\tilde{T}_1 < T_1$, with a completely analogous argument. For instance, in the setting of Figure 4, one gets the regularity under supportedness assumptions at H_1 , just extendibility at $\mathfrak{t}_2 = T_1$, but a priori regularity for $\mathfrak{t}_2 \in (\tilde{T}_1, T_1)$.

Proof. Let $u' \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$ be the unique solution in $H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$ of $\mathcal{P}u' = f$ where $f = \mathcal{P}u \in H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$; we obtain u' by applying the existence part of Corollary 2.11. Then $u, u' \in H_b^{-\infty,-\infty}(\Omega_{[T_0, T_1]})^{\bullet,-}$ and $\mathcal{P}(u - u') = 0$. Applying Lemma 2.9, we conclude that $u = u'$, which completes the proof of the first part.

For the second part, let $\chi \in C^\infty(\mathbb{R})$ be supported in (T_0, ∞) , identically 1 near $[\tilde{T}_0, \infty)$, and consider $u' = (\chi \circ \mathfrak{t}_j)u \in H_b^{1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$, with the support property arising from the vanishing of χ near T_0 . Then $\mathcal{P}u' = [\mathcal{P}, (\chi \circ \mathfrak{t}_j)]u + (\chi \circ \mathfrak{t}_j)\mathcal{P}u$. Now the first term on the right hand side is in $H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$ as on the support of $d\chi$, which is in $\Omega_{[T_0, T_1]} \setminus \Omega_{(\tilde{T}_0, T_1)}$, u is in $H_b^{s,r}$, and the commutator is first order, while the second term is in the desired space since $\mathcal{P}u \in H_b^{s-1,r}(\Omega_{[T_0, T_1]})^{-,-}$, and as for u itself, the cutoff improves the support property. Thus, the first part of the lemma is applicable, giving that $\chi u \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{\bullet,-}$. Since $(1 - \chi)u \in H_b^{s,r}(\Omega_{[T_0, T_1]})^{-,-}$ by the a priori assumption, the conclusion follows. \square

We take $T_0 = 0$ and thus consider, for $s \geq 0$,

$$\mathcal{P} : H_b^{s,r}(\Omega)^{\bullet,-} \rightarrow H_b^{s-2,r}(\Omega)^{\bullet,-} \quad (2.28)$$

and

$$\mathcal{P}^* : H_b^{s,r}(\Omega)^{-,\bullet} \rightarrow H_b^{s-2,r}(\Omega)^{-,\bullet}. \quad (2.29)$$

In combination with the real principal type propagation results and Proposition 2.1 this yields under the non-trapping assumptions, much as in the complex absorbing case, that²⁶

$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}} + C\|u\|_{H_b^{0,r}(\Omega)^{\bullet,-}}, \quad \beta r < -1/2, \quad s > 0, \quad (2.30)$$

and

$$\|u\|_{H_b^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C\|\mathcal{P}^*u\|_{H_b^{s-1,\tilde{r}}(\Omega)^{-,\bullet}} + C\|u\|_{H_b^{0,\tilde{r}}(\Omega)^{-,\bullet}}, \quad \beta\tilde{r} > s-1/2, \quad s > 0. \quad (2.31)$$

We could proceed as in the complex absorption case to make the space on the left hand side include compactly into the ‘error term’ on the right hand using the normal operators. As this imposes some constraints, cf. (2.13), which together with the requirements of the energy estimates, namely that the Sobolev order is ≥ 0 , mean that we would get slightly too strong restrictions on s , see Remark 2.19, we proceed instead with a direct energy estimate. We thus assume that Ω is sufficiently small so that there is a boundary defining function τ of M with $\frac{d\tau}{\tau}$ timelike on Ω , of the same timelike character as \mathbf{t}_2 , opposite to \mathbf{t}_1 . As explained in [41, Section 7], in this case there is $C > 0$ such that for $\text{Im } \sigma > C$, $\hat{P}(\sigma)$ is necessarily invertible.

The energy estimate is:

Lemma 2.15. *There exists $r_0 < 0$ such that for $r \leq r_0$, $-\tilde{r} \leq r_0$, there is $C > 0$ such that for $u \in H_b^{2,r}(\Omega)^{\bullet,-}$, $v \in H_b^{2,\tilde{r}}(\Omega)^{-,\bullet}$, one has*

$$\begin{aligned} \|u\|_{H_b^{1,r}(\Omega)^{\bullet,-}} &\leq C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega)^{\bullet,-}}, \\ \|v\|_{H_b^{1,\tilde{r}}(\Omega)^{-,\bullet}} &\leq C\|\mathcal{P}^*v\|_{H_b^{0,\tilde{r}}(\Omega)^{-,\bullet}}. \end{aligned} \quad (2.32)$$

Proof. We run the argument of Lemma 2.4 globally on Ω using a timelike vector field (e.g. starting with $W = G(\frac{d\tau}{\tau}, \cdot)$) that has, as a multiplier, a sufficiently large positive power $\alpha = -2r$ of τ , i.e. replacing (2.16) by

$$V = -i\tau^\alpha W.$$

Then the term with τ^α differentiated (which in (2.17) is included in the \tilde{R}^\sharp term), and thus possessing a factor of α , is used to dominate the other, ‘error’, terms in (2.17), completing the proof of the lemma as in Lemma 2.4. \square

This can be used as in Lemma 2.7 to provide solvability, and using the propagation of singularities, which in this case includes the use of Proposition 2.1, noting that $s - 1/2 > \beta r$ is automatically satisfied, improved regularity. In particular, we obtain the following analogues of Corollaries 2.10-2.11.

Corollary 2.16. *There is $r_0 < 0$ such that for $r \leq r_0$ and for $s \geq 0$ there is $C > 0$ with the following property.*

If $f \in H_b^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_b^{s,r}(\Omega)^{\bullet,-}$ such that $\mathcal{P}u = f$. Further, this unique u satisfies

$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \leq C\|f\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}}.$$

²⁶In fact, the error term on the right hand side can be taken to be supported in a smaller region, since at H_1 in the first case and at H_2 in the second, there are no error terms due to the energy estimates (2.20), applied with \mathcal{P}^* in place of \mathcal{P} in the second case.

Corollary 2.17. *There is $r_0 < 0$ such that if $r < r_0$, $-\tilde{r} < r_0$ and $s \geq 0$ then there is $C > 0$ such that the following holds.*

For $u \in H_b^{s,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u \in H_b^{s-1,r}(\Omega)^{\bullet,-}$, one has

$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \leq C \|\mathcal{P}u\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}}, \quad (2.33)$$

*while for $v \in H_b^{s,\tilde{r}}(\Omega)^{-,\bullet}$ with $\mathcal{P}^*v \in H_b^{s-1,\tilde{r}}(\Omega)^{-,\bullet}$, one has*

$$\|v\|_{H_b^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C \|\mathcal{P}^*v\|_{H_b^{s-1,\tilde{r}}(\Omega)^{-,\bullet}}. \quad (2.34)$$

We restate Corollary 2.16 as an invertibility statement.

Theorem 2.18. *There is $r_0 < 0$ with the following property. Suppose $s \geq 0$, $r \leq r_0$, and let²⁷*

$$\mathcal{Y}^{s,r} = H_b^{s,r}(\Omega)^{\bullet,-}, \quad \mathcal{X}^{s,r} = \{u \in H_b^{s,r}(\Omega)^{\bullet,-} : \mathcal{P}u \in H_b^{s-1,r}(\Omega)^{\bullet,-}\},$$

where \mathcal{P} is a priori a map $\mathcal{P} : H_b^{s,r}(\Omega)^{\bullet,-} \rightarrow H_b^{s-2,r}(\Omega)^{\bullet,-}$. Then

$$\mathcal{P} : \mathcal{X}^{s,r} \rightarrow \mathcal{Y}^{s-1,r}$$

is a continuous, invertible map, with continuous inverse.

Remark 2.19. Using normal operators as in the discussion leading to Proposition 2.3, one would get the following statement: Suppose $s > 1$, $s - 3/2 > \beta r$. Then with $\mathcal{X}^{s,r}$, $\mathcal{Y}^{s,r}$ as above, $\mathcal{P} : \mathcal{X}^{s,r} \rightarrow \mathcal{Y}^{s,r}$ is Fredholm. Here the main loss is that one needs to assume $s > 1$; this is done since in the argument one needs to take s' with $s' + 1 < s$ in order to transition the normal operator estimates from $N(\mathcal{P})u$ to $\mathcal{P}u$ and still have a compact inclusion, but the normal operator estimates need $s' \geq 0$ as, due to the boundary H_2 , they are again based on energy estimates. Using the direct global energy estimate eliminates this loss, which is an artifact of combining local energy estimates with the b-theory. In particular, in the complex absorption setting, this problem does not arise, but on the other hand, one need not have the forward support property of the solution.

The results of [41] then are immediately applicable to obtain an expansion of the solutions; the main point of the following theorem being the elimination of the losses in differentiability in [41, Proposition 3.5] due to Proposition 2.1.

Theorem 2.20. *(Strengthened version of [41, Proposition 3.5].) Let M be a manifold with a non-trapping b-metric g as above, with boundary X and let τ be a boundary defining function, \mathcal{P} as in (2.14). Suppose the domain Ω is as defined above, and $\frac{d\tau}{\tau}$ timelike.*

Let σ_j be the poles of $\widehat{\mathcal{P}}^{-1}$, and let ℓ be such that $\text{Im } \sigma_j + \ell \notin \mathbb{N}$ for all j . Let $\phi \in C^\infty(\mathbb{R})$ be such that $\text{supp } \phi \subset (0, \infty)$, and $\phi \circ \mathbf{t}_1 \equiv 1$ near $X \cap \Omega$. Then for $s > 1/2 + \beta\ell$, there are $m_{j\ell} \in \mathbb{N}$ such that solutions of $\mathcal{P}u = f$ with $f \in H_b^{s-1,\ell}(\Omega)^{\bullet,-}$, and with $u \in H_b^{s_0,r_0}(\Omega)^{\bullet,-}$, $s \geq s_0 \geq 1$, $s_0 - 1/2 > \beta r_0$ satisfy that for some $a_{j\ell\kappa} \in C^\infty(X \cap \Omega)$,

$$u' = u - \sum_j \sum_{l \in \mathbb{N}} \sum_{\kappa \leq m_{j\ell}} \tau^{i\sigma_j + l} (\log \tau)^\kappa (\phi \circ \mathbf{t}_1) a_{j\ell\kappa} \in H_b^{s,\ell}(\Omega)^{\bullet,-}, \quad (2.35)$$

²⁷Note that $\mathcal{Y}^{s,r}$, $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being $\|u\|_{\mathcal{X}^{s,r}}^2 = \|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}}^2 + \|\mathcal{P}u\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}}^2$, as follows by the continuity of \mathcal{P} as a map $H_b^{s,r}(\Omega)^{\bullet,-} \rightarrow H_b^{s-2,r}(\Omega)^{\bullet,-}$ and the completeness of the b-Sobolev spaces $H_b^{s,r}(\Omega)^{\bullet,-}$.

where the sum is understood to be over a finite set with $-\operatorname{Im} \sigma_j + l < \ell$.

Here the (semi)norms of both $a_{j\kappa}$ in $C^\infty(X \cap \Omega)$ and u' in $H_b^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_b^{s-1,\ell}(\Omega)^{\bullet,-}$.

The analogous result also holds if f possesses an expansion modulo $H_b^{s-1,\ell}(\Omega)^{\bullet,-}$, namely

$$f = f' + \sum_j \sum_{\kappa \leq m'_j} \tau^{\alpha_j} (\log \tau)^\kappa (\phi \circ \mathbf{t}_1) a_{j\kappa},$$

with $f' \in H_b^{s-1,\ell}(\Omega)^{\bullet,-}$ and $a_{j\kappa} \in C^\infty(X \cap \Omega)$, where terms corresponding to the expansion of the f are added to (2.35) in the sense of the extended union of index sets [32, Section 5.18], recalled below in Definition 2.30.

Remark 2.21. Here the factor $\phi \circ \mathbf{t}_1$ is added to cut off the expansion away from H_1 , thus assuring that u' is in the indicated space (a supported distribution).

Also, the sum over l is generated by the lack of dilation invariance of \mathcal{P} . If we take ℓ such that $-\operatorname{Im} \sigma_j > \ell - 1$ for all j then all the terms in the expansion arise directly from the resonances, thus $l = 0$ and $m_{j0} + 1$ is the order of the pole of $\widehat{\mathcal{P}}^{-1}$ at σ_j , with the $a_{j0\kappa}$ being resonant states.

Proof. First assume that $-\operatorname{Im} \sigma_j > \ell$ for every j ; thus there are no terms subtracted from u in (2.35). We proceed as in [41, Proposition 3.5], but use the propagation of singularities, in particular Propositions 2.1 and 2.13, to eliminate the losses. First, by the propagation of singularities, using $s_0 - 1/2 > \beta r_0$ and $s \geq s_0$, $s \geq 0$,

$$u \in H_b^{s,r_0}(\Omega)^{\bullet,-}.$$

Thus, as $\mathcal{P} - N(\mathcal{P}) \in \tau \operatorname{Diff}_b^2(M)$,

$$N(\mathcal{P})u = f - \tilde{f}, \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_b^{s-2,r_0+1}(\Omega)^{\bullet,-} \quad (2.36)$$

Applying²⁸ [41, Lemma 3.1] (using $s \geq s_0 \geq 1$), which is the lossless version of [41, Proposition 3.5] in the dilation invariant case, one obtains (2.35) with ℓ replaced by $\ell' = \min(\ell, r_0 + 1)$ except that $u = u' \in H_b^{s-1,\ell'}(\Omega)^{\bullet,-}$ corresponding to the \tilde{f} term in $N(\mathcal{P})u$ rather than $u = u' \in H_b^{s,\ell'}(\Omega)^{\bullet,-}$ as desired. However, using $\mathcal{P}u = f \in H_b^{s-1,\ell'}(\Omega)^{\bullet,-}$, we deduce by the propagation of singularities, using $s > \beta \ell' + 1/2$, $s \geq 0$, that $u = u' \in H_b^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \leq r_0 + 1$, we have proved (2.35). Otherwise we iterate, replacing r_0 by $r_0 + 1$. We thus reach the conclusion, (2.35), in finitely many steps.

If there are j such that $-\operatorname{Im} \sigma_j < \ell$, then in the first step, when using [41, Lemma 3.1], we obtain the partial expansion u_1 corresponding to $\ell' = \min(\ell, r_0 + 1)$ in place of ℓ ; here we may need to decrease ℓ' by an arbitrarily small amount to make sure that ℓ' is not $-\operatorname{Im} \sigma_j$ for any j . Further, the terms of the partial expansion are annihilated by $N(\mathcal{P})$, so u' satisfies

$$\mathcal{P}u' = \mathcal{P}u - N(\mathcal{P})u_1 - (\mathcal{P} - N(\mathcal{P}))u_1 \in H_b^{s-1,\ell'}(\Omega)^{\bullet,-}$$

as $(\mathcal{P} - N(\mathcal{P}))u_1 \in H_b^{\infty,r_0+1}(\Omega)^{\bullet,-}$ in fact due to the conormality of u_1 and $\mathcal{P} - N(\mathcal{P}) \in \tau \operatorname{Diff}_b^2(M)$. Correspondingly, the propagation of singularities result is

²⁸In [41], Lemma 3.1 is stated on the normal operator space M_∞ , which does not have a boundary face corresponding to H_2 , i.e. $S_2 \times (0, \infty)$, with complex absorption instead. However, given the analysis on $X \cap \Omega$ discussed above, all the arguments go through essentially unchanged: this is a Mellin transform and contour deformation argument.

applicable as above to conclude that $u' \in H_b^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \leq r_0 + 1$ we are done. Otherwise we have better information on \tilde{f} in the next step, namely

$$\tilde{f} = (\mathcal{P} - N(\mathcal{P}))u = (\mathcal{P} - N(\mathcal{P}))u' + (\mathcal{P} - N(\mathcal{P}))u_1$$

with the first term in $H_b^{s-2,r_0+1}(\Omega)^{\bullet,-}$ (same as in the case first considered above, without relevant resonances), while the expansion of u_1 shows that $(\mathcal{P} - N(\mathcal{P}))u_1$ has a similar expansion, but with an extra power of τ (i.e. $\tau^{i\sigma_j}$ is shifted to $\tau^{i\sigma_j+1}$). We can now apply [41, Lemma 3.1] again; in the case of the terms arising from the partial expansion, u_1 , there are now new terms corresponding to shifting the powers $\tau^{i\sigma_j}$ to $\tau^{i\sigma_j+1}$, as stated in the referred Lemma, and possibly causing logarithmic terms if $\sigma_j - i$ is also a pole of $\widehat{\mathcal{P}}^{-1}$. Iterating in the same manner proves the theorem when $f \in H_b^{s-1,\ell}(\Omega)^{\bullet,-}$. When f has an expansion modulo $H_b^{s-1,\ell}(\Omega)^{\bullet,-}$, the same argument works; [41, Lemma 3.1] gives the terms with the extended union, which then further generate additional terms due to $\mathcal{P} - N(\mathcal{P})$, just as the resonance terms did. \square

There is one problem with this theorem for the purposes of semilinear equations: the resonant terms with $\text{Im } \sigma_j \geq 0$ which give rise to unbounded, or at most just bounded, terms in the expansion which become larger when one takes powers of these, or when one iteratively applies the \mathcal{P}^{-1} (with the latter being the only issue if $\text{Im } \sigma_j = 0$ and the pole is simple).

As an example of the limitations this causes, consider an asymptotically de Sitter space (\tilde{M}, \tilde{g}) . We then blow up a point p at the future boundary \tilde{X}_+ , as discussed in the introduction, to obtain the analogue of the static model of de Sitter space $M = [\tilde{M}; p]$ with the pulled back metric g , which is a b-metric near the front face (but away from the side face); let $\mathcal{P} = \square_g - \lambda$. If \tilde{M} is actual de Sitter space, then M is the actual static model; otherwise the metric of the asymptotically de Sitter space, frozen at p , induces a de Sitter metric, g_0 , which is well defined at the front face of the blow up M (but away from its side faces) as a b-metric. In particular, the resonances in the ‘static region’ of any asymptotically de Sitter space are the same as in the static model of actual de Sitter space.

On actual de Sitter space, the poles of $\widehat{\mathcal{P}}^{-1}$ are those on the hyperbolic space in the interior of the light cone equipped by a potential, as described in [43, Lemma 7.11], or indeed in [41, Proposition 4.2] where essentially the present notation is used.²⁹ As shown in Corollary 7.18 of [43], converted to our notation, the only possible poles are in

$$i\hat{s}_\pm(\lambda) - i\mathbb{N}, \quad \hat{s}_\pm(\lambda) = -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}. \quad (2.37)$$

In particular, when $\lambda = m^2$, $m > 0$, then we conclude:

Lemma 2.22. *For $m > 0$, $\mathcal{P} = \square_g - m^2$, all poles of $\widehat{\mathcal{P}}^{-1}$ have strictly negative imaginary part.*

In other words, for small mass $m > 0$, there are no resonances σ of the Klein-Gordon operator with $\text{Im } \sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (2.35) no longer has a constant term. Let us fix such $m > 0$ and $\epsilon_0 > 0$, which ensures that for $0 < \epsilon < \epsilon_0$, the only term in the asymptotic expansion (2.35), when

²⁹In [43, Lemma 7.11] $-\sigma^2$ plays the same role as σ^2 here or in [41, Proposition 4.2].

$s > 1/2 + \epsilon$ and $f \in H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$, is the ‘remainder’ term $u' \in H_b^{s, \epsilon}(\Omega)^{\bullet, -}$. Here we use that $\beta = 1$ in de Sitter space, hence on an asymptotically de Sitter space, see [41, Section 4.4], in particular the second displayed equation after Equation (4.16) there which computes β in accordance with Remark 2.2.

Being interested in finding *forward solutions* to (non-linear) wave equations on *asymptotically de Sitter spaces*, we now define the forward solution operator

$$S_{\text{KG}}: H_b^{s-1, \epsilon}(\Omega)^{\bullet, -} \rightarrow H_b^{s, \epsilon}(\Omega)^{\bullet, -} \quad (2.38)$$

using Theorems 2.18 and 2.20.

Remark 2.23. If $\tilde{M} \subset M$ is an asymptotically de Sitter space with global time function t , $\tau = e^{-t}$ is the defining function for future infinity, and the domain Ω is such that $\Omega \cap \tilde{M} = \{\tau < \tau_0\}$, then S_{KG} in fact restricts to a forward solution operator on \tilde{M} itself; indeed, if $E: H_b^{s-1, \epsilon}(\{\tau < \tau_0\}) \rightarrow H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$ is an extension operator, then the forward solution operator on $\{\tau < \tau_0\}$ is given by extending $f \in H_b^{s-1, \epsilon}(\{\tau < \tau_0\})$ using E , finding the forward solution on Ω using S_{KG} , and restricting back to $\{\tau < \tau_0\}$. The result is independent of the extension operator, as is easily seen from standard energy estimates; see in particular [41, Proposition 3.9].

2.2. A class of semilinear equations. Let us fix $m > 0$ and $\epsilon_0 > 0$ as above for statements about semilinear equations involving the Klein-Gordon operator; for equations involving the wave operator only, let $-\epsilon_0$ be equal to the largest imaginary part of all non-zero resonances of \square_g .

Theorem 2.24. *Let $0 \leq \epsilon < \epsilon_0$ and $s > 1/2 + \epsilon$, $s \geq 1$. Moreover, let³⁰ $q: H_b^{s, \epsilon}(\Omega)^{\bullet, -} \times H_b^{s-1, \epsilon}(\Omega; {}^bT^*\Omega)^{\bullet, -} \rightarrow H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$ be a continuous function with $q(0, 0) = 0$ such that there exists a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfying*

$$\|q(u, {}^bdu) - q(v, {}^bdv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R,$$

where we use the norms corresponding to the map q . Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, {}^bdu) \quad (2.39)$$

has a unique solution $u \in H_b^{s, \epsilon}(\Omega)^{\bullet, -}$, with norm $\leq R$, that depends continuously on f .

More generally, suppose

$$q: H_b^{s, \epsilon}(\Omega)^{\bullet, -} \times H_b^{s-1, \epsilon}(\Omega; {}^bT^*\Omega)^{\bullet, -} \times H_b^{s-1, \epsilon}(\Omega)^{\bullet, -} \rightarrow H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$$

satisfies $q(0, 0, 0) = 0$ and

$$\|q(u, {}^bdu, w) - q(u', {}^bdu', w')\| \leq L(R)(\|u - u'\| + \|w - w'\|)$$

provided $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, where we use the norms corresponding to the map q , for a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, {}^bdu, \square_g u) \quad (2.40)$$

³⁰Here and below in the subsequent sections bundles like ${}^bT^*\Omega$ refer to ${}^bT^*_\Omega M$; the boundaries H_j of Ω are regarded as artificial, and do not affect the cotangent bundle or the corresponding vector fields.

has a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_b^{s,\epsilon}} + \|\square_g u\|_{H_b^{s-1,\epsilon}} \leq R$, that depends continuously on f .

Further, if the non-linearity is of the form $q(\mathbf{b}du)$, with $q: H_b^{s-1,\epsilon}(\Omega; \mathbf{b}T^*\Omega)^{\bullet,-} \rightarrow H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$ having a small Lipschitz constant near 0, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$\square_g u = f + q(\mathbf{b}du)$$

has a unique solution u with $u - (\phi \circ \mathbf{t}_1)c = u' \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, where $c \in \mathbb{C}$, that depends continuously on f .³¹ Here, $\phi \in C^\infty(\mathbb{R})$ with support in $(0, \infty)$ and \mathbf{t}_1 are as in Theorem 2.20.

Proof. To prove the first part, let S_{KG} be the forward solution operator for $\square_g - m^2$ as in (2.38). We want to apply the Banach fixed point theorem to the operator $T_{\text{KG}}: H_b^{s,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, $T_{\text{KG}}u = S_{\text{KG}}(f + q(u, \mathbf{b}du))$.

Let $C_L = \|S_{\text{KG}}\|^{-1}$, then we have the estimate

$$\|T_{\text{KG}}u - T_{\text{KG}}v\| \leq \|S_{\text{KG}}\|L(R')\|u - v\| \leq C_0\|u - v\| \quad (2.41)$$

for $\|u\|, \|v\| \leq R'$ and a constant $C_0 < 1$, granted that $L(R') \leq C_0\|S_{\text{KG}}\|^{-1}$, which holds for small $R' > 0$ by assumption on L . Then, T_{KG} maps the R' -ball in $H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ into itself if $\|S_{\text{KG}}\|(\|f\| + L(R')R') \leq R'$, i.e. if $\|f\| \leq R'(\|S_{\text{KG}}\|^{-1} - L(R'))$. Put

$$R = R'/3, \quad C = R(\|S_{\text{KG}}\|^{-1} - L(R)).$$

Then the existence of a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq R$ to the PDE (2.39) with $\|f\|_{H_b^{s-1,\epsilon}} \leq C$ follows from the Banach fixed theorem.³²

To prove the continuous dependence of u on f , suppose we are given $u_j \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, $j = 1, 2$, with norms $\leq R$, $f_j \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norms $\leq C$, such that

$$(\square_g - m^2)u_j = f_j + q(u_j, \mathbf{b}du_j), \quad j = 1, 2.$$

Put $v := u_2 - u_1$, $g = f_2 - f_1$, then $v \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ has norm $\leq 2R = 2R'/3$ and satisfies the equation

$$(\square_g - m^2)v = g + (q(u_1 + v, \mathbf{b}d(u_1 + v)) - q(u_1, \mathbf{b}du_1)). \quad (2.42)$$

On the other hand, defining a map

$$\begin{aligned} T' &: H_b^{s,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_b^{s,\epsilon}(\Omega)^{\bullet,-}, \\ T'w &= S_{\text{KG}}(g + (q(u_1 + w, \mathbf{b}d(u_1 + w)) - q(u_1, \mathbf{b}du_1))), \end{aligned}$$

we obtain

$$\begin{aligned} \|T'w - T'z\| &\leq \|S_{\text{KG}}\|(q(u_1 + w, \mathbf{b}d(u_1 + w)) - q(u_1 + z, \mathbf{b}d(u_1 + z))) \\ &\leq \|S_{\text{KG}}\|L(R')\|w - z\| \end{aligned}$$

for $\|w\|, \|z\| \leq 2R'/3$, and for $\|w\| \leq 2R'/3$,

$$\|T'w\| \leq \|S_{\text{KG}}\|(\|g\| + 2L(R')R'/3) \leq 2R'/3$$

³¹That is, $c \in \mathbb{C}$ and $u' \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ depend continuously on f .

³²Of course, for this existence statement, we can also work with R' , resp. C' , instead of R , resp. C .

if $\|g\| = \|f_2 - f_1\|$ is small enough; therefore, the Banach fixed point theorem implies that v is in fact the unique solution of (2.42) with norm $\leq 2R'/3$, and we have the estimate

$$\|v\| = \|T'v\| \leq \|S_{\text{KG}}\|(\|g\| + L(R')\|v\|),$$

thus

$$\|u_1 - u_2\| \leq \frac{\|S_{\text{KG}}\|}{1 - \|S_{\text{KG}}\|L(R')} \|f_2 - f_1\|,$$

which completes the proof of the first part.

For the more general statement, we use that one can think of \square_g in the non-linearity as a first order operator. Concretely, we work on the coisotropic space

$$\mathcal{X} = \{u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-} : \square_g u \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_b^{s,\epsilon}(\Omega)^{\bullet,-}} + \|\square_g u\|_{H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}}.$$

This is a Banach space: Indeed, if (u_k) is a Cauchy sequence in \mathcal{X} , then $u_k \rightarrow u$ in $H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, and $\square_g u_k \rightarrow v$ in $H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$; in particular, $\square_g u_k \rightarrow \square_g u$ and $\square_g u_k \rightarrow v$ in $\tau^\epsilon H_b^{s-2}(\Omega)^{\bullet,-}$, thus $\square_g u = v \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$, which was to be shown. We then define $T_{\text{KG}}: \mathcal{X} \rightarrow \mathcal{X}$ by $T_{\text{KG}}u = S_{\text{KG}}(f + q(u, {}^b du, \square_g u))$ and obtain the estimate

$$\begin{aligned} \|T_{\text{KG}}u - T_{\text{KG}}v\|_{\mathcal{X}} &= \|T_{\text{KG}}u - T_{\text{KG}}v\|_{H_b^{s,\epsilon}} + \|q(u, {}^b du, \square_g u) - q(v, {}^b dv, \square_g v)\|_{H_b^{s-1,\epsilon}} \\ &\leq (\|S_{\text{KG}}\| + 1)L(R')(\|u - v\|_{H_b^{s,\epsilon}} + \|\square_g u - \square_g v\|_{H_b^{s-1,\epsilon}}) \\ &= (\|S_{\text{KG}}\| + 1)L(R')\|u - v\|_{\mathcal{X}} \leq C_0\|u - v\|_{\mathcal{X}} \end{aligned}$$

for $u, v \in \mathcal{X}$ with norms $\leq R'$, with $C_0 < 1$ if $R' > 0$ is small enough, provided we require $L(0) < C_L := (\|S_{\text{KG}}\| + 1)^{-1}$. Then, for $u \in \mathcal{X}$ with norm $\leq R'$,

$$\|T_{\text{KG}}u\|_{\mathcal{X}} \leq (\|S_{\text{KG}}\| + 1)(\|f\|_{H_b^{s-1,\epsilon}} + L(R')R') \leq R'$$

if $\|f\| \leq C'$, $C' > 0$ small. Thus, T_{KG} is a contraction on \mathcal{X} , and we obtain the solvability of equation (2.40). The continuous dependence of the solution on the right hand side, replacing R' and C' by smaller constants R and C , is proved as above.

For the third part, we use the forward solution operator $S: H_b^{s-1,\epsilon}(\Omega)^{\bullet,-} \rightarrow \mathcal{Y} := \mathbb{C} \oplus H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ ³³ for \square_g . We will apply the Banach fixed point theorem to the operator $T: \mathcal{Y} \rightarrow \mathcal{Y}$, $Tu = S(f + q({}^b du))$: We again have an estimate like (2.41), since ${}^b du \in H_b^{s-1,\epsilon}(\Omega; {}^b T^* \Omega)^{\bullet,-}$ for $u \in \mathcal{Y}$, and for small $R' > 0$, T maps the R' -ball around 0 in \mathcal{Y} into itself if the norm of f in $H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is small, as above. The continuous dependence of the solution on the right hand side on a smaller ball is proved as above. \square

The following basic statement ensures that there are interesting non-linearities q that satisfy the requirements of the theorem; see also Section 2.3.

Lemma 2.25. *Let $s > n/2$, then $H_b^s(\mathbb{R}_\pm^n)$ is an algebra. In particular, $H_b^s(N)$ is an algebra on any compact n -dimensional manifold N with boundary which is equipped with a b -metric.*

³³ \mathcal{Y} is a Banach space with norm $\|(c, u')\|_{\mathcal{Y}} = |c| + \|u'\|_{H_b^{s,\epsilon}(\Omega)^{\bullet,-}}$. See Section 2.3 for related and more general statements.

Proof. The first statement is the special case $k = 0$ of Lemma 4.4 after a logarithmic change of coordinates, which gives an isomorphism $H_b^s(\mathbb{R}_+^n) \cong H^s(\mathbb{R}^n)$; the lemma is well-known in this case, see e.g. [38, Chapter 13.3]. The second statement follows by localization and from the coordinate invariance of H_b^s . \square

More and related statements will be given in Section 4.2.

Remark 2.26. The algebra property of $H_b^s(N)$ for $s > \dim(N)/2$ is a special case of the fact that for any $F \in C^\infty(\mathbb{R})$, for real valued u , or $F \in C^\infty(\mathbb{C})$, for complex valued u , with $F(0) = 0$, the composition map $H_b^s(N) \ni u \mapsto F \circ u \in H_b^s(N)$ is well-defined and continuous, see for example [38, Chapter 13.10]. In the real valued u case, if $F(0) \neq 0$, then writing $F(t) = F(0) + tF_1(t)$ shows that $F \circ u \in \mathbb{C} + H_b^s(N)$. If $r > 0$, then $H_b^{s,r}(N) \subset H_b^s(N)$ shows that $F_1(u) \in H_b^s(N)$, thus $F \circ u = F(0) + uF_1(u) \in \mathbb{C} + H_b^{s,r}(N)$; and if F vanishes to order k at 0 then $F(t) = t^k F_k(t)$, so $F \circ u = u^k (F_k \circ u)$, and the multiplicative properties of $H_b^{s,r}(N)$ show that $F \circ u \in H_b^{s,kr}(N)$. The argument is analogous for complex valued u , indeed for \mathbb{R}^L -valued u , using Taylor's theorem on F at the origin.

As a corollary we have

Corollary 2.27. *If $s > n/2$, the hypotheses of Theorem 2.24 hold for non-linearities $q(u) = cu^p$, $p \geq 2$ integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_b^s(M)$.*

If $s - 1 > n/2$, the hypotheses of Theorem 2.24 hold for non-linearities q

$$q(u, {}^b du) = \sum_{2 \leq j + |\alpha| \leq d} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \quad (2.43)$$

$q_{j,\alpha} \in \mathbb{C} + H_b^s(M)$, $X_{\alpha,k} \in \mathcal{V}_b(M)$.

Thus, in either case, for $m > 0$, $0 \leq \epsilon < \epsilon_0$, $s > 1/2 + \epsilon$ and for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, {}^b du) \quad (2.44)$$

has a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

The analogous conclusion also holds for $\square_g u = f + q({}^b du)$ provided $\epsilon > 0$, with the solution being in $\mathbb{C}(\phi \circ \mathbf{t}_1) \oplus H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, $\phi \circ \mathbf{t}_1$ identically 1 near $X \cap \Omega$, vanishing near H_1 .

Remark 2.28. For such polynomial non-linearities, the Lipschitz constant $L(R)$ in the statement of Theorem 2.24 actually satisfies $L(0) = 0$. To give an example where $L(0) > 0$, let us consider the situation where we are given a solution u to $(\square_g - m^2)u = f + u^2$ and a small perturbation g of f ; we want to solve $(\square - m^2)(u + v) = (f + g) + (u + v)^2$ for v , which simplifies to $(\square_g - m^2)v = g + 2uv + v^2$. Thus, if u is small (in an appropriate norm), which is reasonable to assume given that we only solve semilinear equations for small data, then $q(v) = 2uv + v^2$ indeed has a small non-zero Lipschitz constant near 0, and thus Theorem 2.24 is applicable.

Since in Theorem 2.24 we allow q to depend on $\square_g u$, we can in particular solve certain quasilinear equations if $s > \max\{1/2 + \epsilon, n/2 + 1\}$: Suppose for example that $q' : H_b^{s,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_b^{s-1}(\Omega)^{\bullet,-}$ is continuous with $\|q'(u) - q'(v)\| \leq L'(R)\|u - v\|$ for $u, v \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$ with norms $\leq R$, where $L' : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is locally bounded, then we can solve the equation

$$(1 + q'(u))(\square_g - m^2)u = f \in H_b^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

provided the norm of f is small. Indeed, put $q(u, w) = -q'(u)(w - m^2u)$, then $q(u, \square_g u) = -q'(u)(\square_g - m^2)u$ and the PDE becomes

$$(\square_g - m^2)u = f + q(u, \square_g u),$$

which is solvable by Theorem 2.24, since, with $\|\cdot\| = \|\cdot\|_{H_b^{s-1, \epsilon}}$, for $u, u' \in H_b^{s, \epsilon}(\Omega)^{\bullet, -}$, $w, w' \in H_b^{s-1, \epsilon}(\Omega)^{\bullet, -}$ with $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, we have

$$\begin{aligned} \|q(u, w) - q(u', w')\| &\leq \|q'(u) - q'(u')\| \|w - m^2u\| + \|q'(u')\| \|w - w' - m^2(u - u')\| \\ &\leq L'(R)((1 + m^2)R + m^2R)\|u - u'\| + L'(R)R\|w - w'\| \\ &\leq L(R)(\|u - u'\| + \|w - w'\|) \end{aligned}$$

with $L(R) \rightarrow 0$ as $R \rightarrow 0$.

By a similar argument, one can also allow q' to depend on ${}^b du$ and $\square_g u$.

Remark 2.29. Recalling the discussion following Theorem 2.20, let us emphasize the importance of $\widehat{P}(\sigma)^{-1}$ having no poles in the closed upper half plane by looking at the explicit example of the operator $\mathcal{P} = \partial_x$ in 1 dimension. In terms of $\tau = e^{-x}$, we have $\mathcal{P} = -\tau \partial_\tau$, thus $\widehat{P}(\sigma) = -i\sigma$, considered as an operator on the boundary (which is a single point) at $+\infty$ of the radial compactification of \mathbb{R} ; hence $\widehat{P}(\sigma)^{-1}$ has a simple pole at $\sigma = 0$, corresponding to constants being annihilated by \mathcal{P} . Now suppose we want to find a forward solution of $u' = u^2 + f$, where $f \in C_c^\infty(\mathbb{R})$. In the first step of the iterative procedure described above, we will obtain a constant term; the next step gives a term that is linear in x (x being the antiderivative of 1), i.e. in $\log \tau$, then we get quadratic terms and so on, therefore the iteration does not converge (for general f), which is of course to be expected, since solutions to $u' = u^2 + f$ in general blow up in finite time. On the other hand, if $\mathcal{P} = \partial_x + 1$, then $\widehat{P}(\sigma)^{-1} = (1 - i\sigma)^{-1}$, which has a simple pole at $\sigma = -i$, which means that forward solutions u of $u' + u = u^2 + f$ with f as above can be constructed iteratively, and the first term of the expansion of u at $+\infty$ is $c\tau^{i(-i)} = ce^{-x}$, $c \in \mathbb{C}$.

2.3. Semilinear equations with polynomial non-linearity. With polynomial non-linearities as in (2.43), we can use the second part of Theorem 2.20 to obtain an asymptotic expansion for the solution; see Remark 2.36 and, in a slightly different setting, Section 3.2 for details on this. Here, we instead define a space that encodes asymptotic expansions directly in such a way that we can run a fixed point argument directly.

To describe the exponents appearing in the expansion, we use index sets as introduced by Melrose, see [32].

Definition 2.30.

- (1) An *index set* is a discrete subset \mathcal{E} of $\mathbb{C} \times \mathbb{N}_0$ satisfying the conditions
 - (i) $(z, k) \in \mathcal{E} \Rightarrow (z, j) \in \mathcal{E}$ for $0 \leq j \leq k$, and
 - (ii) If $(z_j, k_j) \in \mathcal{E}$, $|z_j| + k_j \rightarrow \infty \Rightarrow \operatorname{Re} z_j \rightarrow \infty$.
- (2) For any index set \mathcal{E} , define

$$w_{\mathcal{E}}(z) = \begin{cases} \max\{k \in \mathbb{N}_0 : (z, k) \in \mathcal{E}\}, & (z, 0) \in \mathcal{E} \\ -\infty & \text{otherwise.} \end{cases}$$

- (3) For two index sets $\mathcal{E}, \mathcal{E}'$, define their extended union by

$$\mathcal{E} \cup \mathcal{E}' = \mathcal{E} \cup \mathcal{E}' \cup \{(z, l + l' + 1) : (z, l) \in \mathcal{E}, (z, l') \in \mathcal{E}'\}$$

and their product by $\mathcal{E}\mathcal{E}' = \{(z + z', l + l') : (z, l) \in \mathcal{E}, (z', l') \in \mathcal{E}'\}$. We shall write \mathcal{E}^k for the k -fold product of \mathcal{E} with itself.

- (4) A *positive index set* is an index set \mathcal{E} with the property that $\operatorname{Re} z > 0$ for all $z \in \mathbb{C}$ with $(z, 0) \in \mathcal{E}$.

Remark 2.31. To ensure that the class of polyhomogeneous conormal distributions with a given index set \mathcal{E} is invariantly defined, Melrose [32] in addition requires that $(z, k) \in \mathcal{E}$ implies $(z + j, k) \in \mathcal{E}$ for all $j \in \mathbb{N}_0$. In particular, this is a natural condition in non dilation-invariant settings as in Theorem 2.20. A convenient way to enforce this condition in all relevant situations is to enlarge the index set corresponding to the poles of the inverse of the normal operator accordingly; see the statement of Theorem 2.35.

Observe though that this condition is not needed in the dilation-invariant cases of the solvability statements below.

Since we want to capture the asymptotic behavior of solutions near $X \cap \Omega$, we fix a cutoff $\phi \in C^\infty(\mathbb{R})$ with support in $(0, \infty)$ such that $\phi \circ \mathbf{t}_1 \equiv 1$ near $X \cap \Omega$ (we already used such a cutoff in Theorem 2.20), and make the following definition.

Definition 2.32. Let \mathcal{E} be an index set, and let $s, r \in \mathbb{R}$. For $\epsilon > 0$ with the property that there is no $(z, 0) \in \mathcal{E}$ with $\operatorname{Re} z = \epsilon$, define the space $\mathcal{X}_{\mathcal{E}}^{s, r, \epsilon}$ to consist of all tempered distributions v on M with support in $\bar{\Omega}$ such that

$$v' = v - \sum_{\substack{(z, k) \in \mathcal{E} \\ \operatorname{Re} z < \epsilon}} \tau^z (\log \tau)^k (\phi \circ \mathbf{t}_1) v_{z, k} \in H_b^{s, \epsilon}(\Omega)^{\bullet, -} \quad (2.45)$$

for certain $v_{z, k} \in H^r(X \cap \Omega)$.

Observe that the terms $v_{z, k}$ in the expansion (2.45) are uniquely determined by v , since $\epsilon > \operatorname{Re} z$ for all $z \in \mathbb{C}$ for which $(z, 0)$ appears in the sum (2.45); then also v' are uniquely determined by v . Therefore, we can use the isomorphism

$$\mathcal{X}_{\mathcal{E}}^{s, r, \epsilon} \cong \left(\bigoplus_{\substack{(z, k) \in \mathcal{E} \\ \operatorname{Re} z < \epsilon}} H^r(X \cap \Omega) \right) \oplus H_b^{s, \epsilon}(\Omega)^{\bullet, -}$$

to give $\mathcal{X}_{\mathcal{E}}^{s, r, \epsilon}$ the structure of a Banach space.

Lemma 2.33. *Let \mathcal{P}, \mathcal{F} be positive index sets, and let $\epsilon > 0$. Define $\mathcal{E}'_0 = \mathcal{P} \cup \mathcal{F}$ and recursively $\mathcal{E}'_{N+1} = \mathcal{P} \cup (\mathcal{F} \cup \bigcup_{k \geq 2} (\mathcal{E}'_N)^k)$; put $\mathcal{E}_N = \{(z, k) \in \mathcal{E}'_N : 0 < \operatorname{Re} z \leq \epsilon\}$. Then there exists $N_0 \in \mathbb{N}$ such that $\mathcal{E}_N = \mathcal{E}_{N_0}$ for all $N \geq N_0$; moreover, the limiting index set $\mathcal{E}_\infty(\mathcal{P}, \mathcal{F}, \epsilon) := \mathcal{E}_{N_0}$ is finite.*

Proof. Writing $\pi_1 : \mathbb{C} \times \mathbb{N}_0 \rightarrow \mathbb{C}$ for the projection, one has

$$\pi_1 \mathcal{E}'_1 = \left\{ z : 0 < \operatorname{Re} z \leq \epsilon, z = \sum_{j=1}^k z_j : k \geq 1, z_j \in \pi_1 \mathcal{E}'_0 \right\},$$

and it is then clear that $\pi_1 \mathcal{E}'_N = \pi_1 \mathcal{E}'_1$ for all $N \geq 1$. Since \mathcal{E}'_0 is a positive index set, there exists $\delta > 0$ s.t. $\operatorname{Re} z \geq \delta$ for all $z \in \mathcal{E}'_0$; hence $\pi_1 \mathcal{E}'_\infty = \pi_1 \mathcal{E}'_1$ is finite.

To finish the proof, we need to show that for all $z \in \mathbb{C}$, the number $w_{\mathcal{E}'_N}(z)$ stabilizes. Defining $p(z) = w_{\mathcal{P}}(z) + 1$ for $z \in \pi_1 \mathcal{P}$ and $p(z) = 0$ otherwise, we have a recursion relation

$$w_{\mathcal{E}'_N}(z) = p(z) + \max \left\{ w_{\mathcal{F}}(z), \max_{\substack{z = z_1 + \dots + z_k \\ k \geq 2, z_j \in \pi_1 \mathcal{E}'_\infty}} \left\{ \sum_{j=1}^k w_{\mathcal{E}'_{N-1}}(z_j) \right\} \right\}, \quad N \geq 1. \quad (2.46)$$

For each z_j appearing in the sum, we have $\text{Im } z_j \leq \text{Im } z - \delta$. Thus, we can use (2.46) with z replaced by such z_j and N replaced by $N - 1$ to express $w_{\mathcal{E}_N}(z)$ in terms of a finite number of $p(z_\alpha)$ and $w_{\mathcal{F}}(z_\alpha)$, $\text{Im } z_\alpha \leq \text{Im } z$, and a finite number of $w_{\mathcal{E}_{N-2}}(z_\beta)$, $z_\beta \leq \text{Im } z - 2\delta$. Continuing in this way, after $N_0 = \lfloor (\text{Im } z)/\delta \rfloor + 1$ steps we have expressed $w_{\mathcal{E}_N}(z)$ in terms of a finite number of $p(z_\gamma)$ and $w_{\mathcal{F}}(z_\gamma)$, $\text{Im } z_\gamma \leq \text{Im } z$, only, and this expression is independent of N as long as $N \geq N_0$. \square

Definition 2.34. Let \mathcal{P}, \mathcal{F} be positive index sets, and let $\epsilon > 0$ be such that there is no $(z, 0) \in \mathcal{E}_\infty(\mathcal{P}, \mathcal{F}, \epsilon)$ with $\text{Re } z = \epsilon$, with $\mathcal{E}_\infty(\mathcal{P}, \mathcal{F}, \epsilon)$ as defined in the statement of Lemma 2.33. Then for $s, r \in \mathbb{R}$, define the Banach spaces

$$\begin{aligned} \mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, r, \epsilon} &:= \mathcal{X}_{\mathcal{E}_\infty(\mathcal{P}, \mathcal{F}, \epsilon)}^{s, r, \epsilon}, \\ {}^0\mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, r, \epsilon} &:= \mathcal{X}_{\mathcal{E}_\infty(\mathcal{P}, \mathcal{F}, \epsilon) \cup \{(0, 0)\}}^{s, r, \epsilon}. \end{aligned}$$

Note that the spaces $({}^0)\mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$ are Banach algebras³⁴ for $s > n/2$. Moreover, $\mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$ interacts well with the forward solution operator S_{KG} of $\square_g - m^2$ in the sense that $u \in \mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$, $k \geq 2$, with \mathcal{P} being related to the poles of $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \square_g - m^2$, as will be made precise in the statement of Theorem 2.35 below, implies $S_{\text{KG}}(u^k) \in \mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$.

We can now state the result giving an asymptotic expansion of the solution of $(\square_g - m^2)u = f + q(u, {}^bdu)$ for polynomial non-linearities q .

Theorem 2.35. *Let $\epsilon > 0$, $s > \max\{1/2 + \epsilon, n/2 + 1\}$, and q as in (2.43). Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \square_g - m^2$, and $m_j + 1$ is the order of the pole of $\widehat{\mathcal{P}}(\sigma)^{-1}$ at σ_j , let $\mathcal{P} = \{(i\sigma_j + k, \ell) : 0 \leq \ell \leq m_j, k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \text{Re}(i\sigma_j)$ for all j , and that moreover $m > 0$, which implies that \mathcal{P} is a positive index set.³⁵ Finally, let \mathcal{F} be a positive index set.*

Then for small enough $R > 0$, there exists $C > 0$ such that for all $f \in \mathcal{X}_{\mathcal{F}}^{s-1, s-1, \epsilon}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, {}^bdu)$$

has a unique solution $u \in \mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$, with norm $\leq R$, that depends continuously on f ; in particular, u has an asymptotic expansion with remainder term in $H_b^{s, \epsilon}(\Omega)^{\bullet, -}$.

Further, if the polynomial non-linearity is of the form $q({}^bdu)$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in \mathcal{X}_{\mathcal{F}}^{s-1, s-1, \epsilon}$ with norm $\leq C$, the equation

$$\square_g u = f + q({}^bdu)$$

has a unique solution $u \in {}^0\mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$, with norm $\leq R$, that depends continuously on f .

Proof. By Theorem 2.20 and the definition of the space $\mathcal{X} = \mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$, we have a forward solution operator $S_{\text{KG}}: \mathcal{X} \rightarrow \mathcal{X}$ of $\square_g - m^2$. Thus, we can apply the Banach fixed point theorem to the operator $T: \mathcal{X} \rightarrow \mathcal{X}$, $Tu = S_{\text{KG}}(f + q(u, {}^bdu))$, where we note that $q: \mathcal{X} \rightarrow \mathcal{X}$, which follows from the definition of \mathcal{X} and the fact that q is a polynomial only involving terms of the form $u^j \prod_{k \leq |\alpha|} X_{\alpha, k} u$ for $j + |\alpha| \geq 2$. This condition on q also ensures that T is a contraction on a sufficiently small ball in \mathcal{X}_+ .

³⁴In the sense that there is a constant $C > 0$ such that $\|uv\| \leq C\|u\|\|v\|$ for all $u, v \in ({}^0)\mathcal{X}_{\mathcal{P}, \mathcal{F}}^{s, s, \epsilon}$.

³⁵See Lemma 2.22.

For the second part, writing ${}^0\mathcal{X} = {}^0\mathcal{X}_{\mathcal{F}, \mathcal{F}}^{s, s, \epsilon}$, we have a forward solution operator $S: \mathcal{X} \rightarrow {}^0\mathcal{X}$. But $q({}^b d u): {}^0\mathcal{X} \rightarrow \mathcal{X}$, since ${}^b d$ annihilates constants, and we can thus finish the proof as above.

The continuous dependence of the solution on the right hand side is proved as in the proof of Theorem 2.24. \square

Note that $\epsilon > 0$ is (almost) unrestricted here, and thus we can get arbitrarily many terms in the asymptotic expansion if we work with arbitrarily high Sobolev spaces.

The condition that the polynomial $q(u, {}^b d u)$ does not involve a linear term is very important as it prevents terms of low order stacking up in the iterative process used to solve the equation. Also, adding a term νu to $q(u, {}^b d u)$ effectively changes the Klein-Gordon parameter from $-m^2$ to $\nu - m^2$, which will change the location of the poles of $\hat{P}(\sigma)^{-1}$; in the worst case, if $\nu > m^2$, this would even cause a pole to move to $\text{Im } \sigma > 0$, corresponding to a resonant state that blows up exponentially in time.

Remark 2.36. Instead of working with the spaces $({}^0)\mathcal{X}_{\mathcal{F}, \mathcal{F}}^{s, s, \epsilon}$, which have the expansion built in, one could alternatively first prove the existence of a solution u in a (slightly) decaying b-Sobolev space, which then allows one to regard the polynomial non-linearity as a perturbation of the linear operator $\square_g - m^2$; then an iterative application of the dilation-invariant result [41, Lemma 3.1] gives an expansion of the solution to the non-linear equation. We will follow this idea in the discussion of polynomial non-linearities on asymptotically Kerr-de Sitter spaces in the next section.

3. KERR-DE SITTER SPACE

3.1. Linear Fredholm theory. The linear theorem in the case of normally hyperbolic trapping for dilation-invariant operators $\mathcal{P} = \square_g - \lambda$ is the following:

Theorem 3.1. (See [41, Theorem 1.4].) *Let M be a manifold with a b-metric g as above, with boundary X , and let τ be the boundary defining function, \mathcal{P} as in (2.14). If g has normally hyperbolic trapping, \mathfrak{t}_1, Ω are as above, $\phi \in C^\infty(\mathbb{R})$ as in Theorem 2.20, then there exist $C' > 0$, $\varkappa > 0$, $\beta \in \mathbb{R}$ such that for $0 \leq \ell < C'$ and $s > 1/2 + \beta\ell$, $s \geq 0$, solutions $u \in H_b^{-\infty, -\infty}(\Omega)^{\bullet, -}$ of $(\square_g - \lambda)u = f$ with $f \in H_b^{s-1+\varkappa, \ell}(\Omega)^{\bullet, -}$ satisfy that for some $a_{j\kappa} \in C^\infty(\Omega \cap X)$ (which are the resonant states) and $\sigma_j \in \mathbb{C}$ (which are the resonances),*

$$u' = u - \sum_j \sum_{\kappa \leq m_j} \tau^{i\sigma_j} (\log \tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa} \in H_b^{s, \ell}(\Omega)^{\bullet, -}. \quad (3.1)$$

Here the (semi)norms of both $a_{j\kappa}$ in $C^\infty(\Omega \cap X)$ and u' in $H_b^{s, \ell}(\Omega)^{\bullet, -}$ ³⁶ are bounded by a constant times that of f in $H_b^{s-1+\varkappa, \ell}(\Omega)^{\bullet, -}$. The same conclusion holds for sufficiently small perturbations of the metric as a symmetric bilinear form on ${}^b T M$ provided the trapping is normally hyperbolic.

In order to state the analogue of Theorems 2.18-2.20 when one has *normally hyperbolic trapping* at $\Gamma \subset {}^b S_X^* M$, we need some notation in addition to that in

³⁶Note that, as in the proof of Theorem 2.18, u' is the restriction to Ω of a distribution in $H_{b, \text{loc}}^{s, \ell}(\tilde{\Omega})^{\bullet, -}$, where $\tilde{\Omega}$ is a slightly bigger domain, thus u' lies in the indicated space.

Section 2. We defer the details of the local nature of the normally hyperbolic trapping at Γ to the Appendix to avoid interrupting the main argument, and here describe what it means for the global flow. In the setting of Section 2, as leading up to Theorem 2.18, see the discussion above Figure 3, we define

- (1) the *forward trapped set* in Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ the bicharacteristics through which do not flow (within Σ_Ω) to ${}^bS_{H_1}^*M \cup L_+$ in the forward direction (i.e. they do not reach ${}^bS_{H_1}^*M$ in finite time and they do not tend to L_+),
- (2) the *backward trapped set* in Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ the bicharacteristics through which do not flow to ${}^bS_{H_2}^*M \cup L_+$ in the backward direction,
- (3) the *forward trapped set* in Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ the bicharacteristics through which do not flow to ${}^bS_{H_2}^*M \cup L_-$ in the forward direction,
- (4) the *backward trapped set* in Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ the bicharacteristics through which do not flow to ${}^bS_{H_1}^*M \cup L_-$ in the backward direction.

The *forward trapped set* is the union of the forward trapped sets in Σ_\pm , and analogously for the *backward trapped set*. The *trapped set* Γ is the intersection of the forward and backward trapped sets. We say that \mathcal{P} is *normally hyperbolic trapping*, or has normally hyperbolic trapping, if $\Gamma \subset {}^bS_X^*M$ is b-normally hyperbolic in the sense discussed in the Appendix in the paragraph of (A.14)-(A.16).

We introduce replacements for the b-Sobolev spaces used in Section 2 which are called *normally isotropic* at Γ ; these spaces $\mathcal{H}_{b,\Gamma}^s$, see also (A.19), and dual spaces $\mathcal{H}_{b,\Gamma}^{*,-s}$ are just the standard b-Sobolev spaces $H_b^s(M)$, resp. $H_b^{-s}(M)$, microlocally away from Γ . Note that ${}^bT^*M$ is *not* a symplectic manifold (in the natural way) since the symplectic form on ${}^bT_{M^\circ}^*M$ does not extend smoothly to ${}^bT^*M$. Thus, the word ‘normally isotropic’ is not completely justified; we use it since it reflects that in the analogous semiclassical setting, see [47], the set Γ is symplectic, and the origin in the symplectic orthocomplement $(T_\alpha\Gamma)^\perp$ of $T_\alpha\Gamma$, which is also symplectic, is isotropic within $(T_\alpha\Gamma)^\perp$.

Concretely, if Γ is locally (in a neighborhood U_0 of Γ) defined by $\tau = 0$, $\phi_+ = \phi_- = 0$, $\hat{p} = 0$ in ${}^bS^*M$, with $d\tau, d\phi_+, d\phi_-, d\hat{p}$, $\hat{p} = \tilde{\rho}^m p$, linearly independent at Γ , then taking any $Q_\pm \in \Psi_b^0(M)$ with principal symbol ϕ_\pm , $\hat{P} \in \Psi_b^0(M)$ with principal symbol \hat{p} , and $Q_0 \in \Psi_b^0(M)$ elliptic on U_0^c with $\text{WF}_b'(Q_0) \cap \Gamma = \emptyset$, we define the (global) b-normally isotropic spaces at Γ of order s , $\mathcal{H}_{b,\Gamma}^s = \mathcal{H}_{b,\Gamma}^s(M)$, by the norm

$$\|u\|_{\mathcal{H}_{b,\Gamma}^s}^2 = \|Q_0 u\|_{H_b^s}^2 + \|Q_+ u\|_{H_b^s}^2 + \|Q_- u\|_{H_b^s}^2 + \|\tau^{1/2} u\|_{H_b^s}^2 + \|\hat{P} u\|_{H_b^s}^2 + \|u\|_{H_b^{s-1/2}}^2, \quad (3.2)$$

and let $\mathcal{H}_{\mathfrak{b},\Gamma}^{*, -s}$ be the dual space relative to L^2 which is thus³⁷

$$Q_0 H_{\mathfrak{b}}^{-s} + Q_+ H_{\mathfrak{b}}^{-s} + Q_- H_{\mathfrak{b}}^{-s} + \tau^{1/2} H_{\mathfrak{b}}^{-s} + \hat{P} H_{\mathfrak{b}}^{-s} + H_{\mathfrak{b}}^{-s+1/2}.$$

In particular,

$$\begin{aligned} H_{\mathfrak{b}}^s(M) &\subset \mathcal{H}_{\mathfrak{b},\Gamma}^s(M) \subset H_{\mathfrak{b}}^{s-1/2}(M) \cap H_{\mathfrak{b}}^{s,-1/2}(M), \\ H_{\mathfrak{b}}^{s+1/2}(M) + H_{\mathfrak{b}}^{s,1/2}(M) &\subset \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s}(M) \subset H_{\mathfrak{b}}^s(M). \end{aligned} \quad (3.3)$$

Microlocally away from Γ , $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$ is indeed just the standard $H_{\mathfrak{b}}^s$ space while $\mathcal{H}_{\mathfrak{b},\Gamma}^{*, -s}$ is $H_{\mathfrak{b}}^{-s}$ since at least one of $Q_0, Q_{\pm}, \tau, \hat{P}$ is elliptic; the space is independent of the choice of Q_0 satisfying the criteria since at least one of Q_{\pm}, τ, \hat{P} is elliptic on $U_0 \setminus \Gamma$. Moreover, $\Psi_{\mathfrak{b}}^k(M) : \mathcal{H}_{\mathfrak{b},\Gamma}^s(M) \rightarrow \mathcal{H}_{\mathfrak{b},\Gamma}^{s-k}(M)$ is continuous as for $A \in \Psi_{\mathfrak{b}}^k(M)$, $Q_+ A u = A Q_+ u + [Q_+, A]u$ and $[Q_+, A] \in \Psi_{\mathfrak{b}}^{k-1}(M)$; the analogous statement also holds for the dual spaces. The notation $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$ is justified for the space is independent of the particular defining functions ϕ_{\pm} chosen; near Γ any other choice would replace ϕ_{\pm} by smooth non-degenerate linear combinations plus a multiple of τ and of \hat{p} , denote these by $\tilde{\phi}_{\pm}$, and thus the corresponding \tilde{Q}_{\pm} can be expressed as

$$B_+ Q_+ + B_- Q_- + B_{\partial} \tau + \hat{B} \hat{P} + B_0 Q_0 + R, \quad B_{\pm}, B_0, B_{\partial}, \hat{B} \in \Psi_{\mathfrak{b}}^0(M), \quad R \in \Psi_{\mathfrak{b}}^{-1}(M),$$

so the new norm can be controlled by the old norm, and conversely in view of the non-degeneracy. We also remark that in fact the term $\|u\|_{H_{\mathfrak{b}}^{s-1/2}}^2$ in the norm (3.2) can be replaced by $\|u\|_{H_{\mathfrak{b}}^{s-1}}^2$, see the discussion preceding Proposition A.2 in the Appendix: this arises from the Poisson bracket property of ϕ_+ and ϕ_- , corresponding to the (partially) isotropic nature of Γ .

Further, if $\Omega \subset M$, as in Section 2, is such that ${}^b S_{H_j}^* \Omega \cap \Gamma = \emptyset$, $j = 1, 2$, then spaces such as

$$\mathcal{H}_{\mathfrak{b},\Gamma}^{*,s}(\Omega)^{\bullet,-}$$

are not only well-defined, but are standard $H_{\mathfrak{b}}^s$ -spaces near the H_j . The inclusions analogous to (3.3) also hold for the corresponding spaces over Ω .

Notice that elements of $\Psi_{\mathfrak{b}}^p(M)$ only map $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$ to $\mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-p-1}(M)$, with the issues being at Γ corresponding to (3.3) (thus there is no distinction between the behavior on the Ω vs. the M -based spaces). However, if $A \in \Psi_{\mathfrak{b}}^p(M)$ has principal symbol vanishing on Γ then

$$A : \mathcal{H}_{\mathfrak{b},\Gamma}^s(M) \rightarrow H_{\mathfrak{b}}^{s-p}(M), \quad A : H_{\mathfrak{b}}^s(M) \rightarrow \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-p}(M), \quad (3.4)$$

as A can be expressed as $A_+ Q_+ + A_- Q_- + A_{\partial} \tau + \hat{A} \hat{P} + A_0 Q_0 + R$, $A_{\pm}, A_0, A_{\partial}, \hat{A} \in \Psi_{\mathfrak{b}}^0(M)$, $R \in \Psi_{\mathfrak{b}}^{-1}(M)$, with the second mapping property following by duality as $\Psi_{\mathfrak{b}}^p(M)$ is closed under adjoints, and the principal symbol of the adjoint vanishes wherever that of the original operator does. Correspondingly, if $A_j \in \Psi_{\mathfrak{b}}^{m_j}(M)$ have principal symbol vanishing at Γ then $A_1 A_2 u : \mathcal{H}_{\mathfrak{b},\Gamma}^s(M) \rightarrow \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-m_1-m_2}(M)$.

³⁷We refer to [31, Appendix A] for a general discussion of the underlying functional analysis. In particular, Lemma A.3 there essentially gives the density of $\dot{C}^{\infty}(M)$ in $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$: one can simply drop the subscript ‘e’ in the statement of that lemma to conclude that $H_{\mathfrak{b}}^{\infty}(M)$ (so in particular $H_{\mathfrak{b}}^{s-1/2}(M)$) is dense in $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$, and then the density of $\dot{C}^{\infty}(M)$ in $H_{\mathfrak{b}}^{s'}(M)$ for any s' completes the argument. The completeness of $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$ follows from the continuity of $\Psi_{\mathfrak{b}}^0(M)$ on $H_{\mathfrak{b}}^{s-1/2}(M)$.

We consider \mathcal{P} as a map

$$\mathcal{P} : \mathcal{H}_{\mathfrak{b},\Gamma}^s(\Omega)^{\bullet,-} \rightarrow \mathcal{H}_{\mathfrak{b},\Gamma}^{s-2}(\Omega)^{\bullet,-},$$

and let

$$\mathcal{Y}_{\Gamma}^s = \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s}(\Omega)^{\bullet,-}, \quad \mathcal{X}_{\Gamma}^s = \{u \in \mathcal{H}_{\mathfrak{b},\Gamma}^s(\Omega)^{\bullet,-} : \mathcal{P}u \in \mathcal{Y}_{\Gamma}^{s-1}\}.$$

While \mathcal{X}_{Γ}^s is complete,³⁸ it is a slightly exotic space, unlike \mathcal{X}^s in Theorem 2.18 which is a coisotropic space depending on Σ (and thus the principal symbol of \mathcal{P}) only, since elements of $\Psi_{\mathfrak{b}}^p(M)$ only map $\mathcal{H}_{\mathfrak{b},\Gamma}^s(M)$ to $\mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-p-1}(M)$ as remarked earlier. Correspondingly, \mathcal{X}_{Γ}^s actually depends on \mathcal{P} modulo $\Psi_{\mathfrak{b}}^0(M)$ plus first order pseudodifferential operators of the form $A_1 A_2$, $A_1 \in \Psi_{\mathfrak{b}}^0(M)$, $A_2 \in \Psi_{\mathfrak{b}}^1(M)$, both with principal symbol vanishing at Γ – here the operators should have Schwartz kernels supported away from the H_j ; near H_j (but away from Γ), one should say \mathcal{P} matters modulo $\text{Diff}_{\mathfrak{b}}^1(M)$, i.e. only the principal symbol of \mathcal{P} matters.

We then have:

Theorem 3.2. *Suppose $s \geq 3/2$, and that the inverse of the Mellin transformed normal operator $\widehat{\mathcal{P}}(\sigma)^{-1}$ has no poles with $\text{Im } \sigma \geq 0$. Then*

$$\mathcal{P} : \mathcal{X}_{\Gamma}^s \rightarrow \mathcal{Y}_{\Gamma}^{s-1}$$

is invertible, giving the forward solution operator.

Proof. First, with $r < -1/2$, thus with dual spaces having weight $\tilde{r} > 1/2$, Theorem 2.18 holds without changes as Proposition A.2 gives non-trapping estimates in this case on the standard b-Sobolev spaces. In particular, if $r \ll 0$, $\text{Ker } \mathcal{P}$ is trivial even on $H_{\mathfrak{b}}^{s-1/2,r}(\Omega)^{\bullet,-}$, hence certainly on its subspace $\mathcal{H}_{\mathfrak{b},\Gamma}^s(\Omega)^{\bullet,-}$. Similarly, $\text{Ker } \mathcal{P}^*$ is trivial on $H_{\mathfrak{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}$, $\tilde{r} \gg 0$, and thus with $r < -1/2$, for $f \in H_{\mathfrak{b}}^{-1,r}(\Omega)^{\bullet,-}$ there exists $u \in H_{\mathfrak{b}}^{0,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u = f$. Further, making use of the non-trapping estimates in Proposition A.2, if $r < 0$ then $f \in H_{\mathfrak{b}}^{s-1,r}(\Omega)^{\bullet,-}$, so the argument of Theorem 2.20 improves this statement to $u \in H_{\mathfrak{b}}^{s,r}(\Omega)^{\bullet,-}$.

In particular, if $f \in \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_{\mathfrak{b}}^{s-1,0}(\Omega)^{\bullet,-}$, then $u \in H_{\mathfrak{b}}^{s,r}(\Omega)^{\bullet,-}$ for $r < 0$. This can be improved using the argument of Theorem 2.20. Indeed, with $-1 \leq r < 0$ arbitrary, $\mathcal{P} - N(\mathcal{P}) \in \tau\text{Diff}_{\mathfrak{b}}^2(M)$ implies as in (2.36) that

$$N(\mathcal{P})u = f - \tilde{f}, \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_{\mathfrak{b}}^{s-2,r+1}(\Omega)^{\bullet,-}, \quad (3.5)$$

and thus as $f \in \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_{\mathfrak{b}}^{s-1,0}(\Omega)^{\bullet,-}$, so the right hand side is in $H_{\mathfrak{b}}^{s-2,0}(\Omega)^{\bullet,-}$, the dilation-invariant result, [41, Lemma 3.1], gives $u \in H_{\mathfrak{b}}^{s-1,0}(\Omega)^{\bullet,-}$. This can then be improved further since in view of $\mathcal{P}u = f \in \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-}$, propagation of singularities, most crucially Proposition A.2, yields $u \in \mathcal{H}_{\mathfrak{b},\Gamma}^s(\Omega)^{\bullet,-}$. This completes the proof of the theorem. \square

³⁸ Also, elements of $C^\infty(\Omega)$ vanishing to infinite order at H_1 and $X \cap \Omega$ are dense in \mathcal{X}_{Γ}^s . Indeed, in view of [31, Lemma A.3] the only possible issue is at Γ , thus the distinction between Ω and M may be dropped. To complete the argument, one proceeds as in the quoted lemma, using the ellipticity of σ at Γ , letting $\Lambda_n \in \Psi_{\mathfrak{b}}^{-\infty}(M)$, $n \in \mathbb{N}$, be a quantization of $\phi(\sigma/n)a$, $a \in C^\infty({}^bS^*M)$ supported in a neighborhood of Γ , identically 1 near Γ , $\phi \in C_c^\infty(\mathbb{R})$, noting that $[\Lambda_n, \mathcal{P}] \in \Psi_{\mathfrak{b}}^{-\infty}(M)$ is uniformly bounded in $\Psi_{\mathfrak{b}}^0(M) + \tau\Psi_{\mathfrak{b}}^1(M)$ in view of (2.1), and thus for $u \in \mathcal{X}_{\Gamma}^s$, $\mathcal{P}\Lambda_n u = \Lambda_n \mathcal{P}u + [\mathcal{P}, \Lambda_n]u \rightarrow \mathcal{P}u$ in $\mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-1}$ since $[\mathcal{P}, \Lambda_n]$ is uniformly bounded $H_{\mathfrak{b}}^{s-1/2} \cap H_{\mathfrak{b}}^{s-1,1/2} \rightarrow H_{\mathfrak{b}}^{s-1/2} \cap H_{\mathfrak{b}}^{s-1,1/2}$, and thus $\mathcal{H}_{\mathfrak{b},\Gamma}^s \rightarrow \mathcal{H}_{\mathfrak{b},\Gamma}^{*,s-1}$ by (3.3).

This result shows the importance of controlling the resonances in $\text{Im } \sigma \geq 0$. For the wave operator on exact Kerr-de Sitter space, Dyatlov's analysis [16, 15] shows that the zero resonance of \square_g is the only one in $\text{Im } \sigma \geq 0$, the residue at 0 having constant functions as its range. For the Klein-Gordon operator $\square_g - m^2$, the statement is even better from our perspective as there are no resonances in $\text{Im } \sigma \geq 0$ for $m > 0$ small. This is pointed out in [16]; we give a direct proof based on perturbation theory.

Lemma 3.3. *Let $\mathcal{P} = \square_g$ on exact Kerr-de Sitter space. Then for small $m > 0$, all poles of $(\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$ have strictly negative imaginary part.*

Proof. By perturbation theory, the inverse family of $\widehat{\mathcal{P}}(\sigma) - \lambda$ has a simple pole at $\sigma(\lambda)$ coming with a single resonant state $\phi(\lambda)$ and a dual state $\psi(\lambda)$, with analytic dependence on λ , where $\sigma(0) = 0, \phi(0) \equiv 1, \psi(0) = 1_{\{\mu > 0\}}$, where we use the notation of [41, Section 6]. Differentiating $\widehat{\mathcal{P}}(\sigma(\lambda))\phi(\lambda) = \lambda\phi(\lambda)$ with respect to λ and evaluating at $\lambda = 0$ gives

$$\sigma'(0)\widehat{\mathcal{P}}'(0)\phi(0) + \widehat{\mathcal{P}}(0)\phi'(0) = \phi(0).$$

Pairing this with $\psi(0)$, which is orthogonal to $\text{Ran } \widehat{\mathcal{P}}(0)$, yields

$$\sigma'(0) = \frac{\langle \psi(0), \phi(0) \rangle}{\langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0) \rangle},$$

Since $\phi(0) = 1$ and $\psi(0) = 1_{\{\mu > 0\}}$, this implies

$$\text{sgn } \text{Im } \sigma'(0) = -\text{sgn } \text{Im} \langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0) \rangle. \quad (3.6)$$

To find the latter quantity, we note that the only terms in the general form of the d'Alembertian that could possibly yield a non-zero contribution here are terms involving τD_τ and either D_r, D_ϕ or D_θ . Concretely, using the explicit form of the dual metric G ³⁹ in the new coordinates $t = \tilde{t} + h(r), \phi = \tilde{\phi} + P(r), \tau = e^{-t}$, with $h(r), P(r)$ as in [41, Equation (6.5)],

$$G = -\rho^{-2} \left(\tilde{\mu}(\partial_r - h'(r)\tau\partial_\tau + P'(r)\partial_\phi)^2 + \frac{(1+\gamma)^2}{\varkappa \sin^2 \theta} (-a \sin^2 \theta \tau \partial_\tau + \partial_\phi)^2 + \varkappa \partial_\theta^2 - \frac{(1+\gamma)^2}{\tilde{\mu}} (-(r^2 + a^2)\tau\partial_\tau + a\partial_\phi)^2 \right),$$

and its determinant $|\det G|^{1/2} = (1+\gamma)^2 \rho^{-2} (\sin \theta)^{-1}$, we see that the only non-zero contribution to the right hand side of (3.6) comes from the term

$$\begin{aligned} & (1+\gamma)^2 \rho^{-2} (\sin \theta)^{-1} D_r ((1+\gamma)^{-2} \rho^2 \sin \theta \rho^{-2} \tilde{\mu} h'(r)) \tau D_\tau \\ & = -i \rho^{-2} \partial_r (\tilde{\mu} h'(r)) \tau D_\tau \end{aligned}$$

of the d'Alembertian. Mellin transforming this amounts to replacing τD_τ by σ ; then differentiating the result with respect to σ gives

$$\begin{aligned} \langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0) \rangle & = -i \int_{\tilde{\mu} > 0} \rho^{-2} \partial_r (\tilde{\mu} h'(r)) \, d\text{vol} \\ & = -i \int_0^\pi \int_0^{2\pi} \int_{r_-}^{r_+} (1+\gamma)^{-2} \sin \theta \partial_r (\tilde{\mu} h'(r)) \, dr \, d\phi \, d\theta \end{aligned}$$

³⁹See Equation (6.1) in [41].

$$= -\frac{4\pi i}{(1+\gamma)^2}[(\tilde{\mu}h'(r))|_{r_+} - (\tilde{\mu}h'(r))|_{r_-}]. \quad (3.7)$$

Since the singular part of $h'(r)$ at r_{\pm} (which are the roots of $\tilde{\mu}$) is $h'(r) = \mp \frac{1+\gamma}{\tilde{\mu}}(r^2 + a^2)$, the right hand side of (3.7) is positive up to a factor of i ; thus $\text{Im } \sigma'(0) < 0$ as claimed. \square

In other words, for small mass $m > 0$, there are no resonances σ of the Klein-Gordon operator with $\text{Im } \sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (3.1) no longer has a constant term. Correspondingly, for $\epsilon \in \mathbb{R}$, $\epsilon \leq \epsilon_0$, Theorem 3.1 gives the forward solution operator

$$S_{\text{KG},1}: H_{\text{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_{\text{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \quad (3.8)$$

in the dilation-invariant case.

Further, Theorem 3.2 is applicable and gives the forward solution operator

$$S_{\text{KG}}: \mathcal{H}_{\text{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \rightarrow \mathcal{H}_{\text{b},\Gamma}^s(\Omega)^{\bullet,-} \quad (3.9)$$

on the normally isotropic spaces.

For the semilinear application, for non-linearities without derivatives, it is important that the loss of derivatives \varkappa in the space $H_{\text{b}}^{s-1+\varkappa,\epsilon}$ is ≤ 1 . This is not explicitly specified in the paper of Wunsch and Zworski [47], though their proof directly⁴⁰ gives that, for small $\epsilon > 0$, \varkappa can be taken proportional to ϵ , and there is $\epsilon'_0 > 0$ such that $\varkappa \in (0, 1]$ for $\epsilon < \epsilon'_0$. We reduce $\epsilon_0 > 0$ above if needed so that $\epsilon_0 \leq \epsilon'_0$; then (3.8) holds with $\varkappa = c\epsilon \in (0, 1]$ if $\epsilon < \epsilon_0$, where $c > 0$.

In fact, one does not need to go through the proof of [47], for the Phragmén-Lindelöf theorem allows one to obtain the same conclusion from their final result:

Lemma 3.4. *Suppose $h: \Omega \rightarrow X$ is a holomorphic function on the half strip $\Omega = \{z \in \mathbb{C}: 0 \leq \text{Im } z \leq c, \text{Re } z \geq 1\}$ which is continuous on $\bar{\Omega}$, with values in a Banach space X , and suppose moreover that there are constants $A, C > 0$ such that*

$$\begin{aligned} \|h(z)\| &\leq C|z|^{k_1}, \quad \text{Im } z = 0, \\ \|h(z)\| &\leq C|z|^{k_2}, \quad \text{Im } z = c, \\ \|h(z)\| &\leq C \exp(A|z|), \quad z \in \bar{\Omega}. \end{aligned} \quad (3.10)$$

Then there is a constant $C' > 0$ such that

$$\|h(z)\| \leq C'|z|^{k_1(1-\frac{\text{Im } z}{c})+k_2\frac{\text{Im } z}{c}}$$

for all $z \in \bar{\Omega}$.

Proof. Consider the function $f(z) = z^{k_1-i\frac{k_2-k_1}{c}z}$, which is holomorphic on a neighborhood of $\bar{\Omega}$. Writing $z \in \bar{\Omega}$ as $z = x + iy$ with $x, y \in \mathbb{R}$, one has

$$\begin{aligned} |f(z)| &= |z|^{k_1} \exp\left(\text{Im}\left(\frac{k_2-k_1}{c}z \log z\right)\right) \\ &= |z|^{k_1} |z|^{\frac{k_2-k_1}{c}\text{Im } z} \exp\left(\frac{k_2-k_1}{c}x \arctan(y/x)\right). \end{aligned}$$

Noting that $|x \arctan(y/x)| = y|(x/y) \arctan(y/x)|$ is bounded by c for all $x+iy \in \bar{\Omega}$, we conclude that

$$e^{-|k_2-k_1||z|^{k_1(1-\frac{\text{Im } z}{c})+k_2\frac{\text{Im } z}{c}}} \leq |f(z)| \leq e^{|k_2-k_1||z|^{k_1(1-\frac{\text{Im } z}{c})+k_2\frac{\text{Im } z}{c}}}.$$

⁴⁰See especially the part before Section 4.4 of [47].

Therefore, $f(z)^{-1}h(z)$ is bounded by a constant C' on $\partial\bar{\Omega}$, and satisfies an exponential bound for $z \in \Omega$. By the Phragmén-Lindelöf theorem, $\|f(z)^{-1}h(z)\|_X \leq C'$, and the claim follows. \square

Since for any $\delta > 0$, we can bound $|\log z| \leq C_\delta |z|^\delta$ for $|\operatorname{Re} z| \geq 1$, we obtain that the inverse family $R(\sigma) = \widehat{\mathcal{P}}(\sigma)^{-1}$ of the normal operator of \square_g on (asymptotically) Kerr-de Sitter spaces as in [41], here in the setting of artificial boundaries as opposed to complex absorption, satisfies a bound

$$\|R(\sigma)\|_{|\sigma|^{-(s-1)}H_{|\sigma|^{-1}}^{s-1}(X \cap \Omega) \rightarrow |\sigma|^{-s}H_{|\sigma|^{-1}}^s(X \cap \Omega)} \leq C_\delta |\sigma|^{-1+\varkappa'+\delta} \quad (3.11)$$

for any $\delta > 0$, $\operatorname{Im} \sigma \geq -c\varkappa'$ and $|\operatorname{Re} \sigma|$ large. Therefore, as mentioned above, by the proof of Theorem 3.1, i.e. [41, Theorem 1.4], in particular using [41, Lemma 3.1], we can assume $\varkappa \in (0, 1]$ in the dilation-invariant result, Theorem 3.1, if we take $C' > 0$ small enough, i.e. if we do not go too far into the lower half plane $\operatorname{Im} \sigma < 0$, which amounts to only taking terms in the expansion (3.1) which decay to at most some fixed order, which we may assume to be less than $-\operatorname{Im} \sigma_j$ for all resonances σ_j .

3.2. A class of semilinear equations; equations with polynomial non-linearity. In the following semilinear applications, let us fix $\varkappa \in (0, 1]$ and ϵ_0 as explained before Lemma 3.4, so that we have the forward solution operator $S_{\text{KG}, \mathbb{I}}$ as in (3.8)

We then have statements paralleling Theorems 2.24, 2.35 and Corollary 2.27, namely Theorems 3.5, 3.9 and Corollary 3.8, respectively.

Theorem 3.5. *Suppose (M, g) is dilation-invariant. Let $-\infty < \epsilon < \epsilon_0$, $s > 1/2 + \beta\epsilon$, $s \geq 1$, and let $q: H_b^{s, \epsilon}(\Omega)^{\bullet, -} \rightarrow H_b^{s-1+\varkappa, \epsilon}(\Omega)^{\bullet, -}$ be a continuous function with $q(0) = 0$ such that there exists a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1+\varkappa, \epsilon}(\Omega)^{\bullet, -}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_b^{s, \epsilon}(\Omega)^{\bullet, -}$, with norm $\leq R$, that depends continuously on f .

More generally, suppose

$$q: H_b^{s, \epsilon}(\Omega)^{\bullet, -} \times H_b^{s-1+\varkappa, \epsilon}(\Omega)^{\bullet, -} \rightarrow H_b^{s-1+\varkappa, \epsilon}(\Omega)^{\bullet, -}$$

satisfies $q(0, 0) = 0$ and

$$\|q(u, w) - q(u', w')\| \leq L(R)(\|u - u'\| + \|w - w'\|)$$

provided $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, where we use the norms corresponding to the map q , for a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{s-1+\varkappa, \epsilon}(\Omega)^{\bullet, -}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, \square_g u)$$

has a unique solution $u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_b^{s,\epsilon}} + \|\square_g u\|_{H_b^{s-1+\varkappa,\epsilon}} \leq R$, that depends continuously on f .

Proof. We use the proof of the first part of Theorem 2.24, where in the current setting the solution operator $S_{\text{KG},\text{I}}$ maps $H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, and the contraction map is $T: H_b^{s,\epsilon}(\Omega)^{\bullet,-} \rightarrow H_b^{s,\epsilon}(\Omega)^{\bullet,-}$, $Tu = S_{\text{KG},\text{I}}(f + q(u))$.

For the general statement, we follow the proof of the second part of Theorem 2.24, where we now instead use the Banach space

$$\mathcal{X} = \{u \in H_b^{s,\epsilon}(\Omega)^{\bullet,-} : \square_g u \in H_b^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_b^{s,\epsilon}} + \|\square_g u\|_{\tau^\epsilon H_b^{s-1+\varkappa}}.$$

which is a Banach space by the same argument as in the proof of Theorem 2.24. \square

We have a weaker statement in the general, non dilation-invariant case, where we work in unweighted spaces.

Theorem 3.6. *Let $s \geq 1$, and suppose $q: H_b^s(\Omega)^{\bullet,-} \rightarrow H_b^s(\Omega)^{\bullet,-}$ is a continuous function with $q(0) = 0$ such that there exists a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^s(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_b^s(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

An analogous statement holds for non-linearities $q = q(u, \square_g u)$ which are continuous maps $q: H_b^s(\Omega)^{\bullet,-} \times H_b^s(\Omega)^{\bullet,-} \rightarrow H_b^s(\Omega)^{\bullet,-}$, vanish at $(0, 0)$ and have a small Lipschitz constant near 0.

Proof. Since

$$S_{\text{KG}}: H_b^s(\Omega)^{\bullet,-} \subset \mathcal{H}_{\bar{h},\Gamma}^{*,s-1/2}(\Omega)^{\bullet,-} \rightarrow \mathcal{H}_{\bar{h},\Gamma}^{s+1/2}(\Omega)^{\bullet,-} \subset H_b^s(\Omega)^{\bullet,-},$$

by (3.3) and (3.9), this follows again from the Banach fixed point theorem. \square

Remark 3.7. The proof of Theorem 3.2 shows that equations on spaces with negative weights (i.e. growing near infinity) behave as nicely as equations on the static part of de Sitter space, discussed in Section 2. However, naturally occurring non-linearities (e.g., polynomials) will not be continuous non-linear operators on such growing spaces.

Corollary 3.8. *If $s > n/2$, the hypotheses of Theorem 3.6 hold for non-linearities $q(u) = cu^p$, $p \geq 2$ integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_b^s(M)$.*

Thus for small $m > 0$ and $R > 0$, there exists $C > 0$ such that for all $f \in H_b^s(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_b^s(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

If f satisfies stronger decay assumptions, then u does as well. More precisely, denoting the inverse family of the normal operator of the Klein-Gordon operator with (small) mass m by $R_m(\sigma) = (\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$, which has poles only in $\text{Im } \sigma < 0$ (cf. Lemma 3.3 and [16, 41]), and moreover defining the spaces $\mathcal{X}_{\mathcal{F}}^{s,r,\epsilon}$ and $\mathcal{X}_{\mathcal{P},\mathcal{F}}^{s,r,\epsilon}$ completely analogously to the corresponding spaces in Section 2.3, we have the following result:

Theorem 3.9. *Fix $0 < \epsilon < \min\{C', 1/2\}$ and let $s \gg s' \geq \max\{1/2 + \beta\epsilon, n/2, 1 + \varkappa\}$.⁴¹ Let*

$$q(u) = \sum_{p=2}^d q_p u^p, \quad q_p \in H_b^s(M).$$

Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $R_m(\sigma)$, and $m_j + 1$ is the order of the pole of $R_m(\sigma)$ at σ_j , let $\mathcal{P} = \{(i\sigma_j + k, \ell) : 0 \leq \ell \leq m_j, k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \text{Re}(i\sigma_j)$ for all j , and that $m > 0$ is so small that \mathcal{P} is a positive index set. Finally, let \mathcal{F} be a positive index set.

Then for small enough $R > 0$, there exists $C > 0$ such that for all $f \in \mathcal{X}_{\mathcal{F}}^{s,s,\epsilon}$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u) \tag{3.12}$$

has a unique solution $u \in \mathcal{X}_{\mathcal{P},\mathcal{F}}^{s',s',\epsilon}$, with norm $\leq R$, that depends continuously on f ; in particular, u has an asymptotic expansion with remainder term in $H_b^{s,\epsilon}(\Omega)^{\bullet,-}$.

Proof. Let us write $\mathcal{P} = \square_g - m^2$. Let $\delta < 1/2$ be such that $0 < 2\delta < \text{Re } z$ for all $(z, 0) \in \mathcal{F}$, then $f \in H_b^{s,2\delta}(\Omega)^{\bullet,-}$. Now, for $u \in H_b^{s,\delta}(\Omega)^{\bullet,-}$, consider $Tu := S_{\text{KG}}(f + q(u))$. First of all, $f + q(u) \in H_b^{s,2\delta}(\Omega)^{\bullet,-} \subset H_b^s(\Omega)^{\bullet,-}$, thus the proof of Theorem 3.2 shows that we also have $u \in H_b^{s+1,r}(\Omega)^{\bullet,-}$, $r < 0$ arbitrary. Therefore,

$$N(\mathcal{P})u = f + q(u) + (N(\mathcal{P}) - \mathcal{P})u \in H_b^{s,2\delta}(\Omega)^{\bullet,-} + H_b^{s-1,r+1}(\Omega)^{\bullet,-} \subset H_b^{s-1,2\delta}(\Omega)^{\bullet,-},$$

and thus if $\delta > 0$ is sufficiently small, namely, $\delta < \inf\{-\text{Im } \sigma_j\}/2$, Theorem 3.1 implies $u \in H_b^{s-\varkappa,2\delta}(\Omega)^{\bullet,-}$. Since we can choose $\varkappa = c\delta$ for some constant $c > 0$, we obtain

$$u \in \bigcap_{r>0} H_b^{s+1,r}(\Omega)^{\bullet,-} \cap H_b^{s-c\delta,2\delta}(\Omega)^{\bullet,-} \subset \bigcap_{r'>0} H_b^{s,2\delta-2c\delta^2/(1+c\delta)-r'}(\Omega)^{\bullet,-}$$

by interpolation. In particular, choosing $\delta > 0$ even smaller if necessary, we obtain $Tu \in H_b^{s,\delta}(\Omega)^{\bullet,-}$. Applying the Banach fixed point theorem to the map T thus gives a solution $u \in H_b^{s,\delta}(\Omega)^{\bullet,-}$ to the equation (3.12).

For this solution u , we obtain

$$N(\mathcal{P})u = \mathcal{P}u + (N(\mathcal{P}) - \mathcal{P})u \in H_b^{s,2\delta} + H_b^{s-2,\delta+1} \subset H_b^{s-2,2\delta}$$

since q only has quadratic and higher terms. Hence Theorem 3.1 implies that $u = u_1 + u'$, where u_1 is an expansion with terms coming from poles of $\widehat{\mathcal{P}}^{-1}$ whose decay order lies between δ and 2δ , and $u' \in H_b^{s-1-\varkappa,2\delta}(\Omega)^{\bullet,-}$. This in turn implies

⁴¹A concrete bound for s will be given in the course of the proof, but it will depend on an unspecified constant; see Footnote 42.

that $f + q(u)$ has an expansion with remainder term in $H_b^{s-1-\varkappa, \min\{4\delta, \epsilon\}}(\Omega)^{\bullet, -}$, thus

$$N(\mathcal{P})u \in H_b^{s-3-\varkappa, \min\{4\delta, \epsilon\}}(\Omega)^{\bullet, -} \text{ plus an expansion,}$$

and we proceed iteratively, until, after k more steps, we have $4 \cdot 2^k \delta \geq \epsilon$, and then u has an expansion with remainder term $H_b^{s-3-2k-\varkappa, \epsilon}(\Omega)^{\bullet, -}$ provided we can apply Theorem 3.1 in the iterative procedure, i.e. provided $s - 3 - 2k - \varkappa =: s' > \max\{1/2 + \beta\epsilon, n/2, 1 + \varkappa\}$.⁴² \square

3.3. Semilinear equations with derivatives in the non-linearities. Theorem 3.2 allows one to solve even semilinear equations with derivatives in some cases. For instance, in the case of de Sitter-Schwarzschild space, within $\Sigma \cap^b S_X^* M$, Γ is given by $r = r_c$, $\sigma_1(D_r) = 0$, where $r_c = \frac{3}{2}r_s$ is the radius of the photon sphere, see e.g. [41, Section 6.4]. Thus, non-linear terms such as $(r - r_c)(\partial_r u)^2$ are allowed for $s > \frac{n}{2} + 1$ since $\partial_r : \mathcal{H}_{b, \Gamma}^s(M) \rightarrow H_b^{s-1}(M)$, with the latter space being an algebra, while multiplication by $r - r_c$ maps this space to $\mathcal{H}_{b, \Gamma}^{*, s-1}$ by (3.4). Thus, a straightforward modification of Theorem 3.6, applying the fixed point theorem on the normally isotropic spaces directly, gives well-posedness.

4. DE SITTER SPACE

We can approach the problem of solving non-linear wave equations on global de Sitter space in two ways: Either, we proceed as in the previous two sections, first showing invertibility of the linear operator on suitable spaces and then applying the contraction mapping principle to solve the non-linear problem; or we use the solvability results from Section 2 for static parts of de Sitter space and glue the solutions on all static parts together to obtain a global solution. The first approach, which we will follow in Sections 4.1-4.4, has the disadvantage that the conditions on the non-linearity that guarantee the existence of solutions are quite restrictive, however in case the conditions are met, one has good decay estimates for solutions. The second approach on the other hand, detailed in Section 4.5, allows many of the non-linearities, suitably reinterpreted, that work on static parts of de Sitter space, but the decay estimates for solutions are quite weak relative to the decay of the forcing term because of the gluing process.

4.1. The linear framework. Let g be the metric on an n -dimensional asymptotically de Sitter space X with global time function t [43]. Then, following [41, Section 4], the operator⁴³

$$P_\sigma = \mu^{-1/2} \mu^{i\sigma/2 - (n+1)/4} \left(\square_g - \left(\frac{n-1}{2} \right)^2 - \sigma^2 \right) \mu^{-i\sigma/2 + (n+1)/4} \mu^{-1/2} \quad (4.1)$$

extends non-degenerately to an operator on a closed manifold \tilde{X} which contains the compactification \bar{X} of de Sitter space as a submanifold with boundary Y , where $Y = Y_- \cup Y_+$ has two connected components, which we call the boundary of X at past, resp. future, infinity.⁴⁴ Here, $\mu = 0$ is the defining function of Y , and $\mu > 0$ is the interior of de Sitter space. Moreover, null-bicharacteristics of P_σ tend to Y_\pm as $t \rightarrow \pm\infty$.

⁴²This is satisfied if $s > \max\{1/2 + \beta\epsilon, n/2, 1 + \varkappa\} + 2[\log_2(\epsilon/\delta)] + \varkappa - 1$.

⁴³ P_σ in our notation corresponds to P_σ^* in [41], the latter operator being the one for which one solves the forward problem.

⁴⁴Non-degenerately here means that near Y_\pm , P_σ fits into the framework of [41].

Following Vasy [40], let us in fact assume that $\tilde{X} = \overline{C_-} \cup \overline{X} \cup \overline{C_+}$ is the union of the compactifications of asymptotically de Sitter space X and two asymptotically hyperbolic caps C_\pm .⁴⁵ Then P_σ is the restriction to X of an operator $\tilde{P}_\sigma \in \text{Diff}^2(\tilde{X})$, which is Fredholm as a map

$$\tilde{P}_\sigma: \tilde{\mathcal{X}}^s \rightarrow \tilde{\mathcal{Y}}^{s-1}, \quad \tilde{\mathcal{X}}^s = \{u \in H^s: \tilde{P}_\sigma u \in H^{s-1}\}, \quad \tilde{\mathcal{Y}}^{s-1} = H^{s-1},$$

where⁴⁶ $s \in C^\infty(S^*\tilde{X})$, monotone along the bicharacteristic flow, is such that $s|_{N^*Y_-} > 1/2 - \text{Im } \sigma$, $s|_{N^*Y_+} < 1/2 - \text{Im } \sigma$, and s is constant near S^*Y_\pm .⁴⁷

Restricting our attention to X , we define the space $H^s(X)^{\bullet,-}$ to be the completion in $H^s(X)$ of the space of C^∞ functions that vanish to infinite order at Y_- ; thus the superscripts indicate that distributions in $H^s(X)^{\bullet,-}$ are supported distributions near Y_- and extendible distributions near Y_+ . Then, define the spaces

$$\mathcal{X}^s = \{u \in H^s(X)^{\bullet,-}: P_\sigma u \in H^{s-1}(X)^{\bullet,-}\}, \quad \mathcal{Y}^{s-1} = H^{s-1}(X)^{\bullet,-}.$$

Theorem 4.1. *Fix $\sigma \in \mathbb{C}$ and $s \in C^\infty(S^*\overline{X})$ as above. Then $P_\sigma: \mathcal{X}^s \rightarrow \mathcal{Y}^{s-1}$ is invertible, and $P_\sigma^{-1}: H^{s-1}(X)^{\bullet,-} \rightarrow H^s(X)^{\bullet,-}$ is the forward solution operator of P_σ .*

Proof. First, let us assume $\text{Re } \sigma \gg 0$ so semiclassical/large parameter estimates are applicable to \tilde{P}_σ , and let $T_0 \in \mathbb{R}$ be such that s is constant in $\{t \leq T_0\}$. Then for any $T_1 \leq T_0$, we can paste together microlocal energy estimates for \tilde{P}_σ near $\overline{C_-}$ and standard energy estimates for the wave equation in $\{t \leq T_1\}$ away from Y_- as in the derivation of Equation (3.29) of [41], and thereby obtain

$$\|u\|_{H^1(\{t \leq T_1\})} \lesssim \|\tilde{P}_\sigma u\|_{H^0(\{t \leq T_1\})}; \quad (4.2)$$

thus, $f \in C^\infty(\tilde{X})$, $\text{supp } f \subset \{t \geq T_1\}$ implies $\text{supp } \tilde{P}_\sigma^{-1} f \subset \{t \geq T_1\}$. Choosing $\phi \in C_c^\infty(X)$ with support in $\{t \geq T_1\}$ and $\psi \in C^\infty(\tilde{X})$ with support in $\{t \leq T_1\}$, we therefore obtain $\psi \tilde{P}_\sigma^{-1} \phi = 0$. Since \tilde{P}_σ^{-1} is meromorphic, this continues to hold for all $\sigma \in \mathbb{C}$ such that $\text{Im } \sigma > 1/2 - s$. Since $T_1 \leq T_0$ is arbitrary, this, together with standard energy estimates on the asymptotically de Sitter space X , proves that P_σ^{-1} propagates supports forward, provided P_σ is invertible. Moreover, elements of $\ker \tilde{P}_\sigma$ are supported in $\overline{C_+}$.

The invertibility of P_σ is a consequence of [2, Lemma 8.3], also see Footnote 15 there: Let $E: H^{s-1}(X)^{\bullet,-} \rightarrow H^{s-1}(\tilde{X})$ be a continuous extension operator that extends by 0 in $\overline{C_-}$ and $R: H^s(\tilde{X}) \rightarrow H^s(X)^{\bullet,-}$ the restriction, then $R \circ \tilde{P}_\sigma^{-1} \circ E$ does not have poles; and since

$$\bigcup_{T_1 \leq T_0} H^s(\{t > T_1\})^{\bullet,-} \subset H^s(X)^{\bullet,-}$$

(where \bullet denotes supported distributions at $\{t = T_1\}$, resp. Y_-) is dense, $R \circ \tilde{P}_\sigma^{-1} \circ E$ in fact maps into $H^s(X)^{\bullet,-}$, thus $P_\sigma^{-1} = R \circ \tilde{P}_\sigma^{-1} \circ E$ indeed exists and has the claimed properties. \square

In our quest for finding forward solutions of semilinear equations, we restrict ourselves to a submanifold with boundary $\Omega \subset \overline{X}$ containing and localized near

⁴⁵One might need to take two copies of X to construct \tilde{X} ; see [40].

⁴⁶We are using variable order Sobolev spaces as in [2, Section 1, Appendix].

⁴⁷The choice of signs here is opposite to the one in [40], since here we are going to construct the forward solution operator on X .

future infinity, so that we can work in fixed order Sobolev spaces; moreover, it will be useful to measure the conormal regularity of solutions to the linear equation at the conormal bundle of the boundary of X at future infinity more precisely. So let $H^{s,k}(\tilde{X}, Y_+)$ be the subspace of $H^s(\tilde{X})$ with k -fold regularity with respect to the $\Psi^0(\tilde{X})$ -module \mathcal{M} of first order Ψ DOs with principal symbol vanishing on N^*Y_+ . A result of Haber and Vasy, [20, Theorem 6.3], with $s_0 = 1/2 - \text{Im } \sigma$ in our case, shows that $f \in H^{s-1,k}(\tilde{X}, Y_+)$, $\tilde{P}_\sigma u = f$, u a distribution, in fact imply that $u \in H^{s,k}(\tilde{X}, Y_+)$. So if we let $H^{s,k}(\Omega)^{\bullet,-}$ denote the space of all $u \in H^s(X)^{\bullet,-}$ which are restrictions to Ω of functions in $H^{s,k}(\tilde{X}, Y_+)$, supported in $\Omega \cup \overline{C_+}$, the argument of Theorem 4.1 shows that we have a forward solution operator $S_\sigma: H^{s-1,k}(\Omega)^{\bullet,-} \rightarrow H^{s,k}(\Omega)^{\bullet,-}$, provided

$$s < 1/2 - \text{Im } \sigma. \quad (4.3)$$

4.1.1. *The backward problem.* Another problem that we will briefly consider below is the backward problem, i.e. where one solves the equation on X backward from Y_+ , which is the same, up to relabelling, as solving the equation forward from Y_- . Thus, we have a backward solution operator $S_\sigma^-: H^{s-1,k}(\Omega)^{-,\bullet} \rightarrow H^{s,k}(\Omega)^{-,\bullet}$ (where Ω is chosen as above so that we can use constant order Sobolev spaces), provided $s > 1/2 - \text{Im } \sigma$. Similarly to the above, $-$ denotes extendible distributions at $\partial\Omega \cap X^\circ$ and \bullet supported distributions at Y_+ ; the module regularity is measured at Y_+ .

4.2. **Algebra properties of $H^{s,k}(\Omega)^{\bullet,-}$.** Let us call a polynomially bounded measurable function $w: \mathbb{R}^n \rightarrow (0, \infty)$ a *weight function*. For such a weight function w , we define

$$H^{(w)}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n): w\hat{u} \in L^2(\mathbb{R}^n)\}.$$

The following lemma is similar in spirit to, but different from, Strichartz' result on Sobolev algebras [37]; it is the basis for the multiplicative properties of the more delicate spaces considered below.

Lemma 4.2. *Let w_1, w_2, w be weight functions such that one of the quantities*

$$\begin{aligned} M_+ &:= \sup_{\xi \in \mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi - \eta)} \right)^2 d\eta \\ M_- &:= \sup_{\eta \in \mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi - \eta)} \right)^2 d\xi \end{aligned} \quad (4.4)$$

is finite. Then $H^{(w_1)}(\mathbb{R}^n) \cdot H^{(w_2)}(\mathbb{R}^n) \subset H^{(w)}(\mathbb{R}^n)$.

Proof. For $u, v \in S(\mathbb{R}^n)$, we use Cauchy-Schwarz to estimate

$$\begin{aligned} \|uv\|_{H^{(w)}}^2 &= \int w(\xi)^2 |\widehat{uv}(\xi)|^2 d\xi \\ &\leq \int w(\xi)^2 \left(\int w_1(\eta) |\hat{u}(\eta)| w_2(\xi - \eta) |\hat{v}(\xi - \eta)| w_1(\eta)^{-1} w_2(\xi - \eta)^{-1} d\eta \right)^2 d\xi \\ &\leq \int \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi - \eta)} \right)^2 d\eta \right) \\ &\quad \times \left(\int w_1(\eta)^2 |\hat{u}(\eta)|^2 w_2(\xi - \eta)^2 |\hat{v}(\xi - \eta)|^2 d\eta \right) d\xi \end{aligned}$$

$$\leq M_+ \|u\|_{H^{(w_1)}}^2 \|v\|_{H^{(w_2)}}^2$$

as well as

$$\begin{aligned} \|uv\|_{H^{(w)}}^2 &\leq \int \left(\int w_2(\xi - \eta)^2 |\hat{v}(\xi - \eta)|^2 d\eta \right) \\ &\quad \times \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi - \eta)} \right)^2 w_1(\eta)^2 |\hat{u}(\eta)|^2 d\eta \right) d\xi \\ &= \|v\|_{H^{(w_2)}}^2 \int w_1(\eta)^2 |\hat{u}(\eta)|^2 \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi - \eta)} \right)^2 d\xi \right) d\eta \\ &\leq M_- \|u\|_{H^{(w_1)}}^2 \|v\|_{H^{(w_2)}}^2. \end{aligned}$$

Since $S(\mathbb{R}^n)$ is dense in $H^{(w_1)}(\mathbb{R}^n)$ and $H^{(w_2)}(\mathbb{R}^n)$, the lemma follows. \square

In particular, if

$$\left\| \frac{w(\xi)}{w(\eta)w(\xi - \eta)} \right\|_{L_\xi^\infty L_\eta^2} < \infty, \quad (4.5)$$

then $H^{(w)}$ is an algebra.

For example, the weight function $w(\xi) = \langle \xi \rangle^s$ for $s > n/2$ satisfies (4.5) as we will check below, which implies that $H^s(\mathbb{R}^n)$ is an algebra for $s > n/2$; this is the special case $k = 0$ of Lemma 4.4 below, and is well-known, see e.g. [38, Chapter 13.3]. Also, product-type weight functions $w_d(\xi) = \langle \xi' \rangle^s \langle \xi'' \rangle^k$ (where $\xi = (\xi', \xi'') \in \mathbb{R}^{d+(n-d)}$) for $s > d/2, k > (n-d)/2$ satisfy (4.5).

The following lemma, together with the triangle inequality $\langle \xi \rangle^\alpha \lesssim \langle \eta \rangle^\alpha + \langle \xi - \eta \rangle^\alpha$ for $\alpha \geq 0$, will often be used to check conditions like (4.4).

Lemma 4.3. *Suppose $\alpha, \beta \geq 0$ are such that $\alpha + \beta > n$. Then*

$$\int_{\mathbb{R}^n} \frac{d\eta}{\langle \eta \rangle^\alpha \langle \xi - \eta \rangle^\beta} \in L^\infty(\mathbb{R}_\xi^n).$$

Proof. Splitting the domain of integration into the two regions $\{\langle \eta \rangle < \langle \xi - \eta \rangle\}$ and $\{\langle \eta \rangle \geq \langle \xi - \eta \rangle\}$, we obtain the bound

$$\int_{\mathbb{R}^n} \frac{d\eta}{\langle \eta \rangle^\alpha \langle \xi - \eta \rangle^\beta} \leq 2 \int_{\mathbb{R}^n} \frac{d\eta}{\langle \eta \rangle^{\alpha+\beta}},$$

which is finite in view of $\alpha + \beta > n$. \square

Another important consequence of Lemma 4.2 is that $H^{s'}(\mathbb{R}^n)$ is an $H^s(\mathbb{R}^n)$ -module provided $|s'| \leq s, s > n/2$, which follows for $s' \geq 0$ from $M_+ < \infty$, and for $s' < 0$ either by duality or from $M_- < \infty$ (with M_\pm as in the statement of the lemma, with the corresponding weight functions).

Lemma 4.4. *Write $x \in \mathbb{R}^n$ as $x = (x', x'') \in \mathbb{R}^{d+(n-d)}$. For $s \in \mathbb{R}, k \in \mathbb{N}_0$, let*

$$\mathcal{Y}_d^{s,k}(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : D_{x''}^k u \in H^s(\mathbb{R}^n)\}.$$

Then for $s > d/2, s + k > n/2$, $\mathcal{Y}_d^{s,k}(\mathbb{R}^n)$ is an algebra.

Proof. Using the Leibniz rule, we see that it suffices to show: If $u, v \in \mathcal{Y}_d^{s,k}$, then $D_{x''}^\alpha u D_{x''}^\beta v \in H^s$, provided $|\alpha| + |\beta| \leq k$. Since $D_{x''}^\alpha u \in \mathcal{Y}_d^{s,k-|\alpha|}$ and $D_{x''}^\beta v \in \mathcal{Y}_d^{s,k-|\beta|}$, this amounts to showing that

$$\mathcal{Y}_d^{s,a} \cdot \mathcal{Y}_d^{s,b} \subset H^s \text{ if } a + b \geq k. \quad (4.6)$$

Using the characterization $\mathcal{Y}_d^{s,a} = H^{(w)}$ for $w(\xi) = \langle \xi \rangle^s \langle \xi'' \rangle^k$, Lemma 4.2 in turn reduces this to the estimate

$$\begin{aligned} & \int \frac{\langle \xi \rangle^{2s}}{\langle \eta \rangle^{2s} \langle \eta'' \rangle^{2a} \langle \xi - \eta \rangle^{2s} \langle \xi'' - \eta'' \rangle^{2b}} d\eta \\ & \lesssim \int \frac{d\eta}{\langle \eta'' \rangle^{2a} \langle \xi - \eta \rangle^{2s} \langle \xi'' - \eta'' \rangle^{2b}} + \int \frac{d\eta}{\langle \eta \rangle^{2s} \langle \eta'' \rangle^{2a} \langle \xi'' - \eta'' \rangle^{2b}} \\ & \leq \int \frac{d\eta'}{\langle \xi' - \eta' \rangle^{2s'}} \int \frac{d\eta''}{\langle \eta'' \rangle^{2a} \langle \xi'' - \eta'' \rangle^{2b+2(s-s')}} \\ & \quad + \int \frac{d\eta'}{\langle \eta' \rangle^{2s'}} \int \frac{d\eta''}{\langle \eta'' \rangle^{2a+2(s-s')} \langle \xi'' - \eta'' \rangle^{2b}}, \end{aligned}$$

where we choose $d/2 < s' < s$ such that $a + b + s - s' > (n - d)/2$, which holds if $k + s > (n - d)/2 + s'$, which is possible by our assumptions on s and k . The integrals are uniformly bounded in ξ : For the η' -integrals, this follows from $s' > d/2$; for the η'' -integrals, we use Lemma 4.3. \square

We shall now use this (non-invariant) result to prove algebra properties for spaces with iterated module regularity: Consider a compact manifold without boundary X and a submanifold Y . Let $\mathcal{M} \supset \Psi^0(X)$ be the $\Psi^0(X)$ -module of first order PDOs whose principal symbol vanishes on N^*Y . For $s \in \mathbb{R}, k \in \mathbb{N}_0$, define

$$H^{s,k}(X, Y) = \{u \in H^s(X) : \mathcal{M}^k u \in H^s(X)\}.$$

Proposition 4.5. *Suppose $\dim(X) = n$ and $\text{codim}(Y) = d$. Assume that $s > d/2$ and $s + k > n/2$. Then $H^{s,k}(X, Y)$ is an algebra.*

Proof. Away from Y , $H^{s,k}(X, Y)$ is just $H^{s+k}(X)$, which is an algebra since $s + k > \dim(X)/2$. Thus, since the statement is local, we may assume that we have a product decomposition near Y , namely $X = \mathbb{R}_{x'}^d \times \mathbb{R}_{x''}^{n-d}$, $Y = \{x' = 0\}$, and that we are given arbitrary $u, v \in H^{s,k}(X, Y)$ with compact support close to $(0, 0)$ for which we have to show $uv \in H^{s,k}(X, Y)$. Notice that for $f \in H^s(X)$ with such small support, $f \in H^{s,k}(X, Y)$ is equivalent to $\mathcal{M}' f \in H^s(X)$, where \mathcal{M}' is the $C^\infty(M)$ -module of differential operators generated by $\text{Id}, \partial_{x'_i}, x'_j \partial_{x'_k}$, where $1 \leq i \leq n - d, 1 \leq j, k \leq d$.

Thus the proposition follows from the following statement: For s, k as in the statement of the proposition,

$$H^{s,k}(\mathbb{R}^n, \mathbb{R}^{n-d}) := \{u \in H^s(\mathbb{R}^n) : (x')^{\tilde{\alpha}} D_{x'}^\alpha D_{x''}^\beta u \in H^s(\mathbb{R}^n), |\tilde{\alpha}| = |\alpha|, |\alpha| + |\beta| \leq k\}$$

is an algebra. Using the Leibniz rule, we thus have to show that

$$((x')^{\tilde{\alpha}} D_{x'}^\alpha D_{x''}^\beta u) ((x')^{\tilde{\gamma}} D_{x'}^\gamma D_{x''}^\delta v) \in H^s, \quad (4.7)$$

provided $|\tilde{\alpha}| = |\alpha|, |\tilde{\gamma}| = |\gamma|, |\alpha| + |\beta| + |\gamma| + |\delta| \leq k$. Since the two factors in (4.7) lie in $H^{s,k-|\alpha|-|\beta|}$ and $H^{s,k-|\gamma|-|\delta|}$, respectively, this amounts to showing that $H^{s,a} \cdot H^{s,b} \subset H^s$ for $a + b \geq k$. This however is easy to see, since $H^{s,c} \subset \mathcal{Y}_d^{s,c}$ for all $c \in \mathbb{N}_0$, and $\mathcal{Y}_d^{s,a} \cdot \mathcal{Y}_d^{s,b} \subset H^s$ was proved in (4.6). \square

In order to be able to obtain sharper results for particular non-linear equations in Section 4.3, we will now prove further results in the case $\text{codim}(Y) = 1$, which we will assume to hold from now on; also, we fix $n = \dim(X)$.

Proposition 4.6. *Assume that $s > 1/2$ and $k > (n-1)/2$. Then $H^{s,k}(X, Y) \cdot H^{s-1,k}(X, Y) \subset H^{s-1,k}(X, Y)$.*

Proof. Using the Leibniz rule, this follows from $\mathcal{Y}_1^{s,a} \cdot \mathcal{Y}_1^{s-1,b} \subset H^{s-1}$ for $a + b \geq k$. This, as before, can be reduced to the local statement on $\mathbb{R}^n = \mathbb{R}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ with $Y = \{x_1 = 0\}$. We write $\xi = (\xi_1, \xi') \in \mathbb{R}^{1+(n-1)}$ and $\eta = (\eta_1, \eta') \in \mathbb{R}^{1+(n-1)}$. By Lemma 4.2, the case $s \geq 1$ follows from the estimate

$$\begin{aligned} & \int \frac{\langle \xi \rangle^{2(s-1)}}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} d\eta \\ & \lesssim \int \frac{d\eta}{\langle \eta \rangle^2 \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} + \int \frac{d\eta}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \\ & \leq 2 \int \frac{d\eta_1}{\langle \eta_1 \rangle^{2s}} \int \frac{d\eta'}{\langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \in L_\xi^\infty \end{aligned}$$

by Lemma 4.3.

If $1/2 < s \leq 1$, then ξ_1 and ξ' play different roles.⁴⁸ Let $u \in \mathcal{Y}_1^{s,a}$, $v \in \mathcal{Y}_1^{s-1,b}$, and put

$$u_0(\xi) = \langle \xi \rangle^s \langle \xi' \rangle^a u(\xi) \in L^2(\mathbb{R}^n), \quad v_0(\xi) = \langle \xi \rangle^{s-1} \langle \xi' \rangle^b v(\xi) \in L^2(\mathbb{R}^n).$$

Then

$$\langle \xi \rangle^{s-1} \widehat{uv}(\xi) = \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \eta' \rangle^b \langle \xi - \eta \rangle^s \langle \xi' - \eta' \rangle^a} u_0(\xi - \eta) v_0(\eta) d\eta,$$

hence by Cauchy-Schwartz and Lemma 4.3

$$\begin{aligned} & \int \langle \xi \rangle^{2(s-1)} |\widehat{uv}(\xi)|^2 d\xi \\ & \leq \int \left(\int \frac{d\eta'}{\langle \eta' \rangle^{2b} \langle \xi' - \eta' \rangle^{2a}} \right) \left(\int \left| \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \xi - \eta \rangle^s} u_0(\xi - \eta) v_0(\eta) d\eta_1 \right|^2 d\eta' \right) d\xi \\ & \lesssim \iint \left(\int |u_0(\xi - \eta)|^2 d\eta_1 \right) \left(\int \frac{\langle \eta \rangle^{2(1-s)}}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2s}} |v_0(\eta)|^2 d\eta_1 \right) d\eta' d\xi \\ & \lesssim \iint \|u_0(\cdot, \xi' - \eta')\|_{L^2}^2 |v_0(\eta)|^2 \\ & \quad \times \left(\int \frac{1}{\langle \xi - \eta \rangle^{2s}} + \frac{1}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2(2s-1)}} d\xi_1 \right) d\xi' d\eta \\ & \lesssim \|u\|_{\mathcal{Y}_1^{s,a}}^2 \|v\|_{\mathcal{Y}_1^{s-1,b}}^2, \end{aligned}$$

since $1/2 < s \leq 1$, thus $1-s \geq 0$ and $2s-1 > 0$, and the ξ_1 -integral is thus bounded from above by

$$\int \frac{1}{\langle \xi_1 - \eta_1 \rangle^{2s}} + \frac{1}{\langle \xi_1 \rangle^{2(1-s)} \langle \xi_1 - \eta_1 \rangle^{2(2s-1)}} d\xi_1 \in L_{\eta_1}^\infty.$$

⁴⁸The background regularity to be proved is H^{s-1} , $s-1 \leq 0$, thus the continuity of multiplication in the conormal direction to Y is proved by 'duality' (i.e. using Lemma 4.2 with $M_- < \infty$), whereas the continuity in the tangential (to Y) directions, where both factors have $k > (n-1)/2$ derivatives, is proved directly (i.e. using Lemma 4.2 with $M_+ < \infty$).

The proof is complete. \square

For semilinear equations whose non-linearity does not involve any derivatives, one can afford to lose derivatives in multiplication statements. We give two useful results in this context, the first being a consequence of Proposition 4.6.

Corollary 4.7. *Let $\mu \in C^\infty(X)$ be a defining function for Y , i.e. $\mu|_Y \equiv 0$, $d\mu \neq 0$ on Y , and μ vanishes on Y only. Suppose $s > 1/2$ and $\ell \in \mathbb{C}$ are such that $\operatorname{Re} \ell + 3/2 > s$. Then multiplication by μ_+^ℓ defines a continuous map $H^{s,k}(X, Y) \rightarrow H^{s-1,k}(X, Y)$ for all $k \in \mathbb{N}_0$.*

Proof. By the Leibniz rule, it suffices to prove the statement for $k = 0$. We have $\mu_+^\ell \in H^{\operatorname{Re} \ell + 1/2 - \epsilon; \infty}(X, Y)$ for all $\epsilon > 0$ ⁴⁹, hence the corollary follows from Proposition 4.6, since one has $\operatorname{Re} \ell + 1/2 - \epsilon \geq s - 1$ for some $\epsilon > 0$ provided $\operatorname{Re} \ell + 3/2 > s$. \square

Proposition 4.8. *Let $0 \leq s', s_1, s_2 < 1/2$ be such that $s' < s_1 + s_2 - 1/2$, and let $k > (n-1)/2$. Then $H^{s_1,k}(X, Y) \cdot H^{s_2,k}(X, Y) \subset H^{s',k}(X, Y)$.*

Proof. Using the Leibniz rule, this reduces to the statement that $\mathcal{Y}_1^{s_1,a} \cdot \mathcal{Y}_1^{s_2,b} \subset H^{s'}$ if $a + b \geq k$. Splitting covariables $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, Lemma 4.2 in turn reduces this to the observation that

$$\begin{aligned} & \int \frac{\langle \xi \rangle^{2s'}}{\langle \eta \rangle^{2s_1} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2s_2} \langle \xi' - \eta' \rangle^{2b}} d\eta \\ & \lesssim \left(\int \frac{d\eta_1}{\langle \eta_1 \rangle^{2(s_1-s')} \langle \xi_1 - \eta_1 \rangle^{2s_2}} + \int \frac{d\eta_1}{\langle \eta_1 \rangle^{2s_1} \langle \xi_1 - \eta_1 \rangle^{2(s_2-s')}} \right) \\ & \quad \times \int \frac{d\eta'}{\langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \end{aligned}$$

is uniformly bounded in ξ by Lemma 4.3 in view of $s' < s_1 + s_2 - 1/2 < \min\{s_1, s_2\}$, thus $s_1 - s' > 0$ and $s_2 - s' > 0$, and $s_1 + s_2 - s' > 1/2$, as well as $a + b > (n-1)/2$. \square

Corollary 4.9. *Let $p \in \mathbb{N}$ and $s = 1/2 - \epsilon$ with $0 \leq \epsilon < 1/2p$, and let $k > (n-1)/2$. Then $u \in H^{s,k}(X, Y) \Rightarrow u^p \in H^{0,k}(X, Y)$.*

Proof. Proposition 4.8 gives $u^2 \in H^{1/2-2\epsilon-\epsilon'_2,k}$ for all $\epsilon'_2 > 0$, thus $u^3 \in H^{1/2-3\epsilon-\epsilon'_3,k}$ for all $\epsilon'_3 > 0$, since $\epsilon'_2 > 0$ is arbitrary; continuing in this way gives $u^p \in H^{1/2-p\epsilon-\epsilon'_p,k}$ for all $\epsilon'_p > 0$, and the claim follows. \square

4.3. A class of semilinear equations. Recall that we have a forward solution operator $S_\sigma: H^{s-1,k}(\Omega)^{\bullet,-} \rightarrow H^{s,k}(\Omega)^{\bullet,-}$ of P_σ , defined in (4.1), provided $s < 1/2 - \operatorname{Im} \sigma$. Let us fix such $s \in \mathbb{R}$ and $\sigma \in \mathbb{C}$. Undoing the conjugation, we obtain a forward solution operator

$$\begin{aligned} S &= \mu^{-1/2} \mu^{-i\sigma/2+(n+1)/4} S_\sigma \mu^{i\sigma/2-(n+1)/4} \mu^{-1/2}, \\ S: \mu^{(n+3)/4+\operatorname{Im} \sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} &\rightarrow \mu^{(n-1)/4+\operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-} \end{aligned}$$

of $(\square_g - (n-1)^2/4 - \sigma^2)$. Since g is a 0-metric, the natural vector fields to appear in a non-linear equation are 0-vector fields; see Section 4.5 for a brief discussion

⁴⁹Indeed, the Fourier transform of $\chi(x)x_+^\ell$ on \mathbb{R} , with $\chi \in C_c^\infty(\mathbb{R})$, is bounded by a constant multiple of $\langle \xi \rangle^{-\operatorname{Re} \ell - 1}$, which is an element of $\langle \xi \rangle^{-r} L_\xi^2$ if and only if $r - \operatorname{Re} \ell - 1 < -1/2$, i.e. if $\operatorname{Re} \ell + 1/2 > r$.

of these concepts. However, since the analysis is based on ordinary Sobolev spaces relative to which one has b-regularity (regularity with respect to the module \mathcal{M}), we consider b-vector fields in the non-linearities. In case one does use 0-vector fields, the solvability conditions can be relaxed; see Section 4.4.

Theorem 4.10. *Suppose $s < 1/2 - \text{Im } \sigma$. Let*

$$\begin{aligned} q: \mu^{(n-1)/4 + \text{Im } \sigma/2} H^{s,k}(\Omega)^{\bullet,-} &\times \mu^{(n-1)/4 + \text{Im } \sigma/2} H^{s,k-1}(\Omega; {}^b T^* \Omega)^{\bullet,-} \\ &\rightarrow \mu^{(n+3)/4 + \text{Im } \sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} \end{aligned}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfying

$$\|q(u, {}^b du) - q(v, {}^b dv)\| \leq L(R) \|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in \mu^{(n+3)/4 + \text{Im } \sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the equation

$$\left(\square_g - \left(\frac{n-1}{2} \right)^2 - \sigma^2 \right) u = f + q(u, {}^b du)$$

has a unique solution $u \in \mu^{(n-1)/4 + \text{Im } \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.24. \square

Remark 4.11. As in Theorem 2.24, we can also allow non-linearities $q(u, {}^b du, \square_g u)$, provided

$$\begin{aligned} q: \mu^{(n-1)/4 + \text{Im } \sigma/2} H^{s,k}(\Omega)^{\bullet,-} &\times \mu^{(n-1)/4 + \text{Im } \sigma/2} H^{s-1,k}(\Omega; {}^b T^* \Omega)^{\bullet,-} \\ &\times \mu^{(n+3)/4 + \text{Im } \sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} \\ &\rightarrow \mu^{(n+3)/4 + \text{Im } \sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} \end{aligned}$$

is continuous, $q(0,0,0) = 0$ and q has a small Lipschitz constant near 0.

4.4. Semilinear equations with polynomial non-linearity. Next, we want to find a forward solution of the semilinear PDE

$$\left(\square_g - \left(\frac{n-1}{2} \right)^2 - \sigma^2 \right) u = f + c \mu^A u^p X(u) \tag{4.8}$$

where $c \in C^\infty(\tilde{X})$, and $X(u) = \prod_{j=1}^q X_j u$ is a q -fold product of derivatives of u along vector fields $X_j \in \mathcal{M}$.⁵⁰ Note that the derivatives in the non-linearity lie in the module \mathcal{M} (in coordinates: $\mu \partial_\mu, \partial_y$), whereas, as mentioned above, the natural vector fields are 0-derivatives (in coordinates: $x \partial_x = 2\mu \partial_\mu$ and $x \partial_y = \mu^{1/2} \partial_y$), but since it does not make the computation more difficult, we consider module instead of 0-derivatives and compensate this by allowing any weight μ^A in front of the non-linearity.

⁵⁰What follows is of course just a computation in the course of which we will obtain conditions on A, p, q which guarantee that the map $u \mapsto c \mu^A u^p X(u)$ satisfies the conditions of the map q in Theorem 4.10.

Rephrasing the PDE in terms of P_σ using $\tilde{u} = \mu^{i\sigma/2-(n+1)/4+1/2}u$ and $\tilde{f} = \mu^{-1/2+i\sigma/2-(n+1)/4}f$, we obtain

$$\begin{aligned} P_\sigma \tilde{u} &= \tilde{f} + c\mu^A \mu^{-1/2+i\sigma/2-(n+1)/4} \mu^{(p+q)(-i\sigma/2+(n-1)/4)} \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}) \\ &= \tilde{f} + c\mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}), \end{aligned}$$

where $f_j \in C^\infty(\tilde{X})$ and

$$\ell = A + (p+q-1)(-i\sigma/2 + (n-1)/4) - 1. \quad (4.9)$$

Therefore, if $\tilde{u} \in H^{s,k}(\Omega)^{\bullet,-}$, we obtain that the right hand side of the equation lies in $H^{s,k-1}(\Omega)^{\bullet,-}$ if $\tilde{f} \in H^{s,k-1}(\Omega)^{\bullet,-}$, $s > 1/2$, $k > (n+1)/2$, which by Proposition 4.5 implies that $H^{s,k-1}(\Omega)^{\bullet,-}$ is an algebra, and if

$$\operatorname{Re} \ell + 1/2 = A + (p+q-1)(\operatorname{Im} \sigma/2 + (n-1)/4) - 1/2 > s, \quad (4.10)$$

since this condition ensures that $\mu^\ell \in H^{s,\infty}(X)$, which implies that multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \rightarrow H^{s,k-1}(\Omega)^{\bullet,-}$.⁵¹ Given the restriction (4.3) on s and $\operatorname{Im} \sigma$, we see that by choosing $s > 1/2$ close to $1/2$, $\operatorname{Im} \sigma < 0$ close to 0, we obtain the condition

$$p+q > 1 + \frac{4(1-A)}{n-1}. \quad (4.11)$$

If these conditions are satisfied, the right hand side of the re-written PDE lies in $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$, so Theorem 4.10 is applicable, and thus (4.8) is well-posed in these spaces.

From (4.11) with $A = 0$, we see that quadratic non-linearities are fine for $n \geq 6$, cubic ones for $n \geq 4$.

To sum this up, we revert back to $u = \mu^{(n-1)/4-i\sigma/2} \tilde{u}$ and $f = \mu^{(n+3)/4-i\sigma/2} \tilde{f}$:

Theorem 4.12. *Let $s > 1/2$, $k > (n+1)/2$, and assume $A \in \mathbb{R}$ and $p, q \in \mathbb{N}_0$, $p+q \geq 2$ satisfy condition (4.10). Moreover, suppose $\sigma \in \mathbb{C}$ satisfies (4.3), i.e. $\operatorname{Im} \sigma < s - 1/2$. Finally, let $c \in C^\infty(\tilde{M})$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j are vector fields in \mathcal{M} . Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in \mu^{(n+3)/4+\operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the PDE*

$$\left(\square_g - \left(\frac{n-1}{2} \right)^2 - \sigma^2 \right) u = f + c\mu^A u^p X(u)$$

has a unique solution $u \in \mu^{(n-1)/4+\operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

The same conclusion holds if the non-linearity is a finite sum of terms of the form $c\mu^A u^p X(u)$, provided each such term separately satisfies (4.3).

⁵¹If one works in higher regularity spaces, $s \geq 3/2$, we in fact only need $\operatorname{Re} \ell + 3/2 > s$, since then multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-} \rightarrow H^{s-1,k}(\Omega)^{\bullet,-}$. However, the solvability criterion (4.11) would be weaker, namely the role of the dimension n shifts by 2, since in order to use $s \geq 3/2$, we need $\operatorname{Im} \sigma < -1$.

Proof. Reformulating the PDE in terms of \tilde{u} and \tilde{f} as above, this follows from an application of the Banach fixed point theorem to the map

$$H^{s,k}(\Omega)^{\bullet,-} \ni \tilde{u} \mapsto S_\sigma \left(\tilde{f} + \mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}) \right) \in H^{s,k}(\Omega)^{\bullet,-}$$

with ℓ given by (4.9) and $f_j \in C^\infty(\tilde{X})$. Here, $p + q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H^{s,k}(\Omega)^{\bullet,-}$. \square

Remark 4.13. Even though the above conditions force $\text{Im } \sigma < 0$, let us remark that the conditions of the theorem, most importantly (4.10), can be satisfied if $m^2 = (n-1)^2/4 + \sigma^2 > 0$ is real, which thus means that we are in fact considering a non-linear equation involving the Klein-Gordon operator $\square_g - m^2$. Indeed, let $\sigma = i\tilde{\sigma}$ with $\tilde{\sigma} < 0$, then condition (4.10) with $A = 0, p + q = 2$, becomes $\tilde{\sigma} > 2 - (n-1)/2$ (where we accordingly have to choose $s > 1/2$ close, depending on $\tilde{\sigma}$, to $1/2$), and the requirement $\tilde{\sigma} < 0$ forces $n \geq 6$. On the other hand, we want $(n-1)^2/4 - \tilde{\sigma}^2 = m^2 > 0$; we thus obtain the condition

$$0 < m^2 < \left(\frac{n-1}{2} \right)^2 - \left(2 - \frac{n-1}{2} \right)^2$$

for masses m that Theorem 4.12 can handle, which does give a non-trivial range of allowed m for $n \geq 6$.

Remark 4.14. Let us compare the numerology in Theorem 4.12 with the numerology for the static model of de Sitter space in Section 2: First, we can solve fewer equations globally on de Sitter space, and second, we need stronger regularity assumptions in order to make an iterative argument work: In the static model, we needed to be in a b-Sobolev space of order $> (n+2)/2$, which in the non-blown-up picture corresponds to 0-regularity of order $> (n+2)/2$, whereas in the global version, we need a background Sobolev regularity $> 1/2$, relative to which we have ‘b-regularity’ (i.e. regularity with respect to the module \mathcal{M}) of order $> (n+1)/2$. This comparison is of course only a qualitative one, though, since the underlying geometries in the two cases are different.

Using Proposition 4.6 and Corollary 4.7, one can often improve this result. Thus, let us consider the most natural case of equation (4.8) in which we use 0-derivatives X_j , corresponding to the 0-structure on the *not* even-ified manifold X , and no additional weight. The only difference this makes is if there are tangential 0-derivatives (in coordinates: $\mu^{1/2} \partial_y$). For simplicity of notation, let us therefore assume that $X_j = \mu^{1/2} \tilde{X}_j$, $1 \leq j \leq \alpha$, and $X_j = \tilde{X}_j$, $\alpha < j \leq q$, where the \tilde{X}_j are vector fields in \mathcal{M} . Then the PDE (4.8), rewritten in terms of P_σ, \tilde{u} and \tilde{f} , becomes

$$P_\sigma \tilde{u} = \tilde{f} + c \mu^\ell \tilde{u}^p \prod_{j=1}^q (\tilde{f}_j + \tilde{X}_j \tilde{u}) \quad (4.12)$$

with $\tilde{f}_j \in C^\infty(\tilde{X})$, where

$$\ell = \alpha/2 + (p+q-1)(-i\sigma/2 + (n-1)/4) - 1.$$

First, suppose that there are no derivatives in the non-linearity so that $p \geq 2, q = \alpha = 0$. Then $\mu^\ell \tilde{u}^p \in H^{s-1,k}(\Omega)^{\bullet,-}$ provided $\text{Re } \ell + 3/2 > s > 1/2$ by Corollary 4.7;

choosing s arbitrarily close to $1/2$, this is equivalent to

$$\operatorname{Im} \sigma/2 + (n-1)/4 > 0. \quad (4.13)$$

This is a very natural condition: The solution operator for the linear wave equation produces solutions with asymptotics $\mu^{(n-1)/4 \pm i\sigma}$,⁵² and the non-linear equation (4.8) should therefore only be well-behaved if solutions to the linear equation decay at infinity, i.e. if $\pm \operatorname{Im} \sigma + (n-1)/4 \geq 0$. Since we need $\operatorname{Im} \sigma < 0$ to be allowed to take $s > 1/2$, condition (4.13) is equivalent to the (small) decay of solutions to the linear equation at infinity (where $\mu = 0$).

Next, let us assume that $q > 0$. Then the non-linear term in equation (4.12) is an element of

$$\mu^\ell H^{s,k}(\Omega)^{\bullet,-} \cdot H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s,k-1}(\Omega)^{\bullet,-}$$

by Proposition 4.6, provided $\operatorname{Re} \ell + 1/2 > s > 1/2$, which gives the condition

$$\operatorname{Im} \sigma/2 + (n-1)/4 > 1 - \alpha/2$$

where we again choose $s > 1/2$ arbitrarily close to $1/2$, i.e. for $\alpha = 2$, we again get condition (4.13), and for $\alpha > 2$, we get an even weaker one.

Finally, let us discuss a non-linear term of the form $c\mu^A u^p$, $p \geq 2$, in the setting of even lower regularity $0 \leq s < 1/2$, the technical tool here being Corollary 4.9: Rewriting the PDE (4.8) with this non-linearity in terms of P_σ , \tilde{u} and \tilde{f} , we get

$$P_\sigma \tilde{u} = \tilde{f} + c\mu^\ell \tilde{u}^p, \quad \ell = A + (p-1)(-i\sigma/2 + (n-1)/4) - 1.$$

Let $s = 1/2 - \epsilon$ with $0 \leq \epsilon < 1/2p$. Then if $\tilde{u} \in H^{1/2-\epsilon,k}(\Omega)^{\bullet,-}$ with $k > (n-1)/2$, Corollary 4.9 yields $\tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-}$, thus

$$\mu^\ell \tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$$

provided $\operatorname{Re} \ell \geq 0$, i.e.

$$n > 1 + \frac{4(1-A)}{p-1} - 2 \operatorname{Im} \sigma, \quad (4.14)$$

where we still require $\operatorname{Im} \sigma < 1/2 - s = \epsilon$, which in particular allows σ to be real if $\epsilon > 0$.

In summary:

Theorem 4.15. *Let $p \geq 2$ be an integer, $1/2 - 1/2p < s \leq 1/2$, $k > (n-1)/2$, and suppose $\sigma \in \mathbb{C}$ is such that $\operatorname{Im} \sigma < 1/2 - s$. Moreover, assume $A \in \mathbb{R}$ and the dimension n satisfy condition (4.14). Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in \mu^{(n+3)/4 + \operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with norm $\leq C$, the PDE*

$$\left(\square_g - \left(\frac{n-1}{2} \right)^2 - \sigma^2 \right) u = f + c\mu^A u^p$$

has a unique solution $u \in \mu^{(n-1)/4 + \operatorname{Im} \sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with norm $\leq R$, that depends continuously on f .

In particular, if $1/4 < s < 1/2$, $0 < \operatorname{Im} \sigma < 1/2 - s$ and $A = 0$, then quadratic non-linearities are fine for $n \geq 5$; if $\operatorname{Im} \sigma = 0$ and $A = 0$, then they work for $n \geq 6$.

⁵²See (2.37), and recall that we are working with the even-ified manifold with boundary defining function $\mu = x^2$.

4.4.1. *Backward solutions to semilinear equations with polynomial non-linearity.* Recalling the setting of Section 4.1.1, let us briefly turn to the backward problem for (4.8), which we rephrase in terms of P_σ as above. For simplicity, let us only consider the ‘least sophisticated’ conditions, namely $s > 1/2, k > (n+1)/2$,

$$A + (p+q-1)(\operatorname{Im} \sigma/2 + (n-1)/4) - 1/2 > s, \quad (4.15)$$

and, this is the important change compared to the forward problem, $s > 1/2 - \operatorname{Im} \sigma$, where the latter guarantees the existence of the backward solution operator S_σ^- . Thus, if $\operatorname{Im} \sigma > 0$ is large enough and $s > 1/2$ satisfies (4.15), then (4.8) is solvable in any dimension.

In the special case that we only consider 0-derivatives and no extra weight, which corresponds to putting $A = q + \alpha/2$, we obtain the condition

$$\operatorname{Im} \sigma > \frac{4(1-q-\alpha/2) - (p+q-1)(n-1)}{2(p+q+1)}$$

if we choose $s > 1/2 - \operatorname{Im} \sigma$ close to $1/2$, which in particular allows $\operatorname{Im} \sigma \geq 0$, and thus σ^2 arbitrary, if $p > 1 + \frac{4}{n-1}$ (so $p \geq 2$ is acceptable if $n \geq 6$) or $q + \alpha/2 \geq 1$.

4.5. **From static parts to global de Sitter space.** Let us consider the equation

$$(\square_g - m^2)u = f + q(u, {}^0d), \quad (4.16)$$

where the reason for using the 0-differential 0d will be given momentarily. The idea is that every point in X lies in the interior of the backward light cone from some point p at future infinity Y_+ , denoted S_p . Then blowing up \bar{X} at p , we obtain a static part of (asymptotically) de Sitter space where the solvability statements have been explained in Section 2. We then solve equation (4.16) on S_p , thereby obtaining a forward solution u_p , and by local uniqueness for $\square_g - m^2$ in X , all such solutions agree on their overlap, i.e. $u_p \equiv u_q$ on $S_p \cap S_q$. Therefore, we can define a function u by setting $u = u_p$ on S_p , $p \in Y_+$, which then is a solution of (4.16) on X . To make this precise, we need to analyze the relationships between the function spaces on the S_p , $p \in Y_+$, and X . As we will see in Lemma 4.16 below, b-Sobolev spaces on the blow-ups S_p of \bar{X} at boundary points are closely related to 0-Sobolev spaces on X .

Recall the definition of 0-Sobolev spaces on a manifold with boundary M (for us, $M = \bar{X}$) with a 0-metric, i.e. a metric of the form $x^{-2}\hat{g}$ with x a boundary defining function, where \hat{g} extends non-degenerately to the boundary: If $\mathcal{V}_0(M) = x\mathcal{V}(M)$ denotes the Lie algebra of 0-vector fields, where $\mathcal{V}(M)$ are smooth vector fields on M , and $\operatorname{Diff}_0^*(M)$ the enveloping algebra of 0-differential operators, then

$$H_0^k(M) = \{u \in L^2(M, d\operatorname{vol}) : Pu \in L^2(M, d\operatorname{vol}), P \in \operatorname{Diff}_0^k(M)\}.$$

For clarity, we shall write $L_0^2(M) = L^2(M, d\operatorname{vol})$. We also recall the definition of the 0-(co)tangent spaces: If \mathcal{I}_p denotes the ideal of $C^\infty(M)$ functions vanishing at $p \in M$, then the 0-tangent space at p is defined as ${}^0T_pM = \mathcal{V}_0(M)/\mathcal{I}_p \cdot \mathcal{V}_0(M)$, and the 0-cotangent space at p , ${}^0T_p^*M$, as the dual of 0T_pM . In local coordinates $(x, y) \in \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ near the boundary of M , we have $d\operatorname{vol} = f(x, y) \frac{dx}{x} \frac{dy}{x^{n-1}}$ with f smooth and non-vanishing, and $\mathcal{V}_0(M)$ is spanned by $x\partial_x$ and $x\partial_{y_j}$; also $x\partial_x$ and $x\partial_{y_j}$, $j = 2, \dots, n$, is a basis of 0T_pM (for $p \in \partial M$, which is the only place where 0-spaces differ from the standard spaces), and $\frac{dx}{x}, \frac{dy_j}{x}$, $j = 2, \dots, n$, is a basis of ${}^0T_p^*M$.

Now, let $\Omega \subset \overline{X}$ be a domain as in Section 4.1. Moreover, let $\beta_p: S_p \rightarrow X$ be the blow-down map, and let $\Omega_p \subset S_p$ be a domain with corners as in Section 2, whose boundary is the union of ∂S_p and an ‘artificial’ spacelike boundary; let us in fact choose $\Omega_p = \beta_p^{-1}(\Omega)$. We then have:

Lemma 4.16. *Let $k \in \mathbb{N}_0$, $\ell \in \mathbb{R}$. Then there are constants $C > 0$ and $C_\delta > 0$ such that for all $\delta > 0$,*

$$\|f\|_{H_0^{k,\ell-(n-1)/2-\delta}(\Omega)^\bullet} \leq C_\delta \sup_{p \in Y_+} \|\beta_p^* f\|_{H_b^{k,\ell}(\Omega_p)^\bullet} \leq CC_\delta \|f\|_{H_0^{k,\ell}(\Omega)^\bullet}. \quad (4.17)$$

Here, \bullet indicates supported distributions at the ‘artificial’ boundary and $-$ extendible distributions at all other boundary hypersurfaces.

Proof. Let us work locally near a point $p \in Y_+$; since $Y_+ \cong \mathbb{S}^{n-1}$ is compact, all constructions below can be made uniformly in p . The only possible issues are near the boundary $Y_+ = \{x = 0\}$, with x a boundary defining function; hence, let us work in a product neighborhood $Y_+ \times [0, 2\epsilon)_x$, $\epsilon > 0$, of Y_+ , and let us assume u is supported in $Y_+ \times [0, \epsilon]$.

We use coordinates x, y_2, \dots, y_n such that $y_j = 0$ at p . Coordinates on S_p are then x, z_2, \dots, z_n with $z_j = y_j/x$, i.e. $\beta_p(x, z) = (x, xz)$, with the restriction $\sum_{j=2}^n |z_j|^2 \leq 1$. Therefore,

$$\begin{aligned} \|\beta_p^* f\|_{L_b^2}^2 &\approx \int_{S_p} |\beta_p^* f(x, z)|^2 \frac{dx}{x} dz = \int_{\beta_p(S_p)} |f(x, xz)|^2 \frac{dx}{x} dz \\ &\leq \int |f(x, y)|^2 \frac{dx}{x} \frac{dy}{x^{n-1}} \approx \|f\|_{L_0^2}^2. \end{aligned}$$

Adding weights to this estimate is straightforward. Next, we observe

$$\begin{aligned} x\partial_x(\beta_p^* f)(x, z) &= x\partial_x f(x, xz) + zx\partial_y f(x, xz) \\ \partial_z(\beta_p^* f)(x, z) &= x\partial_y f(x, xz), \end{aligned} \quad (4.18)$$

and since $|z| \leq 1$, we conclude that $\beta_p^* f \in H_b^1(S_p)$ is equivalent to $f, x\partial_x f, x\partial_y f \in L_0^2(\beta_p(S_p))$, which proves the second inequality in (4.17) in the case $k = 1$; the general case is similar.

For the first inequality in (4.17), we first note that the additional weight comes from the number of static parts, i.e. interiors of backward light cones from points in Y_+ , that one needs to cover any fixed half space $\{x \geq x_0\}$: Namely, for $0 < x_0 \leq \epsilon$, let $\mathcal{B}(x_0) \subset Y_+$ be a set of points such that every point in $\{x \geq x_0\}$ lies in S_p for some $p \in \mathcal{B}(x_0)$; then we can choose $\mathcal{B}(x_0)$ such that $|\mathcal{B}(x_0)| \leq Cx_0^{-(n-1)}$, where $|\cdot|$ denotes the number of elements in a set. This follows from the observation that the area of the slice $x = x_0$ of S_p within $Y_+ \cong \mathbb{S}^{n-1}$ ⁵³ is bounded from below by cx_0^{n-1} for some p -independent constant $c > 0$. Indeed, note that null-geodesics of the 0-metric g are, up to reparametrization, the same as null-geodesics of the conformally related metric x^2g , which is a non-degenerate Lorentzian metric up to Y_+ . ⁵⁴

Thus, putting $\alpha = (n-1)/2 + \delta$, $\delta > 0$, we estimate

$$\int_{x \leq \epsilon} |x^\alpha f(x, y)| \frac{dx}{x} \frac{dy}{x^{n-1}} = \sum_{j=0}^{\infty} \int_{2^{-j-1}\epsilon < x \leq 2^{-j}\epsilon} |x^\alpha f(x, y)|^2 \frac{dx}{x} \frac{dy}{x^{n-1}}$$

⁵³Keep in mind that we are working in a product neighborhood of Y_+ .

⁵⁴Also see Figure 5.

$$\begin{aligned}
&\lesssim \sum_{j=0}^{\infty} 2^{-2\alpha j} \sum_{p \in \mathcal{B}(2^{-j-1}\epsilon)} \|\beta_p^* f\|_{L_b^2}^2 \lesssim \sum_{j=0}^{\infty} 2^{-2\alpha j} (2^{-j-1}\epsilon)^{-n+1} \sup_{p \in Y_+} \|\beta_p^* f\|_{L_b^2}^2 \\
&\lesssim \sum_{j=0}^{\infty} 2^{-j(2\alpha-n+1)} \sup_{p \in Y_+} \|\beta_p^* f\|_{L_b^2}^2,
\end{aligned}$$

with the sum converging since $2\alpha - n + 1 = 2\delta > 0$. Weights and higher order Sobolev spaces are handled similarly, using (4.18). \square

In particular, this justifies why in equation (4.16), we have to take $d = {}^0d: H_0^{k,\ell}(X) \rightarrow H_0^{k-1,\ell}(X; {}^0T^*X)$ in order to make the global equation interact well with the static patches.

Since we want to consider local problems to solve the global one, the non-linearity q must be local in the sense that $q(u, {}^0du)(p)$ for $p \in X$ only depends on p and its arguments evaluated at p ; let us for simplicity assume that q is in fact a polynomial as in (2.43).

Using Corollary 2.27, we then obtain:

Theorem 4.17. *Let $0 \leq \epsilon < \epsilon_0$ with ϵ_0 as in Section 2.2, and $s > \max\{1/2 + \epsilon, n/2 + 1\}$, $s \in \mathbb{N}$. Let*

$$q(u, {}^0du) = \sum_{2 \leq j + |\alpha| \leq d} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u,$$

$q_{j,\alpha} \in \mathbb{C} + H_0^s(\overline{X})$, $X_{\alpha,k} \in \mathcal{V}_0(M)$. Then there exists $C > 0$ such that for all $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ with norm $\leq C$, the equation

$$(\square_g - m^2)u = f + q(u, {}^0du)$$

has a unique solution $u \in \bigcap_{\delta > 0} H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ that depends continuously on f .

The analogous conclusion also holds for $\square_g u = f + q({}^0du)$ provided $\epsilon > 0$, with the solution u being in $\bigcap_{\delta > 0} H_0^{s,-(n-1)/2-\delta}(\Omega)^\bullet$. Moreover, for all $p \in Y_+$, the limit $u_\partial(p) := \lim_{p' \rightarrow p, p' \in X} u(p')$ exists, $u_\partial \in C^{0,\epsilon}(Y_+)$, and $u - u_\partial(\phi \circ \mathfrak{t}_1) \in x^\epsilon C^0(\overline{X})$, where $\phi \circ \mathfrak{t}_1$ is identically 1 near Y_+ and vanishes near the ‘artificial’ boundary of Ω .

Proof. We start by proving the first part: If $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$, then $f_p = \beta_p^* f \in H_b^{s-1,\epsilon}(S_p)$ is a uniformly bounded family in the respective norms by Lemma 4.16. We can then use Corollary 2.27 to solve

$$(\square_g - m^2)u_p = f_p + q(u_p, {}^bdu_p)$$

in the static part S_p , where we use that q is a polynomial and the fact that ${}^bT_{p'}^* S_p$ naturally injects into ${}^0T_{\beta_p(p')}^* \Omega$ for $p' \in S_p$ to make sense of the non-linearity; we thus obtain a uniformly bounded family $u_p = \tilde{u}_p|_{S_p} \in H_b^{s,\epsilon}(S_p)^{\bullet,-}$. By local uniqueness and since f vanishes near Y_- , we see that the function u , defined by $u(\beta_p(p')) = u_p(p')$ for $p \in Y_+$, $p' \in S_p$, is well-defined, and by Lemma 4.16, we indeed have $u \in H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ for all $\delta > 0$.

For the second part, we follow the same strategy, obtaining solutions $u_p = c_p(\phi \circ \mathfrak{t}_1) + u'_p$ of

$$\square_g u_p = f_p + q({}^bdu_p),$$

where $c_p \in \mathbb{C}$ and $u'_p \in H_b^{s,\epsilon}(S_p)^{\bullet,-}$ are uniformly bounded, thus u_p is uniformly bounded in $H_b^{s,-\delta}(\Omega)^{\bullet}$ for every fixed $\delta > 0$, and therefore the existence of a unique solution u follows as before. Put $u_\partial(p) := c_p$, then $u_\partial(p) = \lim_{p' \rightarrow p, p' \in S_p} u(p')$, since $u'_p \in x^\epsilon C^0(S_p)$ by the Sobolev embedding theorem. We first prove that u_∂ so defined is ϵ -Hölder continuous. Let us work in local coordinates (x, y) near a point $(0, y_0)$ in Y_+ . Now, u'_p is uniformly bounded in $x^\epsilon C^0(S_p)$, and since for $x_0 > 0$ arbitrary, we have $c_{p_1} + u'_{p_1}(x_0, y_*) = c_{p_2} + u'_{p_2}(x_0, y_*)$ for all $p_1, p_2 \in Y_+$, provided $|p_1 - p_2| \leq cx_0$ for some constant $c > 0$, which ensures that $S_{p_1} \cap S_{p_2} \cap \{x = x_0\}$ is non-empty and thus contains a point (x_0, y_*) (see Figure 5), we obtain

$$|c_{p_1} - c_{p_2}| = |u'_{p_1}(x_0, y_*) - u'_{p_2}(x_0, y_*)| \leq Cx_0^\epsilon, \quad |p_1 - p_2| \leq cx_0$$

for all x_0 , thus

$$\frac{|u_\partial(p_1) - u_\partial(p_2)|}{|p_1 - p_2|^\epsilon} \leq C, \quad p_1, p_2 \in Y_+.$$

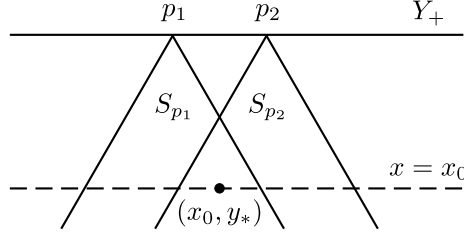


FIGURE 5. Setup for the proof of $u_\partial \in C^{0,\epsilon}(Y_+)$: Shown are the backward light cones from two nearby points $p_1, p_2 \in Y_+$ that intersect within the slice $\{x = x_0\}$ at a point (x_0, y_*) .

This in particular implies that

$$\begin{aligned} |u(x, y) - u_\partial(0, y_0)| &\leq |u(x, y) - u_\partial(0, y)| + |u_\partial(0, y) - u_\partial(0, y_0)| \\ &\leq C(|y - y_0|^\epsilon + x^\epsilon) \xrightarrow{x \rightarrow 0, y \rightarrow y_0} 0, \end{aligned} \quad (4.19)$$

hence we in fact have $u_\partial(p) = \lim_{p' \rightarrow p, p' \in X} u(p')$. Finally, putting $y = y_0$ in (4.19) proves that $u - u_\partial(\phi \circ \mathbf{t}_1) \in x^\epsilon C^0(\bar{X})$. \square

The major lossy part of the argument is the conversion from f to the family $\beta_p^* f$: Even though the second inequality in Lemma 4.16 is optimal (e.g., for functions which are supported in a single static patch), one loses $(n-1)/2$ orders of decay relative to the gluing estimate, i.e. the first inequality in Lemma 4.16, which is used to pass from the family u_p to u .

Observe on the other hand that the decay properties of u , without regard to those of f , in the first part of the theorem are very natural, since the constant function 1 is an element of $\bigcap_{\delta > 0} H_0^{\infty, -(n-1)/2 - \delta}(X)$, thus u has an additional decay of ϵ relative to constants.

Remark 4.18. Notice that for the proof of Theorem 4.17 it is irrelevant whether certain 0-Sobolev spaces are algebras, since the main analysis, Corollary 2.27, is carried out on b-Sobolev spaces.

5. MINKOWSKI SPACE

5.1. The linear Fredholm framework. We now consider n -dimensional non-trapping asymptotically Minkowski spacetimes (M, g) , a notion which includes the radial compactification of Minkowski spacetime. This notion was briefly recalled in the introduction; here we restate this in the notation of [2, Section 3] where this notion was introduced.

Thus, M is compact with smooth boundary, with a boundary defining function ρ (we switch the notation from τ mainly to emphasize that ρ is not everywhere timelike), and scattering vector fields $V \in \mathcal{V}_{\text{sc}}(M)$, introduced by Melrose [29], are smooth vector fields of the form $\rho V'$, $V' \in \mathcal{V}_{\text{b}}(M)$. Hence, if the z_j are local coordinates on ∂M extended to a neighborhood in M , then a local basis of these vector fields over $C^\infty(M)$ is $\rho^2 \partial_\rho, \rho \partial_{z_j}$. Correspondingly, $\mathcal{V}_{\text{sc}}(M)$ is the set of smooth sections of a vector bundle ${}^{\text{sc}}TM$, which is therefore, roughly speaking, $\rho^{\text{b}}TM$. The dual bundle, called the scattering cotangent bundle, is denoted by ${}^{\text{sc}}T^*M$. If M is the radial compactification of \mathbb{R}^n , by gluing a sphere at infinity via the reciprocal polar coordinate map $(r, \omega) \mapsto (r^{-1}, \omega) \in (0, 1)_\rho \times \mathbb{S}_\omega^{n-1}$, i.e. adding $\rho = 0$ to the right hand side (corresponding to ‘ $r = \infty$ ’), then $\mathcal{V}_{\text{sc}}(M)$ is spanned by (the lifts of) the translation invariant vector fields over $C^\infty(M)$, which in turn simply consists of classical symbols on \mathbb{R}^n of order 0.

A *Lorentzian scattering metric* g is a Lorentzian signature, taken to be $(1, n-1)$, metric on ${}^{\text{sc}}TM$, i.e. a smooth symmetric section of ${}^{\text{sc}}T^*M \otimes {}^{\text{sc}}T^*M$ with this signature with the following additional properties:

- (1) there is a real C^∞ function v defined on M with $dv, d\rho$ linearly independent at ‘the light cone at infinity’, $S = \{v = 0, \rho = 0\}$,
- (2) $g(\rho^2 \partial_\rho, \rho^2 \partial_\rho)$ has⁵⁵ the same sign⁵⁶ as v at $\rho = 0$,
- (3) near S ,

$$g = v \frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2} \right) - \frac{\tilde{h}}{\rho^2},$$

where α is a smooth one-form on M ,

$$\alpha = \frac{1}{2} dv + \mathcal{O}(v) + \mathcal{O}(\rho),$$

\tilde{h} is a smooth 2-cotensor on M , which is positive definite on the (codimension two) annihilator of $d\rho$ and dv .

A Lorentzian scattering metric is *non-trapping* if

- (1) $S = S_+ \cup S_-$ (each a disjoint union of connected components), in $X = \partial M$ the open set $\{v > 0\} \cap X$ decomposes as $C_+ \cup C_-$ (disjoint union), with $\partial C_+ = S_+, \partial C_- = S_-$; we write $C_0 = \{v < 0\} \cap X$,
- (2) the projections of all null-bicharacteristics in ${}^{\text{sc}}T^*M \setminus o$ to M tend to S_\pm as their parameter tends to $\pm\infty$ or vice versa.

Since a conformal factor only reparameterizes bicharacteristics, this means that with $\hat{g} = \rho^2 g$, which is a b-metric on M , the projections of all null-bicharacteristics of \hat{g} in ${}^{\text{b}}T^*M \setminus o$ tend to S_\pm . As already pointed out in the introduction, the difference between the de Sitter-type and Minkowski settings is that at the spherical

⁵⁵The vector field $\rho^2 \partial_\rho$ is well-defined up to a positive factor at $\rho = 0$, and is called the scattering normal vector field.

⁵⁶Thus, $\rho^2 \partial_\rho$ is timelike in $v > 0$, spacelike in $v < 0$.

conormal bundle ${}^bSN^*S$ of S the nature of the radial points is source/sink rather than a saddle point of the flow at L_\pm discussed in Section 2.1.

We first state solvability properties, namely we show that under the assumptions of [2, Section 3], the problem of finding a tempered solution to $\square_g w = f$ is a Fredholm problem in suitable weighted Sobolev spaces. In particular, there is only a finite dimensional obstruction to existence. Then we strengthen the assumptions somewhat and show actual solvability in the strong sense that in these spaces the solution w satisfies that if f is vanishing to infinite order near $\overline{C_-}$, then so does w .

Let

$$L = \rho^{-(n-2)/2} \rho^{-2} \square_g \rho^{(n-2)/2} \in \text{Diff}_b^2(M)$$

be the ‘conjugated’ b-wave operator (as in [2, Section 4]), which is formally self-adjoint with respect to the density of the Lorentzian b-metric $\hat{g} = \rho^2 g$, further $L = \square_{\hat{g}} - \gamma$, where $\gamma \in C^\infty(M)$ real valued. With⁵⁷

$$m \in C^\infty({}^bS^*M) \text{ a variable (Sobolev) order function, decreasing along} \quad (5.1)$$

the direction of the Hamilton flow oriented to the future, i.e. towards S_+ ,

and with

$$\mathcal{R}_+ = {}^bSN^*S_+, \text{ resp. } \mathcal{R}_- = {}^bSN^*S_-,$$

the future, resp. past, radial sets in ${}^bS^*M$, see [2, Section 3.6], and with

$$m + l < 1/2 \text{ at } \mathcal{R}_+, \quad m + l > 1/2 \text{ at } \mathcal{R}_-,$$

m constant near $\mathcal{R}_+ \cup \mathcal{R}_-$, one has an estimate

$$\|u\|_{H_b^{m,l}} \leq C \|Lu\|_{H_b^{m-1,l}} + C \|u\|_{H_b^{m',l}}, \quad (5.2)$$

provided one assumes $m' < m$,

$$m' + l > 1/2 \text{ at } \mathcal{R}_-, \quad u \in H_b^{m',l}.$$

To see this, we recall and record a slight improvement of [2, Proposition 4.4]:

Proposition 5.1. *Suppose L is as above.*

*If $m + l < 1/2$, and if $u \in H_b^{-\infty,l}(M)$ then \mathcal{R}_\pm (and thus a neighborhood of \mathcal{R}_\pm) is disjoint from $\text{WF}_b^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \text{WF}_b^{m-1,l}(Lu) = \emptyset$ and a punctured neighborhood of \mathcal{R}_\pm , with \mathcal{R}_\pm removed, in $\Sigma \cap {}^bS^*M$ is disjoint from $\text{WF}_b^{m,l}(u)$.*

*On the other hand, if $m' + l > 1/2$, $m \geq m'$, $u \in H_b^{-\infty,l}(M)$ and if $\text{WF}_b^{m',l}(u) \cap \mathcal{R}_\pm = \emptyset$ then \mathcal{R}_\pm (and thus a neighborhood of \mathcal{R}_\pm) is disjoint from $\text{WF}_b^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \text{WF}_b^{m-1,l}(Lu) = \emptyset$ and a punctured neighborhood of \mathcal{R}_\pm , with \mathcal{R}_\pm removed, in $\Sigma \cap {}^bS^*M$ is disjoint from $\text{WF}_b^{m,l}(u)$.*

Proof. The first statement is proved in [2, Proposition 4.4]. The second statement follows the same way, but in that case the product of the required powers of the boundary defining functions, $\rho^{-2l} \tilde{\rho}^{-2m+1}$, with $\tilde{\rho}$ the defining function of fiber infinity⁵⁸ as in Section 2.1, in the commutant of [2, Proposition 4.4] provides a favorable sign, thus [2, Equation (4.1)] holds without the E term. However, when

⁵⁷In the actual application of asymptotically Minkowski spaces, one can take m to be a function on M rather than ${}^bS^*M$ by making it take constant values near $\overline{C_+}$, resp. $\overline{C_-}$, corresponding to the requirements at \mathcal{R}_+ , resp. \mathcal{R}_- below, and transitioning in between using a time function as in Theorem 5.2, hypothesis (3), i.e. making m of the form $F \circ \tilde{t}$ for appropriate F . Since this simplifies some arguments below, we assume this whenever it is convenient.

⁵⁸This defining function is denoted by ν in [2].

regularizing, the regularizer contributes a term with the opposite sign, exactly as in [41, Proof of Propositions 2.3-2.4]; this forces the requirement on the a priori regularity, namely $\text{WF}_b^{m',l}(u) \cap \mathcal{R}_\pm = \emptyset$, exactly as in the referred results of [41]; see also Proposition 2.1 above. \square

Indeed, due to the closed graph theorem, (5.2) follows immediately from the b-radial point regularity statements of Proposition 5.1 for sources/sinks, and the propagation of b-singularities for variable order Sobolev spaces, which is not proved in [2], but whose analogue in standard Sobolev spaces is proved there in [2, Proposition A.1] (with additional references given to related results in the literature), and as it is a purely symbolic argument, the extension to the b-setting is straightforward.⁵⁹

One also has a similar estimate for L when one replaces m by a weight \tilde{m} which is increasing along the direction of the Hamilton flow oriented towards the past,

$$\tilde{m} + \tilde{l} > 1/2 \text{ at } \mathcal{R}_+, \quad \tilde{m} + \tilde{l} < 1/2 \text{ at } \mathcal{R}_-,$$

provided one assumes $\tilde{m}' < \tilde{m}$,

$$\tilde{m}' + \tilde{l} > 1/2 \text{ at } \mathcal{R}_+, \quad u \in H_b^{\tilde{m}', \tilde{l}}.$$

Further L can be replaced by L^* . Thus,

$$\|u\|_{H_b^{\tilde{m}, \tilde{l}}} \leq C \|L^* u\|_{H_b^{\tilde{m}-1, \tilde{l}}} + C \|u\|_{H_b^{\tilde{m}', \tilde{l}}}. \quad (5.3)$$

Just as in the asymptotically de Sitter/Kerr settings, one wants to improve these estimates so that the space $H_b^{m,l}$, resp. $H_b^{\tilde{m}, \tilde{l}}$, on the left hand side includes compactly into the error term on the right hand side. This argument is completely analogous to Section 2.1 using the Mellin transformed normal operator estimates obtained in [2, Section 5]. We thus further assume that there are no poles of the Mellin conjugate $\hat{L}(\sigma)$ on the line $\text{Im } \sigma = -l$. Then using the Mellin transform and the estimates for $\hat{L}(\sigma)$ (including the high energy estimates, which imply that for all but a discrete set of l the aforementioned lines do not contain such poles), as in Section 2.1, we obtain that on $\mathbb{R}_\rho^+ \times \partial M$

$$\|v\|_{H_b^{\hat{m}, l}} \leq C \|N(L)v\|_{H_b^{\hat{m}-1, l}} \quad (5.4)$$

when $\hat{m} \in C^\infty(S^*\partial M)$ is a variable order function decreasing along the direction of the Hamilton flow oriented to the future, Λ_+ , resp. Λ_- , the future, resp. past, radial sets in $S^*\partial M$, and with

$$\hat{m} + l < 1/2 \text{ at } \Lambda_+, \quad \hat{m} + l > 1/2 \text{ at } \Lambda_-.$$

One can take

$$\hat{m} = m|_{T^*\partial M},$$

for instance, under the identification of $T^*\partial M$ as a subspace of ${}^bT_{\partial M}^*M$, taking account that homogeneous degree zero functions on $T^*\partial M \setminus o$ are exactly functions on $S^*\partial M$, and analogously on ${}^bT_{\partial M}^*M$. However, in the limit $\sigma \rightarrow \infty$, one should use norms depending on σ reflecting the dependence of the semiclassical norm on h . We recall from Footnote 57 that in the main case of interest one can take m to be a pullback from M , and thus the Mellin transformed operator norms are independent of σ . In either case, we simply write m in place of \hat{m} .

⁵⁹Cf. Proposition 2.1 here and [2, Proposition 4.4] extending the radial point results, Propositions 2.3-2.4, of [41], from the boundaryless setting to the b-setting.

Again, we have an analogous estimate for $N(L^*)$:

$$\|v\|_{H_b^{\tilde{m}, \tilde{l}}} \leq C \|N(L^*)v\|_{H_b^{\tilde{m}-1, \tilde{l}}}, \quad (5.5)$$

provided $-\tilde{l}$ is not the imaginary part of a pole of \widehat{L}^* , and provided \tilde{m} satisfies the requirements above. As $\widehat{L}^*(\sigma) = (\widehat{L})^*(\bar{\sigma})$, the requirement on $-\tilde{l}$ is the same as \tilde{l} not being the imaginary part of a pole of \widehat{L} .

At this point the argument of the paragraph of (2.9) in Section 2.1 can be repeated verbatim to yield that for m with $m+l > 3/2$ (with the stronger restriction coming from the requirements on m' at \mathcal{R}_- , \tilde{m}' at \mathcal{R}_+ , and $m' < m-1$, $\tilde{m}' < \tilde{m}-1$; recall that one needs to estimate the normal operator on these primed spaces)

$$\|u\|_{H_b^{m,l}} \leq C \|Lu\|_{H_b^{m-1,l}} + C \|u\|_{H_b^{m'+1,l-1}}, \quad (5.6)$$

where now the inclusion $H_b^{m,l} \rightarrow H_b^{m'+1,l-1}$ is compact (as we choose $m' < m-1$); this argument required m, l, m' satisfied the requirements preceding (5.2), and that $-l$ is not the imaginary part of any pole of \widehat{L} .

Analogous estimates hold for L^* :

$$\|u\|_{H_b^{\tilde{m}, \tilde{l}}} \leq C \|L^*u\|_{H_b^{\tilde{m}-1, \tilde{l}}} + C \|u\|_{H_b^{m'+1, \tilde{l}-1}}, \quad (5.7)$$

provided $\tilde{m}, \tilde{l}, \tilde{m}'$ satisfy the requirements stated before (5.3), $\tilde{m}' < \tilde{m}-1$, and provided $-\tilde{l}$ is not the imaginary part of a pole of \widehat{L}^* (i.e. \tilde{l} of \widehat{L}).

Via the same functional analytic argument as in Section 2.1 we thus obtain Fredholm properties of L , in particular solvability, modulo a (possible) finite dimensional obstruction, in $H_b^{m,l}$ if

$$m+l > 3/2 \text{ at } \mathcal{R}_-, \quad m+l < -1/2 \text{ at } \mathcal{R}_+.$$

More precisely, we take $\tilde{m} = 1 - m$, $\tilde{l} = -l$, so $m+l < -1/2$ at \mathcal{R}_+ means $\tilde{m} + \tilde{l} = 1 - (m+l) > 3/2$, so the space on the left hand side of (5.6) is dual to that in the first term on the right hand side of (5.7), and the same for the equations interchanged. Then the Fredholm statement is for

$$L : \mathcal{X}^{m,l} \rightarrow \mathcal{Y}^{m-1,l},$$

with

$$\mathcal{Y}^{s,r} = H_b^{s,r}, \quad \mathcal{X}^{s,r} = \{u \in H_b^{s,r} : Lu \in H_b^{s-1,r}\}.$$

Note that, by propagation of singularities, i.e. most importantly using Proposition 5.1, with $\text{Ker } L \subset H_b^{m,l}$, $\text{Ker } L^* \subset H_b^{1-m,-l}$ a priori,

$$\begin{aligned} \text{Ker } L &\subset H_b^{m^b,l}, \quad \text{Ker } L^* \subset H_b^{1-m^b,-l} \text{ if} \\ m^b + l &> 1/2 \text{ at } \mathcal{R}_-, \quad m^b + l < 1/2 \text{ at } \mathcal{R}_+. \end{aligned} \quad (5.8)$$

We can improve this further using the propagation of singularities. Namely, suppose one merely has

$$m+l > 3/2 \text{ at } \mathcal{R}_-, \quad m+l < 1/2 \text{ at } \mathcal{R}_+, \quad (5.9)$$

so the requirement at \mathcal{R}_+ is weakened. Then let $m^\sharp = m-1$ near \mathcal{R}_+ , $m^\sharp \leq m$ everywhere, but still satisfying the requirements for the order function along the Hamilton flow, so the Fredholm result is applicable with m^\sharp in place of m . Now, if $u \in \mathcal{X}^{m^\sharp,l}$, $Lu = f$, $f \in \mathcal{Y}^{m-1,l} \subset \mathcal{Y}^{m^\sharp-1,l}$, then Proposition 5.1 gives $u \in \mathcal{X}^{m,l}$.

Further, if $\text{Ker } L$ and $\text{Ker } L^*$ are trivial, this gives that for m, l as in (5.9), satisfying also the conditions along the Hamilton flow, $L : \mathcal{X}^{m,l} \rightarrow \mathcal{Y}^{m-1,l}$ is invertible.

Now, as invertibility (the absence of kernel and cokernel) is preserved under sufficiently small perturbations, it holds in particular for perturbations of the Minkowski metric which are Lorentzian scattering metrics in our sense, with closeness measured in smooth sections of the second symmetric power of ${}^bT^*M$. (Note that non-trapping is also preserved under such perturbations.)

For more general asymptotically Minkowski metrics we note that, due to Theorem 2.20 (which does not have any requirements for the timelike nature of the boundary defining function, and which works locally near $\overline{C_-}$ either by working on (extendible) function spaces or by using the localization given by wave propagation as in Section 3.3 of [41] or Section 4.1 here) elements of $\text{Ker } L$ on $H_b^{m,l}$, with m, l as above, lie in $\dot{C}^\infty(M)$ locally near $\overline{C_-}$ provided all resonances, i.e. poles of $\hat{L}(\sigma)$, in $\text{Im } \sigma < -l$ have polar parts (coefficients of the Laurent series) that map into distributions supported on $\overline{C_+}$. As shown in [40, Remark 4.17] when $\hat{L}(\sigma)$ arises from a Lorentzian conic metric as in⁶⁰ [40, Equation (3.5)], but with the arguments applicable without significant changes in our more general case, see also [2, Section 7] for our general setting, and [41, Remark 4.6] for a related discussion with complex absorption, the resonances of $\hat{L}(\sigma)$ consist of the resonances of the asymptotically hyperbolic resolvents on the caps, namely $\mathcal{R}_{C_+}(\sigma)$, $\mathcal{R}_{C_-}(-\sigma)$, as well as possibly imaginary integers, $\sigma \in i\mathbb{Z} \setminus \{0\}$, with resonant states when $\text{Im } \sigma < 0$ being differentiated delta distributions at $S_+ = \partial C_+$ while the dual states are differentiated delta distributions at $S_- = \partial C_-$ when $\text{Im } \sigma > 0$; the latter arise, e.g. as poles on even dimensional Minkowski space. More generally, when composed with extension of $C_c^\infty(\overline{C_-} \cup C_0)$ by zero to $C^\infty(X)$ from the right and with restriction to $\overline{C_-} \cup C_0$ from the left, the only poles of $\hat{L}(\sigma)$ are those of $\mathcal{R}_{C_-}(-\sigma)$ as well as the possible $\sigma \in i\mathbb{N}_+$. Thus, fixing⁶¹ $l > -1$, one can conclude that elements of $\text{Ker } L$ are in $\dot{C}^\infty(M)$ locally near $\overline{C_+}$ provided $\mathcal{R}_{C_-}(\tilde{\sigma})$ has no poles in $\text{Im } \tilde{\sigma} > l$. The analogous statement for $\text{Ker } L^*$ on $H_b^{\tilde{m},\tilde{l}}$ is that fixing $\tilde{l} > -1$, elements are in $\dot{C}^\infty(M)$ near $\overline{C_+}$ provided $\mathcal{R}_{C_+}(\tilde{\sigma})$ has no poles in $\text{Im } \tilde{\sigma} > \tilde{l}$. As $\tilde{l} = -l$ for our duality arguments, the weakest symmetric assumption (in terms of strength at C_+ and C_-) is that \mathcal{R}_{C_\pm} do not have any poles in the closed upper half plane; here the closure is added to make sure L is actually Fredholm on $H_b^{m,l}$ with $l = 0$. In general, if one wants to use other values of l , one needs to assume the absence of poles in $\text{Im } \sigma \geq -|l|$ (if one wants to keep the hypotheses symmetric).

Note that assuming $\frac{d\rho}{\rho}$ is timelike (with respect to \hat{g}) near $\overline{C_-}$, one *automatically* has the absence of poles of \mathcal{R}_{C_-} in an upper half plane, and the finiteness (with multiplicity) of the number of poles in any upper half plane, by the semiclassical estimates of [41], see Section 3.2 and 7.2 (one can ignore the complex absorption

⁶⁰In [40], the boundary defining function used to define the Mellin transform is replaced by its reciprocal, which effectively switches the sign of σ in the operator, but also the backward propagator is considered (propagating toward the past light cone), which reverses the role of σ and $-\sigma$ again, so in fact, the signs in [40] and [2] agree for the formulae connecting the asymptotically hyperbolic resolvents and the global operator, $\hat{L}(\sigma)$.

⁶¹The only change for $l \leq -1$ is that one needs to exclude the potential pure imaginary integer poles as well.

discussion there), so in this case the issue is that of a possible finite number of resonances. There is an analogous statement if $\frac{d\rho}{\rho}$ is timelike near $\overline{C_+}$ for \mathcal{R}_{C_+} .

Now, assuming still that $\frac{d\rho}{\rho}$ is timelike at, hence near $\overline{C_-}$, it is easy to construct a function \mathfrak{t} which has a timelike differential near $\overline{C_-}$, and appropriate sublevel sets are small neighborhoods of $\overline{C_-}$. Once one has such a function \mathfrak{t} , energy estimates can be used to conclude that rapidly vanishing, in such a neighborhood, solutions of $Lu = 0$ actually vanish in this neighborhood, so elements of $\text{Ker } L$ have support disjoint from $\overline{C_-}$; similarly elements of $\text{Ker } L^*$ have support disjoint from $\overline{C_+}$.

Concretely, with \hat{G} the dual b-metric of \hat{g} , let $0 < \epsilon_0 < \epsilon_1$, $\tilde{\epsilon} > 0$, $\delta > 0$ be such that

$$\begin{aligned} \rho < \tilde{\epsilon}, v > -\epsilon_1 &\Rightarrow \hat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right) > \delta, \\ \rho < \tilde{\epsilon}, -\epsilon_1 < v < -\epsilon_0 &\Rightarrow \hat{G}\left(\frac{d\rho}{\rho}, dv\right) < 0, \hat{G}(dv, dv) > 0. \end{aligned}$$

Such constants exist since $\hat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right)$ is positive on a neighborhood of $\overline{C_-}$ by assumption (thus for any sufficiently small $\epsilon_1, \tilde{\epsilon}$ there is a desired δ so that the first inequality is satisfied), $\hat{G}\left(\frac{d\rho}{\rho}, dv\right)|_{S_-} = -2$ so any sufficiently small ϵ_1 and $\tilde{\epsilon}$ give $\hat{G}\left(\frac{d\rho}{\rho}, dv\right) < 0$ in the desired region, and finally $\hat{G}(dv, dv) > 0$ on C_0 near S_- (as $\hat{G}(dv, dv) = -4v + \mathcal{O}(v^2)$ there), so choosing ϵ_1 sufficiently small, $\epsilon_0 < \epsilon_1$, and then $\tilde{\epsilon}$ sufficiently small satisfies all criteria. Now let ϵ_-, ϵ_+ be such that $0 < \epsilon_- < \epsilon_+ < \tilde{\epsilon}$, and let $\phi \in C^\infty(\mathbb{R})$ have $\phi' \leq 0$, $\phi = 0$ near $[-\epsilon_0, \infty)$, $\phi > \tilde{\epsilon}$ near $(-\infty, -\epsilon_1]$, $\phi' < 0$ when ϕ takes values in $[\epsilon_-, \epsilon_+]$. Then $\mathfrak{t} = \rho + \phi(v)$ has the property that

$$\mathfrak{t} \leq \epsilon_+ \Rightarrow \rho, \phi(v) \leq \epsilon_+ \Rightarrow \rho < \tilde{\epsilon}, v > -\epsilon_1,$$

and

$$v \geq -\epsilon_0 \Rightarrow \mathfrak{t} = \rho.$$

Thus, if $v \geq -\epsilon_0$ and $\mathfrak{t} \leq \epsilon_+$ then $d\mathfrak{t}$ is timelike as $d\rho$ is such, while if $v < -\epsilon_0$, $\mathfrak{t} \leq \epsilon_+$ then

$$\hat{G}(d\mathfrak{t}, d\mathfrak{t}) = \rho^2 \hat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right) + 2\phi'(v)\rho \hat{G}\left(\frac{d\rho}{\rho}, dv\right) + (\phi'(v))^2 \hat{G}(dv, dv)$$

and all terms are ≥ 0 in view of $-\epsilon_1 < v < -\epsilon_0$, $\rho \leq \tilde{\epsilon}$, with the inequality being strict when $\mathfrak{t} \in [\epsilon_-, \epsilon_+]$. Thus, near $\mathfrak{t}^{-1}([\epsilon_-, \epsilon_+])$, \mathfrak{t} is a timelike function. Further, ρ is also timelike, with $\frac{d\rho}{\rho}$ and $d\mathfrak{t}$ in the same component of the timelike cone; see Figure 6. Correspondingly, one can apply energy estimates using the timelike vector field $V = (\chi \circ \mathfrak{t})\rho^{-\ell} \hat{G}\left(\frac{d\rho}{\rho}, \cdot\right)$, where $\chi \in C^\infty(\mathbb{R})$ with $\chi' \leq 0$, $\chi = 1$ near $(-\infty, \epsilon_-]$, $\chi = 0$ near $[\epsilon_+, \infty)$, cf. [41, Section 3.3] leading up to Equation (3.24) and the subsequent discussion, which in turn is based on [44, Sections 3-4]. Here one needs to make both $-\chi'$ large relative to χ and $\ell > 0$ large (making the b-derivative of $\rho^{-\ell}$ large relative to $\rho^{-\ell}$), as discussed in the Mellin transformed setting in [41, Section 3.3], in [44, Sections 3-4], as well as Section 2.1 here (with τ in place of ρ , but with the sign of ℓ reversed due to the difference between b-saddle points and b-sinks/sources). Notice that taking ℓ large is exactly where the rapid decay near $\overline{C_+}$ is used.

While the existence of appropriate timelike functions, such as \mathfrak{t} , in a neighborhood of $\overline{C_+}$ and $\overline{C_-}$ was automatic (in a slightly degenerate sense at $\overline{C_\pm}$ themselves)

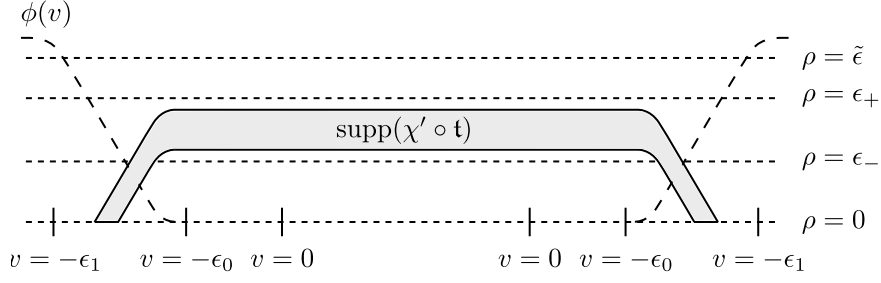


FIGURE 6. Setup for energy estimates near $\overline{C_-}$: The shaded region is the support of $\chi' \circ \mathfrak{t}$, where $-\chi'$ is used to dominate χ to give positivity in the energy estimate; near $\rho = 0$ and on $\text{supp}(\chi \circ \mathfrak{t})$, i.e. in the region between $\rho = 0$ and the shaded region, a sufficiently large weight $\rho^{-\ell}$ gives positivity.

when $\frac{d\rho}{\rho}$ is timelike in these regions, and indeed these functions could be extended to a neighborhood of C_0 if v is appropriately chosen, it does not seem clear how the non-trapping assumption on nullbicharacteristics implies the global existence of a time function. Thus, we *assume* the existence of such a C^∞ function $\tilde{\mathfrak{t}}$ on $M \setminus (\overline{C_+} \cup \overline{C_-})$ (where $\overline{C_+} \cup \overline{C_-}$ could be replaced by a neighborhood). Then the standard energy estimates apply and yield that any element of $\text{Ker } L$ that vanishes near $\overline{C_-}$ vanishes identically, and similarly with $\overline{C_+}$.

In summary:

Theorem 5.2. *If (M, g) is a non-trapping Lorentzian scattering metric in the sense of [2], $|l| < 1$, and*

- (1) *The induced asymptotically hyperbolic resolvents \mathcal{R}_{C_\pm} have no poles in $\text{Im } \sigma \geq -|l|$,*
- (2) *$\frac{d\rho}{\rho}$ is timelike near $\overline{C_+} \cup \overline{C_-}$,*
- (3) *there is a time-function $\tilde{\mathfrak{t}}$ on $M \setminus (\overline{C_+} \cup \overline{C_-})$,*

*then for $l \in \mathbb{R}$ and for order functions $m \in C^\infty({}^bS^*M)$ satisfying (5.1) and (5.9), the forward problem for the conjugated wave operator L , i.e. with L considered as a map*

$$L : \mathcal{X}^{m,l} \rightarrow \mathcal{Y}^{m-1,l},$$

is invertible.

Extending the notation of [2], especially Section 4, we denote by $H_b^{m,l,k}(M)$, where $m, l \in \mathbb{R}, k \in \mathbb{N}_0$, the space of all $u \in H_b^{m,l}(M)$ (i.e. $u \in \rho^l H_b^m(M)$, where ρ is the boundary defining function of M) such that $\mathcal{M}^j u \in H_b^{m,l}(M)$ for all $0 \leq j \leq k$. Here, $\mathcal{M} \subset \Psi_b^0(M)$ is the $\Psi_b^0(M)$ -module of pseudodifferential operators with principal symbol vanishing on the radial set \mathcal{R}_+ of the operator $L = \rho^{-(n-2)/2} \rho^{-2} \square_g \rho^{(n-2)/2}$; in the coordinates ρ, v, y as in [2] (ρ being as above, v a defining function of the light cone at infinity within ∂M , y coordinates within in the light cone at infinity), \mathcal{M} has local generators $\rho \partial_\rho, \rho \partial_v, v \partial_v, \partial_y$. Then the results of [2], concretely Proposition 4.4, extend our theorem to the spaces with module regularity. Namely the reference guarantees the module regularity $u \in H_b^{m,l,k}(M)$

of a solution u of $Lu = f$ if f has matching module regularity⁶² $f \in H_b^{m-1,l,k}(M)$ and if u is in $H_b^{m+k,l}(M)$ near $\overline{C_-}$. If $f \in H_b^{m-1,l,k}(M)$, then in particular f is locally in $H_b^{m+k-1,l}$ near $\overline{C_-}$, thus, taking into account that $m+l > 1/2$ already there, u is in $H_b^{m+k,l}$ in that region by Proposition 5.1 (by the first case there, i.e. in the high regularity regime). Thus, an application of the closed graph theorem gives the following boundedness result:

Theorem 5.3. *Under the assumptions of Theorem 5.2, L^{-1} has the property that it restricts to*

$$L^{-1} : H_b^{m-1,l,k} \rightarrow H_b^{m,l,k}, \quad k \geq 0,$$

as a bounded map.

In particular, letting $\Omega = \{\tilde{t} \geq 0\}$, where we normalized \tilde{t} so that it attains the value 0 within $M \setminus (C_+ \cup C_-)$, we have a forward solution operator S of L which maps $H_b^{m-1,l,k}(\Omega)^\bullet$ into $H_b^{m,l,k}(\Omega)^\bullet$, given that $m+l < 1/2$; let us assume that m is constant in Ω . Here, $H_b^{m,l,k}(\Omega)^\bullet$ consists of supported distributions at $\partial\Omega \cap C_0^c = \{\tilde{t} = 0\}$.

5.2. Algebra properties of $H_b^{m,-\infty,k}$. In order to discuss non-linear wave equations in Minkowski space, we need to discuss the algebra properties of $H_b^{m,-\infty,k} = \bigcup_{l \in \mathbb{R}} H_b^{m,l,k}$. Even though we are only interested in $H_b^{m,-\infty,k}(\Omega)^\bullet$, we consider $H_b^{m,-\infty,k}(M)$, where m is constant on M for notational simplicity, and the results we prove below are valid for $H_b^{m,-\infty,k}(\Omega)^\bullet$ by the same proofs.

We start with the following lemma:

Lemma 5.4. *Let $l_1, l_2 \in \mathbb{R}$, $k > n/2$. Then $H_b^{0,l_1,k} \cdot H_b^{0,l_2,k} \subset H_b^{0,l_1+l_2-1/2,k}$.*

Proof. The generators $\rho\partial_\rho, \rho\partial_v, v\partial_v, \partial_y$ of \mathcal{M} take on a simpler form if we blow up the point $(\rho, v) = (0, 0)$. It is most convenient to use projective coordinates on the blown-up space, namely:

- (1) Near the interior of the front face, we use the coordinates $\tilde{\rho} = \rho \geq 0$ and $s = v/\rho \in \mathbb{R}$. We compute $\rho\partial_\rho = \tilde{\rho}\partial_{\tilde{\rho}} - s\partial_s$, $v\partial_v = s\partial_s$, $\rho\partial_v = \partial_s$; and since $\frac{d\rho}{\rho} dv dy = d\tilde{\rho} ds dy$ (this is the b-density from $H_b^{0,l,k}$), the space $H_b^{0,l,k}$ becomes

$$A^{l,k} := \{u \in \tilde{\rho}^l L^2(d\tilde{\rho} ds dy) : \mathcal{A}^j u \in \tilde{\rho}^l L^2(d\tilde{\rho} ds dy), 0 \leq j \leq k\},$$

where \mathcal{A} is the C^∞ -module of differential operators generated by $\partial_s, \tilde{\rho}\partial_{\tilde{\rho}}, \partial_y$.

Now, observe that $\tilde{\rho}^l L^2(d\tilde{\rho} ds dy) = \tilde{\rho}^{l-1/2} L^2(\frac{d\tilde{\rho}}{\tilde{\rho}} ds dy)$; therefore, we can rewrite

$$\begin{aligned} A^{l,k} &= \{u \in \tilde{\rho}^{l-1/2} L^2(\frac{d\tilde{\rho}}{\tilde{\rho}} ds dy) : \mathcal{A}^j u \in \tilde{\rho}^{l-1/2} L^2(\frac{d\tilde{\rho}}{\tilde{\rho}} ds dy), 0 \leq j \leq k\} \\ &= \tilde{\rho}^{l-1/2} H_b^k(\frac{d\tilde{\rho}}{\tilde{\rho}} ds dy). \end{aligned}$$

In particular, by the Sobolev algebra property, Lemma 2.25, and the locality of the multiplication, choosing $k > n/2$ ensures that $\tilde{\rho}^{l_1-1/2} H_b^k \cdot \tilde{\rho}^{l_2-1/2} H_b^k \subset \tilde{\rho}^{l_1+l_2-1} H_b^k$, which is to say $A^{l_1,k} \cdot A^{l_2,k} \subset A^{l_1+l_2-1/2,k}$.

⁶²This Proposition in [2] is stated making the stronger assumption, $f \in H_b^{m-1+k,l}(M)$. However, the proof goes through for just $f \in H_b^{m-1,l,k}(M)$ in a completely analogous manner to the result of Haber and Vasy [20, Theorem 6.3], where (in the boundaryless setting, for a Lagrangian radial set) the result is stated in this generality.

- (2) Near either corner of the blown-up space, we use $\tilde{v} = v$ and $t = \rho/v$ (say, $\tilde{v} \geq 0, t \geq 0$). We compute $\rho\partial_\rho = t\partial_t$, $v\partial_v = \tilde{v}\partial_{\tilde{v}} - t\partial_t$, $\rho\partial_v = t\tilde{v}\partial_{\tilde{v}} - t^2\partial_t$; and since $\frac{d\rho}{\rho} dv dy = \frac{dt}{t} d\tilde{v} dy$, the space $H_b^{0,l,k}$ becomes

$$B^{l,k} := \{u \in (t\tilde{v})^l L^2(\frac{dt}{t} d\tilde{v} dy) : \mathcal{B}^j u \in (t\tilde{v})^l L^2(\frac{dt}{t} d\tilde{v} dy), 0 \leq j \leq k\},$$

where \mathcal{B} is the C^∞ -module of differential operators generated by $t\partial_t, \tilde{v}\partial_{\tilde{v}}, \partial_y$. Again, we can rewrite this as

$$B^{l,k} = t^l \tilde{v}^{l-1/2} H_b^k(\frac{dt}{t} \frac{d\tilde{v}}{\tilde{v}} dy),$$

which implies that for $k > n/2$,

$$B^{l_1,k} \cdot B^{l_2,k} \subset t^{l_1+l_2} v^{l_1+l_2-1} H_b^k(\frac{dt}{t} \frac{d\tilde{v}}{\tilde{v}} dy) \subset B^{l_1+l_2-1/2,k}.$$

To relate these two statements to the statement of the lemma, we use cutoff functions χ_A, χ_B to localize within the two coordinate systems. More precisely, choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}_s)$ such that $\chi(s) \equiv 1$ near $s = 0$, $\chi(s) = 0$ for $|s| \geq 2$, and $\chi^{1/2} \in C_c^\infty(\mathbb{R}_s)$. Then multiplication with $\chi_A(\rho, v) := \chi(v/\rho)$ is a continuous map $H_b^{0,l,k} \rightarrow A^{l,k}$. Indeed, to check this, one simply observes that $\mathcal{M}^j \chi_A \in L^\infty$ for all $j \in \mathbb{N}_0$. Similarly, letting $\chi_B(\rho, v) := 1 - \chi_A(\rho, v)$, multiplication with χ_B is a continuous map $H_b^{0,l,k} \rightarrow B^{l,k}$. Finally, note that we have $A^{l,k}, B^{l,k} \subset H_b^{0,l,k}$.

To put everything together, take $u_j \in H_b^{0,l_j,k}$ ($j = 1, 2$), then

$$u_1 u_2 = (\chi_A u_1)(\chi_A u_2) + (\chi_B u_1)(\chi_B u_2) + (\chi_A u_1)(\chi_B u_2) + (\chi_B u_1)(\chi_A u_2).$$

The first two terms then lie in $H_b^{0,l_1+l_2-1/2,k}$. To deal with the third term, write

$$(\chi_A u_1)(\chi_B u_2) = (\chi_A^{1/2} u_1)(\chi_A^{1/2} \chi_B u_2) \in A^{l_1,k} \cdot A^{l_2,k} \subset H_b^{0,l_1+l_2-1/2,k},$$

likewise for the fourth term. Thus, $u_1 u_2 \in H_b^{0,l_1+l_2-1/2,k}$, as claimed. \square

Remark 5.5. The proof actually shows more, namely that

$$H_b^{0,l,k} H_b^{0,l',k} \subset \rho_{\text{ff}}^{-1/2} H_b^{0,l+l',k}, \quad (5.10)$$

where ρ_{ff} is the defining function of the front face $\rho = v = 0$, e.g. $\rho_{\text{ff}} = (\rho^2 + v^2)^{1/2}$. The reason for (5.10) to be a natural statement is that module- and b-derivatives are the same away from $\rho = v = 0$, hence regularity with respect to the module \mathcal{M} is, up to a weight, which is a power of ρ_{ff} , the same as b-regularity.

More abstractly speaking, the above proof shows the following: If ρ_b denotes a boundary defining function of the other boundary hypersurface of $[M; S_+]$, i.e. $\partial[M; S_+] \setminus \text{ff}$, then

$$H_b^{0,l,k} \cong \rho_{\text{ff}}^{-1/2} (\rho_{\text{ff}} \rho_b)^l H_b^k([M; S_+]).$$

Note that one can also show this in one step, introducing the coordinates $\rho_{\text{ff}} \geq 0$ and $s = v/(\rho + \rho_{\text{ff}}) \in [-1, 1]$ on $[M; S_+]$ in a neighborhood of ff , and mimicking the above proof, which however is computationally less convenient.

Remark 5.6. We can extend the lemma to $H_b^{m,l,k} H_b^{m,l',k} \subset H_b^{m,l+l'-1/2,k}$ for $m \in \mathbb{N}_0$ using the Leibniz rule to distribute the m b-derivatives among the two factors, and then using the lemma for the case $m = 0$.

The following corollary, which will play an important role in Section 5.5, improves Lemma 5.4 if we have higher b-regularity.

Corollary 5.7. *Let $k > n/2$, $0 \leq \delta < 1/n$ and $l, l' \in \mathbb{R}$. Then*

- (1) $H_b^{1,l,k} H_b^{0,l',k} \subset H_b^{0,l+l'-1/2+\delta,k}$.
(2) $H_b^{1,l,k} H_b^{1,l',k} \subset H_b^{1,l+l'-1/2+\delta,k}$.

Proof. Take $s = 1/(2\delta) > n/2$, then

$$H_b^{s,l,k} H_b^{0,l',k} \subset H_b^{0,l+l',k}; \quad (5.11)$$

indeed, using the Leibniz rule to distribute the k module derivatives among the two factors and cancelling the weights, this amounts to showing that $H_b^{s,0,k_1} H_b^{0,0,k_2} \subset H_b^{0,0,0}$ for $k_1 + k_2 \geq k$; but this is true even for $k_1 = k_2 = 0$, since H_b^s is a multiplier on H_b^0 provided $s > n/2$.

The lemma on the other hand gives

$$H_b^{0,l,k} H_b^{0,l',k} \subset \rho^{-1/2} H_b^{0,l+l',k}. \quad (5.12)$$

Interpolating in the first factor between (5.11) and (5.12) thus gives the first statement.

For the second statement, use the Leibniz rule to distribute the one b -derivative to either factor; then, one has to show $H_b^{1,l,k} H_b^{0,l',k} \subset H_b^{0,l+l'-1/2+\delta,k}$, and the same inclusion with l and l' switched, which is what we just proved. \square

Lemma 5.4 and the remark following it imply that for $u \in H_b^{m,l,k}$, $p \geq 1$, with $m \geq 0, k > n/2$, we have $u^p \in H_b^{m,pl-(p-1)/2,k}$; in fact, $u^p \in \rho_{\text{ff}}^{-(p-1)/2} H_b^{m,pl,k}$, see Remark 5.5. Using Corollary 5.7, we can improve this to the statement $u \in H_b^{m,l,k} \Rightarrow u^p \in H_b^{m,pl-(p-1)/2+(p-1)\delta,k}$ for $m \geq 1$.

For non-linearities that only involve powers u^p , we can afford to lose differentiability, as at the end of Section 4.2, and gain decay in return, as the following lemma shows.

Lemma 5.8. *Let $\alpha > 1/2$, $l \in \mathbb{R}$, $k \in \mathbb{N}_0$. Then $\rho_{\text{ff}}^{-\alpha} H_b^{0,l,k} \subset \rho^{1/2-\alpha} H_b^{-1,l,k}$, where $\rho_{\text{ff}} = (\rho^2 + v^2)^{1/2}$.*

Proof. We may assume $l = 0$, and that u is supported in $|v| < 1$, $\rho < 1$. First, consider the case $k = 0$. Let $u \in \rho_{\text{ff}}^{-\alpha} H_b^0$, and put

$$\tilde{u}(\rho, v, y) = \int_{-\infty}^v u(\rho, w, y) dw,$$

so $\partial_v \tilde{u} = u$. We have to prove $\chi \tilde{u} \in \rho^{1/2-\alpha} H_b^0$ if $\chi \equiv 1$ near $\text{supp } u$, which implies $u \in H_b^{-1}$, as $\partial_v : H_b^0 \rightarrow H_b^{-1}$, and the b -Sobolev space are local spaces. But

$$|\tilde{u}(\rho, v, y)|^2 \leq \left(\int_{-1}^1 \rho_{\text{ff}}(\rho, w)^{2\alpha} |u(\rho, w, y)|^2 dw \right) \int_{-1}^1 \rho_{\text{ff}}(\rho, w)^{-2\alpha} dw; \quad (5.13)$$

now,

$$\int_{-1}^1 \rho_{\text{ff}}^{-2\alpha} dw = \rho^{1-2\alpha} \int_{-1/\rho}^{1/\rho} \frac{dz}{(1+|z|^2)^\alpha} \lesssim \rho^{1-2\alpha}$$

for $\alpha > 1/2$, therefore, with the v integral considered on a fixed interval, say $|v| < 2$ (notice that the right hand side in (5.13) is independent of v !),

$$\iiint \rho^{2\alpha-1} |\tilde{u}(\rho, v, y)|^2 \frac{d\rho}{\rho} dv dy \lesssim \iiint \rho_{\text{ff}}^{2\alpha} |u(\rho, w, y)|^2 \frac{d\rho}{\rho} dw dy,$$

proving the claim for $k = 0$. Now, $\rho \partial_\rho$ and ∂_y just commute with this calculation, so the corresponding derivatives are certainly well-behaved. On the other hand,

$\partial_v \tilde{u} = u$, so the estimates involving at least one v -derivative are just those for u itself. \square

Corollary 5.9. *Let $k, p \in \mathbb{N}$ be such that $k > n/2$, $p \geq 2$. Let $l \in \mathbb{R}$, $u \in H_b^{0,l,k}$. Then $u^p \in H_b^{-1,lp-(p-1)/2+1/2-\delta,k}$ with $\delta = 0$ if $p \geq 3$ and $\delta > 0$ if $p = 2$.*

Proof. This follows from $u^p \in \rho_{\text{ff}}^{-(p-1)/2-\delta} H_b^{0,lp,k}$ and the previous lemma, using that $(p-1)/2 + \delta > 1/2$ with δ as stated. \square

In other words, we gain the decay $\rho^{1/2-\delta}$ if we give up one derivative.

5.3. A class of semilinear equations. We are now set to discuss solutions to non-linear wave equations on Minkowski space time. In fact, we can work in much greater generality, as explained in and after Theorem 5.2, and obtain a forward solution operator $S: H_b^{m-1,l,k}(\Omega)^\bullet \rightarrow H_b^{m,l,k}(\Omega)^\bullet$ of $P = \rho^{-(n-2)/2} \rho^{-2} \square_g \rho^{(n-2)/2}$ provided $|l| < 1$, $m+l < 1/2$ and $k \geq 0$.

Undoing the conjugation, we obtain a forward solution operator

$$\begin{aligned} \tilde{S} &= \rho^{(n-2)/2} S \rho^{-2} \rho^{-(n-2)/2}, \\ \tilde{S}: H_b^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet &\rightarrow H_b^{m,l+(n-2)/2,k}(\Omega)^\bullet \end{aligned}$$

of \square_g .

Since g is a Lorentzian scattering metric, the natural vector fields to appear in a non-linear equation are scattering vector fields; more generally, since the analysis is carried out on b-spaces, we indeed allow b-vector fields in the following statement:

Theorem 5.10. *Let*

$$q: H_b^{m,l+(n-2)/2,k}(\Omega)^\bullet \times H_b^{m-1,l+(n-2)/2,k}(\Omega; {}^bT^*\Omega)^\bullet \rightarrow H_b^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous non-decreasing function $L: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfying

$$\|q(u, {}^bdu) - q(v, {}^bdv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ so that the following holds: If $L(0) < C_L$, then for small $R > 0$, there exists $C > 0$ such that for all $f \in H_b^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ with norm $\leq C$, the equation

$$\square_g u = f + q(u, {}^bdu)$$

has a unique solution $u \in H_b^{m,l+(n-2)/2,k}(\Omega)^\bullet$, with norm $\leq R$, that depends continuously on f .

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.24. \square

Remark 5.11. Here, just as in Theorem 4.10, we can also allow q to depend on $\square_g u$ as well.

5.4. Semilinear equations with polynomial non-linearity. Next, we want to find a forward solution of the semilinear PDE

$$\square_g u = f + cu^p X(u),$$

where $c \in C^\infty(M)$, $p \in \mathbb{N}_0$, and $X(u) = \prod_{j=1}^q \rho V_j(u)$ is a q -fold product of derivatives of u along scattering vector fields; here, V_j are b-vector fields. Let us assume $p+q \geq 2$ in order for the equation to be genuinely non-linear. We rewrite the PDE as

$$\begin{aligned} L(\rho^{-(n-2)/2}u) &= \rho^{-(n-2)/2-2}f + c\rho^{-2}\rho^{(p-1)(n-2)/2}(\rho^{-(n-2)/2}u)^p \\ &\quad \times \prod_{j=1}^q \rho V_j(\rho^{(n-2)/2}\rho^{-(n-2)/2}u). \end{aligned}$$

Introducing $\tilde{u} = \rho^{-(n-2)/2}u$ and $\tilde{f} = \rho^{-(n-2)/2-2}f$ yields the equation

$$\begin{aligned} L\tilde{u} &= \tilde{f} + c\rho^{(p-1)(n-2)/2-2}\tilde{u}^p \prod_{j=1}^q \rho^{n/2}(f_j\tilde{u} + V_j\tilde{u}) \\ &= \tilde{f} + c\rho^{(p-1)(n-2)/2+qn/2-2}\tilde{u}^p \prod_{j=1}^q (f_j\tilde{u} + V_j\tilde{u}), \end{aligned} \quad (5.14)$$

where the f_j are smooth functions. Now suppose that $\tilde{u} \in H_b^{m,l,k}(\Omega)^\bullet$ with $m+l < 1/2$, $m \geq 1$, $k > n/2$ (so that $H_b^{m-1,-\infty,k}(\Omega)^\bullet$ is an algebra), then the second summand of the right hand side of (5.14) lies in $H_b^{m-1,\ell,k}(\Omega)^\bullet$, where

$$\ell = (p-1)(n-2)/2 + qn/2 - 2 + pl - (p-1)/2 + ql - (q-1)/2 - 1/2.$$

For this space to lie in $H_b^{m-1,l,k}(\Omega)^\bullet$ (which we want in order to be able to apply the solution operator S and land in $H_b^{m,l,k}(\Omega)^\bullet$ so that a fixed point argument as in Section 2 can be applied), we thus need $\ell \geq l$, which can be rewritten as

$$(p-1)(l + (n-3)/2) + q(l + (n-1)/2) \geq 2. \quad (5.15)$$

For $m = 1$ and $l < 1/2 - m$ less than, but close to $-1/2$, and $0 \leq \delta < 1/n$, we thus get the condition

$$(p-1)(n-4) + q(n-2) > 4.$$

If there are only non-linearities involving derivatives of u , i.e. $p = 0$, we get the condition $q > 1 + 2/(n-2)$, i.e. quadratic non-linearities are fine for $n \geq 5$, cubic ones for $n \geq 4$.

Note that if $q = 0$, we can actually choose $m = 0$ and $l < 1/2$ close to $1/2$, and we have Corollary 5.9 at hand. Thus we can improve (5.15) to $(p-1)(1/2 + (n-3)/2) > 2 - 1/2$, i.e. $p > 1 + 3/(n-2)$, hence quadratic non-linearities can be dealt with if $n \geq 6$, whereas cubic non-linearities are fine as long as $n \geq 4$. Observe that this condition on p always implies $p > 1$, which is a natural condition, since $p = 1$ would amount to changing \square_g into $\square_g - m^2$ (if one chooses the sign appropriately). But the Klein-Gordon operator naturally fits into a scattering framework, as mentioned in the Introduction, i.e. requires a different analysis; we will not pursue this further in this paper.

To summarize the general case, note that $\tilde{u} \in H_b^{m,l,k}(\Omega)^\bullet$ is equivalent to $u \in H_b^{m,l+(n-2)/2,k}(\Omega)^\bullet$, and $\tilde{f} \in H_b^{m-1,l,k}(\Omega)^\bullet$ to $f \in H_b^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$; thus:

Theorem 5.12. *Let $|l| < 1, m + l < 1/2, k > n/2$, and assume that $p, q \in \mathbb{N}_0$, $p + q \geq 2$, satisfy condition (5.15); let $m \geq 0$ if $q = 0$, otherwise let $m \geq 1$. Moreover, let $c \in C^\infty(M)$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j is a scattering vector field on M . Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in H_b^{m-1, l+(n-2)/2+2, k}(\Omega)^\bullet$ with norm $\leq C$, the equation*

$$\square_g u = f + cu^p X(u)$$

has a unique solution $u \in H_b^{m, l+(n-2)/2, k}(\Omega)^\bullet$, with norm $\leq R$, that depends continuously on f .

The same conclusion holds if the non-linearity is a finite sum of terms of the form $cu^p X(u)$, provided each such term separately satisfies (5.15).

Proof. Reformulating the PDE in terms of \tilde{u} and \tilde{f} as above, this follows from an application of the Banach fixed point theorem to the map

$$H_b^{m, l, k}(\Omega)^\bullet \ni \tilde{u} \mapsto S \left(\tilde{f} + c\rho^{(p-1)(n-2)/2+qn/2-2} \tilde{u}^p \prod_{j=1}^q (f_j \tilde{u} + V_j \tilde{u}) \right) \in H_b^{m, l, k}(\Omega)^\bullet$$

with m, l, k as in the statement of the theorem. Here, $p+q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H_b^{m, l, k}(\Omega)^\bullet$. \square

Remark 5.13. If the derivatives in the non-linearity only involve module derivatives, we get a slightly better result since we can work with $\tilde{u} \in H_b^{0, l, k}(\Omega)^\bullet$: Indeed, a module derivative falling on \tilde{u} gives an element of $H_b^{0, l, k-1}(\Omega)^\bullet$, applied to which the forward solution operator produces an element of $H_b^{1, l, k-1}(\Omega)^\bullet \subset H_b^{0, l, k}(\Omega)^\bullet$.

The numerology works out as follows: In condition (5.15), we now take $l < 1/2$ close to $1/2$, thus obtaining

$$(p-1)(n-2) + qn > 4.$$

Thus, in the case that there are only derivatives in the non-linearity, i.e. $p = 0$, we get $q > 1 + 2/n$, which allows for quadratic non-linearities provided $n \geq 3$.

Remark 5.14. Observe that we can improve (5.15) in the case $p \geq 1, q \geq 1, m \geq 1$ by using the δ -improvement from Corollary 5.7, namely, the right hand side of (5.14) actually lies in $H_b^{m-1, \ell, k}(\Omega)^\bullet$, where now

$\ell = (p-1)(n-2)/2 + qn/2 - 2 + pl - (p-1)/2 + (p-1)\delta + ql - (q-1)/2 - 1/2 + \delta$, which satisfies $\ell \geq l$ if

$$(p-1)(l + (n-3)/2 + \delta) + q(l + (n-1)/2) + \delta \geq 2,$$

which for $l < -1/2$ close to $-1/2$ means: $(p-1)(n-4+2\delta) + q(n-2) + 2\delta > 4$, where $0 < \delta < 1/n$.

Remark 5.15. Let us compare the above result with Christodoulou's [7]. A special case of his theorem states that the Cauchy problem for the wave equation on Minkowski space with small initial data in $H_{k, k-1}(\mathbb{R}^{n-1})$ ⁶³ admits a global solution $u \in H_{\text{loc}}^k(\mathbb{R}^n)$ with decay $|u(x)| \lesssim (1 + (v/\rho)^2)^{-(n-2)/2}$; here, $k = n/2 + 2$, and n is assumed to ≥ 4 and even; in case $n = 4$, the non-linearity is moreover assumed to satisfy the null condition. The only polynomial non-linearity that we cannot deal with using the above argument is thus the null-form non-linearity in 4 dimensions.

⁶³Note that n is the dimension of Minkowski space here, whereas Christodoulou uses $n + 1$.

To make a further comparison possible, we express $H_{k,\delta}(\mathbb{R}^{n-1})$ as a b-Sobolev space on the radial compactification of \mathbb{R}^{n-1} : Note that $u \in H_{k,\delta}(\mathbb{R}^{n-1})$ is equivalent to $(\langle x \rangle D_x)^\alpha u \in \langle x \rangle^{-\delta} L^2(\mathbb{R}^{n-1})$, $|\alpha| \leq k$. In terms of the boundary defining function ρ of $\partial\mathbb{R}^{n-1}$ and the standard measure $d\omega$ on the unit sphere $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, we have $L^2(\mathbb{R}^{n-1}) = L^2(\frac{d\rho}{\rho^2} \frac{dy}{\rho^{n-2}}) = \rho^{(n-1)/2} L^2(\frac{d\rho}{\rho} dy)$, and thus $H_{k,\delta}(\mathbb{R}^{n-1}) = \rho^{(n-1)/2+\delta} H_b^k(\tilde{\mathfrak{t}} = 0)$. Therefore, converting the Cauchy problem into a forward problem, the forcing lies in $H_b^{k,(n-1)/2+k-1,0}(\Omega)^\bullet = H_b^{n/2+2,n+1/2,0}(\Omega)^\bullet$. Comparing this with the space $H_b^{0,l+(n-2)/2+2,n/2+1}$ (with $l < 1/2$) needed for our argument, we see that Christodoulou's result applies to a regime of fast decay which is disjoint from our slow decay (or even mild growth) regime.

Remark 5.16. In the case of non-linearities u^p , the result of Christodoulou [7] implies the existence of global solutions to $\square_g u = f + u^p$ if the spacetime dimension n is *even* and $n \geq 4$ if $p \geq 3$; in even dimensions $n \geq 6$, $p \geq 2$ suffices; the above result extends this to all dimensions satisfying the respective inequalities. In a somewhat similar context, see the work of Chruściel and Łeński [9], it has been proved that $p \geq 2$ in fact works in all dimensions $n \geq 5$.

5.5. Semilinear equations with null condition. With g the Lorentzian scattering metric on Minkowski space as before, define the null form $Q(\text{sc} du, \text{sc} dv) = g^{jk} \partial_j u \partial_k v$, and write $Q(\text{sc} du)$ for $Q(\text{sc} du, \text{sc} du)$. We are interested in solving the PDE

$$\square_g u = Q(\text{sc} du) + f.$$

The previous discussion solves this for $n \geq 5$; thus, let us from now on assume $n = 4$. To make the computations more transparent, we will keep the n in the notation and only substitute $n = 4$ when needed. Rewriting the PDE in terms of the operator $L = \rho^{-2} \rho^{-(n-2)/2} \square_g \rho^{(n-2)/2}$ as above, we get

$$L\tilde{u} = \tilde{f} + \rho^{-(n-2)/2-2} Q(\text{sc} d(\rho^{(n-2)/2} \tilde{u})),$$

where $\tilde{u} = \rho^{-(n-2)/2} u$ and $\tilde{f} = \rho^{-(n-2)/2-2} f$. We can write $Q(\text{sc} du) = \frac{1}{2} \square_g(u^2) - u \square_g u$, thus the PDE becomes

$$\begin{aligned} L\tilde{u} &= \tilde{f} + \rho^{-(n-2)/2-2} \left(\frac{1}{2} \square_g(\rho^{n-2} \tilde{u}^2) - \rho^{(n-2)/2} \tilde{u} \square_g(\rho^{(n-2)/2} \tilde{u}) \right) \\ &= \tilde{f} + \frac{1}{2} L(\rho^{(n-2)/2} \tilde{u}^2) - \rho^{(n-2)/2} \tilde{u} L\tilde{u}. \end{aligned}$$

Since the results of Section 5.2 give small improvements on the decay of products of $H_b^{1,*,*}$ functions with $H_b^{m,*,*}$ functions ($m \geq 0$), one wants to solve this PDE on a function space that keeps track of these small improvements.

Definition 5.17. For $l \in \mathbb{R}, k \in \mathbb{N}_0$ and $\alpha \geq 0$, define the space $\mathcal{X}^{l,k,\alpha} := \{v \in H_b^{1,l+\alpha,k}(\Omega)^\bullet : Lv \in H_b^{0,l,k}(\Omega)^\bullet\}$ with norm

$$\|v\|_{\mathcal{X}^{l,k,\alpha}} = \|v\|_{H_b^{1,l+\alpha,k}(\Omega)^\bullet} + \|Lv\|_{H_b^{0,l,k}(\Omega)^\bullet}. \quad (5.16)$$

By an argument similar to the one used in the proof of Theorem 2.24, we see that $\mathcal{X}^{l,k,\alpha}$ is a Banach space.

On $\mathcal{X}^{l,k,\alpha}$, which $\alpha > 0$ chosen below, we want to run an iteration argument: Start by defining the operator $T: \mathcal{X}^{l,k,\alpha} \rightarrow H_b^{1,-\infty,k}(\Omega)^\bullet$ by

$$T: \tilde{u} \mapsto S(\tilde{f} - \rho^{(n-2)/2} \tilde{u} L\tilde{u}) + \frac{1}{2} \rho^{(n-2)/2} \tilde{u}^2.$$

Note that $\tilde{u} \in \mathcal{X}^{l,k,\alpha}$ implies, using Corollary 5.7 with $\delta < 1/n$,

$$\begin{aligned} \rho^{(n-2)/2} \tilde{u}^2 &\in \rho^{(n-2)/2} H_b^{1,2(l+\alpha)-1/2+\delta,k}(\Omega)^\bullet = H_b^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet, \\ \rho^{(n-2)/2} \tilde{u} L \tilde{u} &\in H_b^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \\ S(\rho^{(n-2)/2} \tilde{u} L \tilde{u}) &\in H_b^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \end{aligned} \quad (5.17)$$

where in the last inclusion, we need to require $1 + (2l + \alpha + (n - 3)/2 + \delta) < 1/2$, which for $n = 4$ means

$$l < -1/2 - (\alpha + \delta)/2; \quad (5.18)$$

let us assume from now on that this condition holds. Furthermore, (5.17) implies $T\tilde{u} \in H_b^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet$. Finally, we analyze

$$L(T\tilde{u}) \in H_b^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + \frac{1}{2}L(\rho^{(n-2)/2} \tilde{u}^2).$$

Using that L is a second-order b-differential operator, we have

$$\begin{aligned} \rho^{(n-2)/2} L(\tilde{u}^2) &\in 2\rho^{(n-2)/2} \tilde{u} L \tilde{u} + \rho^{(n-2)/2} H_b^{0,l+\alpha,k}(\Omega)^\bullet H_b^{0,l+\alpha,k}(\Omega)^\bullet \\ &\subset H_b^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + H_b^{0,2(l+\alpha)+(n-3)/2,k}(\Omega)^\bullet \\ &= H_b^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet, \end{aligned}$$

which gives

$$\begin{aligned} L(\rho^{(n-2)/2} \tilde{u}^2) &\in L(\rho^{(n-2)/2} \tilde{u}^2) + \rho^{(n-2)/2} L(\tilde{u}^2) \\ &\quad + \rho^{(n-2)/2} H_b^{1,l+\alpha,k}(\Omega)^\bullet H_b^{0,l+\alpha,k}(\Omega)^\bullet \\ &\subset H_b^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet + H_b^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet \\ &\quad + H_b^{0,2l+\alpha+(n-3)/2+\delta+\alpha}(\Omega)^\bullet \\ &= H_b^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet. \end{aligned}$$

Hence, putting everything together,

$$L(T\tilde{u}) \in H_b^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet.$$

Therefore, we have $T\tilde{u} \in \mathcal{X}^{l,k,\alpha}$ provided

$$\begin{aligned} 2l + \alpha + (n - 3)/2 + \delta &\geq l + \alpha \\ 2l + \alpha + (n - 3)/2 + \min\{\alpha, \delta\} &\geq l, \end{aligned}$$

which for $0 < \alpha < \delta$ and $n = 4$ is equivalent to

$$l \geq -1/2 - \delta, l \geq -1/2 - 2\alpha. \quad (5.19)$$

This is consistent with condition (5.18) if $-1/2 - (\alpha + \delta)/2 > -1/2 - 2\alpha$, i.e. if $\alpha > \delta/3$.

Finally, for the map T to be well-defined, we need $S\tilde{f} \in \mathcal{X}^{l,k,\alpha}$, hence $\tilde{f} \in \text{Ran}_{\mathcal{X}^{l,k,\alpha}} L$, which is in particular satisfied if $\tilde{f} \in H_b^{0,l+\alpha,k}(\Omega)^\bullet$. Indeed, since $1 + l + \alpha < 1 - 1/2 - (\delta - \alpha)/2 < 1/2$ by condition (5.18), the element $S\tilde{f} \in H_b^{1,l+\alpha,k}(\Omega)^\bullet$ is well-defined.

We have proved:

Theorem 5.18. *Let $c \in \mathbb{C}$, $0 < \delta < 1/4$, $\delta/3 < \alpha < \delta$, and let $-1/2 - 2\alpha \leq l < -1/2 - (\alpha + \delta)/2$. Then for small enough $R > 0$, there exists a constant $C > 0$ such that for all $f \in H_{\mathbb{b}}^{0,l+3+\alpha,k}(\Omega)^\bullet$ with norm $\leq C$, the equation*

$$\square_g u = f + cQ(\text{sc} du)$$

has a unique solution $u \in \mathcal{X}^{l+1,k,\alpha}$, with norm $\leq R$, that depends continuously on f .

APPENDIX A. NON-TRAPPING ESTIMATES NEAR NORMALLY HYPERBOLIC TRAPPING

First consider a semiclassical operator $P_h(z)$ with normally hyperbolic trapping on a manifold without boundary X , with trapped set Γ , as was done by Wunsch and Zworski [47]. Wunsch and Zworski studied this setting, imposing other global assumptions, the most important one being adding complex absorption W in such a way that all bicharacteristics outside Γ either enter the elliptic set of W in finite time or tend to Γ in both the forward and backward directions, and in at least one of the two directions they tend to the elliptic set of W . The bicharacteristics tending to Γ in the forward/backward directions are forward/backward trapped; denote by Γ_- , resp. Γ_+ the forward, resp. backward trapped set⁶⁴, and assume that these are smooth codimension one submanifolds of the semiclassical characteristic set $\Sigma_{h,z}$, intersecting transversally.

In this normally hyperbolic setting Wunsch and Zworski [47] have shown polynomial semiclassical resolvent estimates

$$\|u\| \leq Ch^{-N} \|P_h(z)u\|, \quad 0 < h < h_0, \quad (\text{A.1})$$

in small strips $|\text{Im } z| \leq ch$, $c > 0$ sufficiently small, $N > 1$, and indeed for z real, the loss is merely logarithmic, i.e. one has

$$\|u\| \leq Ch^{-1}(\log h^{-1}) \|P_h(z)u\|, \quad 0 < h < h_0, \quad (\text{A.2})$$

where $\|\cdot\|$ is the L^2 -norm.

In this appendix we are concerned with improved estimates if one localizes u and $P_h(z)u$ away from the trapping Γ in a rather weak sense, such as by applying pseudodifferential operators with symbols vanishing at Γ . To place this into context, recall that Datchev and Vasy [13, 14] have shown that under our assumptions, with $\text{Im } z = \mathcal{O}(h^\infty)$, if $A, B \in \Psi_h(X)$ with $\text{WF}'_h(A) \cap \Gamma = \text{WF}'_h(B) \cap \Gamma = \emptyset$, B elliptic on $\text{WF}'_h(A)$, then for all M there is N such that

$$\|Au\| \leq Ch^{-1} \|BP_h(z)u\| + C'h^M \|u\| + C''h^{-N} \|(\text{Id} - B)P_h(z)u\|, \quad (\text{A.3})$$

i.e. if $P_h(z)u$ is $\mathcal{O}(h^{N-1})$ at Γ , where $\text{Id} - B$ is elliptic, then on the elliptic set of A , hence off Γ by appropriate choice of A , u satisfies non-trapping semiclassical estimates:

$$\|AP_h(z)^{-1}Av\| \leq Ch^{-1} \|v\|,$$

with A as above (take B as above with $\text{WF}'_h(\text{Id} - B) \cap \text{WF}'_h(A) = \emptyset$). Here the $\mathcal{O}(h^\infty)$ bound on $\text{Im } z$ arises from the a priori estimate, (A.1), and if $1 < N < 3/2$, e.g. as is on, and sufficiently near, the real axis⁶⁵, then one can take $\text{Im } z =$

⁶⁴In the notation of Wunsch and Zworski [47], Γ_\pm are the backward/forward trapped sets for all (not necessarily null) bicharacteristics near the, say, zero level set of the semiclassical principal symbol $p_{h,z}$, Γ_\pm^λ are the corresponding sets within the λ -level set of $p_{h,z}$.

⁶⁵In the latter case by the Phragmén-Lindelöf theorem.

$\mathcal{O}(h^{-1+2N})$. The purpose of this appendix is to improve this result by relaxing the conditions on $\text{WF}'_{\hbar}(A)$ and $\text{WF}'_{\hbar}(B)$ in (A.3).

The main point of the next theorem is thus that its estimate degenerates only at, as opposed to near, Γ . The proof given here is closely related to the proof of Wunsch and Zworski [47, Section 4], but being suboptimal in terms of the L^2 -estimate, even though it is optimal (non-trapping) when a pseudodifferential operator with vanishing principal symbol at Γ is applied from both sides, it can take place in a significantly simpler, standard, semiclassical pseudodifferential algebra. To set this up, let $Q_{\pm} \in \Psi_{\hbar}(X)$ be self-adjoint and have symbols which are defining functions of Γ_{\pm} near Γ , *within the characteristic set* $\Sigma_{\hbar,z}$, say on a neighborhood O of Γ . Also let $p = p_{\hbar,z}$ be the semiclassical principal symbol of $P_{\hbar}(z)$. Here we recall from the work of Wunsch and Zworski [47, Lemma 4.1], with a corrected argument in [46], that for defining functions ϕ_{\pm} of Γ_{\pm} (near Γ) one can take ϕ_{\pm} with

$$\mathbf{H}_p \phi_{\pm} = \mp c_{\pm}^2 \phi_{\pm}$$

with $c_{\pm} > 0$ near Γ , and with⁶⁶

$$\{\phi_+, \phi_-\} > 0$$

near Γ . By shrinking O if necessary we may assume that this Poisson bracket as well as c_{\pm} have positive lower bounds on O . Then notice that

$$\mathbf{H}_p \phi_+^2 = -2c_+^2 \phi_+^2, \quad \mathbf{H}_p \phi_-^2 = 2c_-^2 \phi_-^2.$$

Let $Q_0 \in \Psi_{\hbar}(X)$ have $\text{WF}'_{\hbar}(Q_0) \cap \Gamma = \emptyset$ and be elliptic on O^c (and thus on a neighborhood of O^c), with real principal symbol for convenience. One considers *normally isotropic* spaces at Γ , $\mathcal{H}_{\hbar,\Gamma}$, with squared norms given by

$$\|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2 + h\|u\|^2;$$

this is just the standard L^2 -space microlocally away from Γ as one of Q_+ , Q_- or Q_0 is elliptic there, and it does not depend on the choice of Q_0 as on $O \setminus \Gamma$ one of Q_+ and Q_- is elliptic at every point. Notice that in fact

$$(Q_+ - iQ_-)^*(Q_+ - iQ_-) = Q_+^* Q_+ + Q_-^* Q_- - i[Q_+, Q_-]$$

and if $B \in \Psi_{\hbar}(X)$ with $\text{WF}'_{\hbar}(B) \subset O$ then

$$h\|Bv\|^2 \leq C \text{Re}\langle i[Q_+, Q_-]Bv, Bv \rangle + h^{N'} \|Bv\|^2,$$

$C > 0$, in view of $\{\phi_+, \phi_-\} > 0$ on O , so

$$Q_+^* Q_+ + Q_-^* Q_- = \frac{1}{2}(Q_+^* Q_+ + Q_-^* Q_- + (Q_+ - iQ_-)^*(Q_+ - iQ_-) + i[Q_+, Q_-])$$

shows that, for $h > 0$ small, the norm on $\mathcal{H}_{\hbar,\Gamma}$ is equivalent to just the norm

$$\|u\|_{\mathcal{H}_{\hbar,\Gamma},2}^2 = \|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2.$$

The dual space relative to L^2 is then⁶⁷

$$\mathcal{H}_{\hbar,\Gamma}^* = h^{1/2}L^2 + Q_+L^2 + Q_-L^2 + Q_0L^2$$

⁶⁶These defining functions exist globally when Γ_{\pm} is orientable; but even if Γ_{\pm} is not such, the square is globally defined. There is only a minor change required below if ϕ_{\pm} are not well defined; see Footnote 69.

⁶⁷One really has Q_{\pm}^* and Q_0^* in this formula, but the reality of the principal symbols assures that one may replace them by Q_{\pm} and Q_0 modulo hL^2 . See [31, Appendix A] for a general discussion of the underlying functional analysis.

(which is L^2 as a space, but with this norm); $P_h(z)u$ will then be measured in $\mathcal{H}_{h,\Gamma}^*$. Then $\Psi_h(X)$ acts on $\mathcal{H}_{h,\Gamma}$, and thus on $\mathcal{H}_{h,\Gamma}^*$, for $B \in \Psi_h(X)$ preserves $h^{-1/2}L^2$ and gives

$$\|Q_+Bu\| \leq \|BQ_+u\| + \|[Q_+, B]u\| \leq C\|Q_+u\| + h\|u\|_{L^2},$$

with a similar result for Q_- and Q_0 . We remark that the notation $\mathcal{H}_{h,\Gamma}$ is justified as the space depends only on Γ , not on the particular defining functions ϕ_\pm as any other defining functions would change Q_\pm by an elliptic factor modulo an element of $h\Psi_h(X)$, whose contribution to the squared norm can be absorbed into $Ch^2\|u\|_{L^2}^2$, and thus dropped altogether (for h small) in view of the equivalence of the two norms discussed above.

Theorem A.1. *Let $P = P_h(z)$, Q_\pm be as above, z real.⁶⁸ Then*

$$\|Q_+u\| + \|Q_-u\| \leq Ch^{-1}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*} + C'h^{1/2}\|u\|, \quad (\text{A.4})$$

and thus by (A.2),

$$\|u\|_{\mathcal{H}_{h,\Gamma}} \leq Ch^{-1}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*}. \quad (\text{A.5})$$

Note that this theorem in particular implies the main result of [13] in this setting, in that the estimates are of the same kind, except that in [13] Pu is assumed to be microlocalized away from, and u is estimated microlocally away from, Γ .

We also remark that the microlocal version of the two estimates of the theorem is that given any neighborhood O' of Γ with closure in O , there exist $B_0 \in \Psi_h(X)$ elliptic at Γ , $B_1, B_2 \in \Psi_h(X)$ with $\text{WF}'_h(B_2) \cap \Gamma_+ = \emptyset$, $\text{WF}'_h(B_j) \subset O'$ for $j = 0, 1, 2$ such that

$$\|B_0Q_+u\| + \|B_0Q_-u\| \leq h^{-1}\|B_1Pu\|_{\mathcal{H}_{h,\Gamma}^*} + \|B_2u\|_{L^2} + C'h^{1/2}\|u\|_{L^2}, \quad (\text{A.6})$$

respectively

$$\|B_0u\|_{\mathcal{H}_{h,\Gamma}} \leq h^{-1}\|B_1Pu\|_{\mathcal{H}_{h,\Gamma}^*} + \|B_2u\|_{L^2} + C'h\|u\|_{L^2}; \quad (\text{A.7})$$

see (A.13). The theorem is proved by controlling the B_2u term using the backward non-trapped nature of $\Gamma_- \setminus \Gamma$.

Proof. We first prove (A.4), which proves (A.5) by (A.2). Then we modify the proof slightly to show (A.5) directly, and in particular prove a weaker version of the Wunsch-Zworski estimate (A.2), namely

$$\|u\|_{L^2} \leq Ch^{-2}\|Pu\|_{L^2}.$$

Let $\chi_0(t) = e^{-1/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, $\chi \in C_c^\infty([0, \infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi'\chi = -\chi_1^2$, $\chi_1 \geq 0$, $\chi_1 \in C_c^\infty([0, \infty))$, and let $\psi \in C_c^\infty(\mathbb{R})$ be identically 1 near 0. Let

$$a = \chi_0(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)\psi(p),$$

$\kappa > 0$ small. Notice that on $\text{supp } a$, if χ is supported in $[0, R]$,

$$\phi_+^2 \leq R, \quad \phi_-^2 \leq \phi_+^2 + \kappa = R + \kappa,$$

so a is localized near Γ if R and κ are taken sufficiently small. Then

$$\begin{aligned} \frac{1}{4}\mathbf{H}_p(a^2) &= -(c_+^2\phi_+^2 + c_-^2\phi_-^2)(\chi_0\chi_0')(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)^2\psi(p)^2 \\ &\quad - c_+^2\phi_+^2(\chi'\chi)(\phi_+^2)\chi_0(\phi_+^2 - \phi_-^2 + \kappa)^2\psi(p)^2. \end{aligned}$$

⁶⁸An inspection of this argument gives that $\text{Im } z = \mathcal{O}(h^2)$ suffices.

Now $\chi'_0 \geq 0$, so the two terms have opposite signs. Let⁶⁹

$$a_{\pm} = \phi_{\pm} \sqrt{(\chi_0 \chi'_0)(\phi_+^2 - \phi_-^2 + \kappa)} \chi(\phi_+^2) \psi(p),$$

and

$$e_- = c_+ \phi_+ \chi_1(\phi_+^2) \chi_0(\phi_+^2 - \phi_-^2 + \kappa) \psi(p);$$

then

$$\frac{1}{4} \mathbb{H}_p(a^2) = -c_+^2 a_+^2 - c_-^2 a_-^2 + e_-^2. \quad (\text{A.8})$$

Here

$$\begin{aligned} \text{supp } e_- &\subset \text{supp } a, \\ \text{supp } e_- \cap \Gamma_+ &= \emptyset, \end{aligned}$$

with the last statement following from ϕ_+^2 taking values away from 0 on $\text{supp } \chi_1$; see Figure 7.

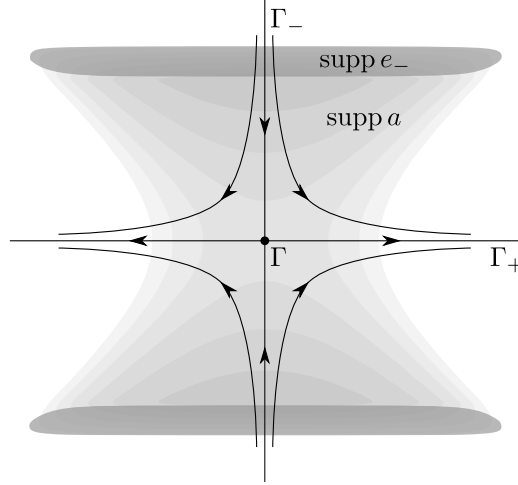


FIGURE 7. Supports of the commutant a and the error term e_- in the positive commutator argument of the non-trapping estimate near the trapped set Γ , Theorem A.1. The support of a is indicated in light gray; on $\text{supp } a \setminus \text{supp } e_-$, darker colors correspond to larger values of a . Also shown are the forward, resp. backward, trapped set Γ_- , resp. Γ_+ , and the bicharacteristic flow nearby. The figure already suggests that $\mathbb{H}_p(a^2)$ is non-positive away from $\text{supp } e_-$, and actually negative away from $\text{supp } e_- \cup \Gamma$; see equation (A.8).

One then takes $A \in \Psi_{\hbar}(X)$ with principal symbol a , and with $\text{WF}'_{\hbar}(A) \subset \text{supp } a$, $A_{\pm} \in \Psi_{\hbar}(X)$ with principal symbols of a_{\pm} , and with $\text{WF}'_{\hbar}(A_{\pm}) \subset \text{supp } a_{\pm}$, C_{\pm}

⁶⁹If ϕ_{\pm} is not defined globally, a_{\pm} are not defined as stated. (The term e_-^2 need not have a sign, so this issue does not arise for it; see the Weyl quantization argument below.) However, a_{\pm} need not be real below, so as long as one can choose ψ_{\pm} complex valued with $|\psi_{\pm}|^2 = \phi_{\pm}^2$, replacing the first factor of ϕ_{\pm} with ψ_{\pm} in the definition of a_{\pm} allows one to complete the argument in general.

have symbol c_{\pm} and with $\text{WF}'_{\hbar}(C_{\pm}) \subset \text{supp } c_{\pm}$; one similarly lets $E_{-} \in \Psi_{\hbar}(X)$ have principal symbol e_{-} , and wave front set in the support of e_{-} . This gives that

$$\frac{i}{4\hbar}[P, A^*A] = -(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) + E_-^*E_- + hF, \quad (\text{A.9})$$

for some $F \in \Psi_{\hbar}(X)$ with

$$\text{WF}'_{\hbar}(F) \subset \text{supp } a.$$

Thus

$$\frac{i}{4\hbar}\langle [P, A^*A]u, u \rangle = -\|C_+A_+u\|^2 - \|C_-A_-u\|^2 + \|E_-u\|^2 + h\langle Fu, u \rangle.$$

Expanding the left hand side gives

$$\begin{aligned} & \langle PA^*Au, u \rangle - \langle A^*APu, u \rangle \\ &= \langle Au, APu \rangle - \langle APu, Au \rangle + \langle (P - P^*)A^*Au, u \rangle. \end{aligned}$$

As we are assuming that $P - P^*$ is $\mathcal{O}(h^2)$ near Γ , we may also assume that this holds on $\text{supp } a$, thus the last term is $\mathcal{O}(h^2)\|u\|^2$. Thus,

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq \|E_-u\|^2 + h^{-1}|\langle APu, Au \rangle| + C_1h\|u\|^2. \quad (\text{A.10})$$

Now, by the duality of $\mathcal{H}_{\hbar, \Gamma}$ and $\mathcal{H}_{\hbar, \Gamma}^*$ relative to the L^2 inner product,

$$|\langle APu, Au \rangle| \leq \|APu\|_{\mathcal{H}_{\hbar, \Gamma}^*} \|Au\|_{\mathcal{H}_{\hbar, \Gamma}} \leq \frac{h\epsilon}{2} \|Au\|_{\mathcal{H}_{\hbar, \Gamma}}^2 + \frac{1}{2h\epsilon} \|APu\|_{\mathcal{H}_{\hbar, \Gamma}^*}^2.$$

Further, $\epsilon > 0$ small, $\epsilon\|Q_+Au\|^2$ can be estimated in terms of $\|C_+A_+u\|^2 + \mathcal{O}(h)\|u\|^2$, as can be seen by comparing the principal symbols, in particular using the ellipticity of C_+ on $\text{supp } a$. One can thus absorb $\frac{\epsilon}{2}\|Au\|_{\mathcal{H}_{\hbar, \Gamma}}^2$ into the left hand side of (A.10). This shows

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq C\|E_-u\|^2 + Ch^{-2}\|APu\|_{\mathcal{H}_{\hbar, \Gamma}^*}^2 + Ch\|u\|^2,$$

which together with the non-trapping control of E_-u (the region $\text{supp } e_-$ is disjoint from Γ_+ , so it is backward non-trapped and thus E_-u is controlled by Pu microlocalized off Γ_+ , thus by Q_+Pu , modulo higher order in h terms in Pu) proves the first part of Theorem A.1. Thus, if we have a bound $\|u\| \leq C'h^{-1-s}\|Pu\|_{L^2}$, $0 < s < 1/2$, and thus $h\|u\|^2 \leq C'h^{-1-2s}\|Pu\|_{L^2}^2 \leq C''h^{-1-2s}\|Pu\|_{\mathcal{H}_{\hbar, \Gamma}^*}^2$, this implies a non-trapping estimate:

$$\|u\|_{\mathcal{H}_{\hbar, \Gamma}} \leq Ch^{-1}\|Pu\|_{\mathcal{H}_{\hbar, \Gamma}^*}.$$

This completes the proof of Theorem A.1.

In fact, as mentioned earlier, a slight change of point of view proves Theorem A.1 directly. To see this, we use the Weyl quantization⁷⁰ when choosing $a, a_{\pm}, c_{\pm}, e_{-}$; since we are on a manifold, this requires identifying functions with half-densities via trivialization of the half-density bundle by the Riemannian metric; this identification preserves self-adjointness. We also write $P_{\hbar, z}$ as the Weyl quantization

⁷⁰In fact, the Weyl quantization is irrelevant. It is straightforward to see that if $A \in \Psi_{\hbar}(X)$ and if the principal symbol of A is real then the real part of the subprincipal symbol is defined independently of choices. This is all that is needed for the argument below.

of $p_0 + hp_1$ with p_0, p_1 real modulo $\mathcal{O}(h^2)$. Then the principal symbol calculation above holds with p_0 in place of p , and with p_1 included it yields additional terms

$$\begin{aligned} \frac{1}{4}\mathbf{H}_p(a^2) &= -(c_+^2\phi_+^2 + c_-^2\phi_-^2 - h\phi_+\mathbf{H}_{p_1}\phi_+ + h\phi_-\mathbf{H}_{p_1}\phi_-) \\ &\quad \times (\chi_0\chi'_0)(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)^2\psi(p)^2 \\ &\quad - (c_+^2\phi_+^2 - h\phi_+\mathbf{H}_{p_1}\phi_+)(\chi'\chi)(\phi_+^2)\chi_0(\phi_+^2 - \phi_-^2 + \kappa)^2\psi(p)^2. \end{aligned}$$

Now, (A.9) becomes

$$\begin{aligned} \frac{i}{4h}[P, A^*A] &= -(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) \\ &\quad + h(A_+^*G_+ + G_+^*A_+ + A_-^*G_- + G_-^*A_-) + E + h^2F, \end{aligned} \quad (\text{A.11})$$

with G_\pm being the Weyl quantization of

$$g_\pm = \pm \frac{1}{2}(\mathbf{H}_{p_1}\phi_\pm)\sqrt{(\chi_0\chi'_0)(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_\pm^2)}\psi(p),$$

and with $F \in \Psi_h(X)$ with

$$\text{WF}'_h(F) \subset \text{supp } a.$$

Correspondingly, (A.10) becomes

$$\begin{aligned} \|C_+A_+u\|^2 + \|C_-A_-u\|^2 &\leq |\langle Eu, u \rangle| + h^{-1}|\langle APu, Au \rangle| \\ &\quad + 2h\|A_+u\|\|G_+u\| + 2h\|A_-u\|\|G_-u\| + C_1h^2\|u\|^2. \end{aligned} \quad (\text{A.12})$$

The terms with G_\pm on the right hand side can be estimated by

$$\epsilon\|A_+u\|^2 + \epsilon^{-1}h^2\|G_+u\|^2 + \epsilon\|A_-u\|^2 + \epsilon^{-1}h^2\|G_-u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_\pm u\|^2$ terms can now be absorbed into the left hand side of (A.12). Proceeding as above yields

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq C|\langle Eu, u \rangle| + Ch^{-2}\|APu\|_{\mathcal{H}_{h,\Gamma}^*}^2 + Ch^2\|u\|^2. \quad (\text{A.13})$$

Together with the non-trapping for the E term this gives the global estimate

$$\|u\|_{\mathcal{H}_{h,\Gamma}^*}^2 \leq Ch^{-2}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*}^2 + Ch^2\|u\|^2,$$

and now the last term on the right hand side can be absorbed in the left hand side for sufficiently small h , giving the estimate (A.5). \square

We now transfer this result into the b-setting; the discussion up to now is essentially the dilation invariant special case of this, though in the b-setting there is additional localization near the boundary X of M .

With the notation of Section 2.1, suppose that $\mathcal{P} \in \Psi_b^m(M)$ with real principal symbol p_0 , regarded as a homogeneous function on ${}^bT^*M \setminus o$, and $\mathcal{P} - \mathcal{P}^* \in \Psi_b^{m-2}(M)$; let $\Sigma \subset {}^bS^*M$ be the characteristic set of \mathcal{P} as before. We first work microlocally near what becomes the trapped set in the application, namely assume that

- (1) $\Gamma \subset \Sigma \cap {}^bS_X^*M$ is a smooth submanifold disjoint from the image of $T^*X \setminus o$ (so τD_τ is elliptic near Γ),
- (2) Γ_+ is a smooth submanifold of $\Sigma \cap {}^bS_X^*M$ in a neighborhood U_1 of Γ ,
- (3) Γ_- is a smooth submanifold of Σ transversal to ${}^bS_X^*M \cap \Sigma$ in U_1 ,
- (4) Γ_+ has codimension 2 in Σ , Γ_- has codimension 1,
- (5) Γ_+ and Γ_- intersect transversally in Σ with $\Gamma_+ \cap \Gamma_- = \Gamma$,

- (6) the rescaled Hamilton vector field $V = \tilde{\rho}^{m-1}H_{p_0}$ is tangent to both Γ_+ and Γ_- , and thus to Γ .

We assume that Γ_+ is backward trapped by Γ for the Hamilton flow (i.e. bicharacteristics in Γ_+ near Γ tend to Γ as the parameter goes to $-\infty$), i.e. is the unstable manifold of Γ , while Γ_- is forward trapped, i.e. is the stable manifold of Γ ; indeed, we assume a quantitative version of this. (There is a completely analogous statement if Γ_+ is forward trapped and Γ_- is backward trapped: replacing \mathcal{P} by $-\mathcal{P}$ preserves all assumptions, but reverses the Hamilton flow.) To state this, let ϕ_- be a defining function of Γ_- , and let $\phi_+ \in C^\infty({}^bS^*M)$ be a defining function of Γ_+ in ${}^bS^*_X M$; thus Γ_+ is defined within ${}^bS^*M$ by $\tau = 0, \phi_+ = 0$. Notice that V being tangent to ${}^bS^*_X M$ (due to (2.1)) implies that $V\tau$ is a multiple of τ ; we assume that, near Γ ,

$$V\tau = -c_\partial^2\tau, \quad c_\partial > 0; \quad (\text{A.14})$$

this is consistent with the stability of Γ_- . By the tangency requirement, with

$$\hat{p}_0 = \tilde{\rho}^m p_0,$$

$V\phi_- = \alpha_- \phi_- + \nu_- \hat{p}_0$, α_- smooth; notice that changing ϕ_- by a smooth non-zero multiple f gives $V(f\phi_-) = \alpha_- f\phi_- + \nu_- f\hat{p}_0 + (Vf)\phi_-$, so α_- depends on the choice of ϕ_- . On the other hand, the tangency requirement gives $V\phi_+ = \alpha_+ \phi_+ + \beta_+ \tau + \nu_+ \hat{p}_0$. For the sake of conciseness, rather than stating the assumptions on the Hamilton flow as in [47], we assume directly that ϕ_\pm satisfy

$$V\phi_- = c_-^2 \phi_- + \nu_- \hat{p}_0, \quad V\phi_+ = -c_+^2 \phi_+ + \beta_+ \tau + \nu_+ \hat{p}_0, \quad (\text{A.15})$$

with $c_\pm > 0$ smooth near Γ , β_+, ν_\pm smooth near Γ and

$$\{\phi_+, \phi_-\} > 0 \quad (\text{A.16})$$

near Γ . However, if we merely assume the normal hyperbolicity within ${}^bS^*_X M$ as in [47, Section 1.2], [47, Lemma 4.1], as corrected in [46], actually gives such defining functions ϕ_\pm^0 *within* ${}^bS^*_X M$ (i.e. letting $\tau = 0$); taking an arbitrary extension in case of ϕ_+ , and an extension which is a defining function in case of Γ_- , all the requirements above are satisfied. Let $U_0 \subset \bar{U}_0 \subset U_1$ be a neighborhood of Γ such that the Poisson bracket in (A.16) as well as c_\pm have positive lower bounds.

There is an asymmetry between the roles of ϕ_\pm and τ , and thus we consider the parabolic defining function

$$\rho_+ = \phi_+^2 + M\tau$$

for Γ_+ , $M > 0$, to be chosen. Then, near Γ ,

$$\begin{aligned} \hat{\rho}_+ &= V\rho_+ = -2c_+^2 \phi_+^2 + 2\beta_+ \phi_+ \tau + 2\nu_+ \phi_+ \hat{p}_0 - M c_\partial^2 \tau \\ &= -2c_+^2 \phi_+^2 - (M c_\partial^2 - 2\beta_+ \phi_+) \tau + 2\nu_+ \phi_+ \hat{p}_0 \\ &\leq -\tilde{c}_+^2 \rho_+ + 2\nu_+ \phi_+ \hat{p}_0, \quad \tilde{c}_+ > 0, \end{aligned} \quad (\text{A.17})$$

if $M > 0$ is chosen sufficiently large, consistently with the forward trapped nature of Γ_- . (Here the term with \hat{p}_0 is considered harmless as one essentially restricts to the characteristic set, $\hat{p}_0 = 0$.) Also, note that one can use⁷¹ the reciprocal $\tilde{\rho} = |\sigma|^{-1}$ of the principal symbol σ of τD_τ as the local defining function of ${}^bS^*M$ as fiber-infinity in ${}^bT^*M$ near Γ ; then

$$V\tilde{\rho} = \tilde{\alpha}\tilde{\rho}\tau \quad (\text{A.18})$$

⁷¹Indeed, in the semiclassical setting, after Mellin transforming this problem, $|\sigma|^{-1}$ plays the role of the semiclassical parameter h , which in that case *commutes* with the operator.

for some $\tilde{\alpha}$ smooth in view of (2.1).

We recall the definition of the b-normally isotropic spaces from Section 3.1. Let $Q_{\pm} \in \Psi_b^0(M)$ have principal symbol ϕ_{\pm} as before, $\hat{P}_0 \in \Psi_b^0(M)$ have principal symbol \hat{p}_0 and let $Q_0 \in \Psi_b^0(M)$ be elliptic, with real principal symbol for convenience, on U_0^c (and thus nearby). Define the (global) b-normally isotropic spaces at Γ of order s , $\mathcal{H}_{b,\Gamma}^s$, by the norm

$$\|u\|_{\mathcal{H}_{b,\Gamma}^s}^2 = \|Q_0 u\|_{H_b^s}^2 + \|Q_+ u\|_{H_b^s}^2 + \|Q_- u\|_{H_b^s}^2 + \|\tau^{1/2} u\|_{H_b^s}^2 + \|\hat{P}_0 u\|_{H_b^s}^2 + \|u\|_{H_b^{s-1/2}}^2, \quad (\text{A.19})$$

and let $\mathcal{H}_{b,\Gamma}^{*, -s}$ be the dual space relative to L^2 which is thus

$$Q_0 H_b^{-s} + Q_+ H_b^{-s} + Q_- H_b^{-s} + \tau^{1/2} H_b^{-s} + \hat{P}_0 H_b^{-s} + H_b^{-s+1/2}.$$

Recall that microlocally away from Γ , $\mathcal{H}_{b,\Gamma}^s$ is just the standard H_b^s space while $\mathcal{H}_{b,\Gamma}^{*, -s}$ is H_b^{-s} since at least one of Q_0, Q_{\pm}, τ is elliptic, $\Psi_b^k(M) \ni A : \mathcal{H}_{b,\Gamma}^s \rightarrow \mathcal{H}_{b,\Gamma}^{s-k}$ is continuous since $[Q_+, A] \in \Psi_b^{k-1}(M)$ etc.; the analogous statement also holds for the dual spaces. Further, the last term in (A.19) can be replaced by $\|u\|_{H_b^{s-1}}^2$ as $i[Q_+, Q_-] = B^* B + R$, $B \in \Psi_b^{-1/2}(M)$, $R \in \Psi_b^{-2}(M)$, using the same argument as in the semiclassical setting (however, it cannot be dropped altogether unlike in the semiclassical setting!). Our result is then:

Proposition A.2. *With $\mathcal{P}, \mathcal{H}_{b,\Gamma}^s, \mathcal{H}_{b,\Gamma}^{*, s}$ as above, for any neighborhood U of Γ and for any N there exist $B_0 \in \Psi_b^0(M)$ elliptic at Γ and $B_1, B_2 \in \Psi_b^0(M)$ with $\text{WF}'_b(B_j) \subset U$, $j = 0, 1, 2$, $\text{WF}'_b(B_2) \cap \Gamma_+ = \emptyset$ and $C > 0$ such that*

$$\|B_0 u\|_{\mathcal{H}_{b,\Gamma}^s} \leq \|B_1 \mathcal{P} u\|_{\mathcal{H}_{b,\Gamma}^{*, s-m+1}} + \|B_2 u\|_{H_b^s} + C \|u\|_{H_b^{-N}}, \quad (\text{A.20})$$

i.e. if all the functions on the right hand side are in the indicated spaces: $B_1 \mathcal{P} u \in \mathcal{H}_{b,\Gamma}^{, s-m+1}$, etc., then $B_0 u \in \mathcal{H}_{b,\Gamma}^s$, and the inequality holds.*

The same conclusion also holds if we assume $\text{WF}'_b(B_2) \cap \Gamma_- = \emptyset$ instead of $\text{WF}'_b(B_2) \cap \Gamma_+ = \emptyset$.

Finally, if $r < 0$, then, with $\text{WF}'_b(B_2) \cap \Gamma_+ = \emptyset$, (A.20) becomes

$$\|B_0 u\|_{H_b^{s,r}} \leq \|B_1 \mathcal{P} u\|_{H_b^{s-m+1,r}} + \|B_2 u\|_{H_b^{s,r}} + C \|u\|_{H_b^{-N,r}}, \quad (\text{A.21})$$

while if $r > 0$, then, with $\text{WF}'_b(B_2) \cap \Gamma_- = \emptyset$,

$$\|B_0 u\|_{H_b^{s,r}} \leq \|B_1 \mathcal{P} u\|_{H_b^{s-m+1,r}} + \|B_2 u\|_{H_b^{s,r}} + C \|u\|_{H_b^{-N,r}}, \quad (\text{A.22})$$

Remark A.3. Note that the weighted versions (A.21)-(A.22) use *standard* weighted b-Sobolev spaces; this corresponds to non-trapping semiclassical estimates if the subprincipal symbol has the correct, definite, sign at Γ .

Proof. We may assume that $U \subset U_0$ is disjoint from a neighborhood of $\text{WF}'_b(Q_0)$, and thus ignore Q_0 in the definition of $\mathcal{H}_{b,\Gamma}^s$ below.

We first prove that there exist B_0, B_1, B_2 as above and $B_3 \in \Psi_b^0(M)$ with $\text{WF}'_b(B_3) \subset U$ such that

$$\|B_0 u\|_{\mathcal{H}_{b,\Gamma}^s} \leq \|B_1 \mathcal{P} u\|_{\mathcal{H}_{b,\Gamma}^{*, s-m+1}} + \|B_2 u\|_{H_b^s} + \|B_3 u\|_{H_b^{s-1}} + C \|u\|_{H_b^{-N}}. \quad (\text{A.23})$$

An iterative argument will then prove the proposition.

The proof is a straightforward modification of the construction in the semiclassical setting above, replacing ϕ_+^2 by $\phi_+^2 + M\tau$, $M > 0$ large, in accordance with (A.17).

We start by pointing out that for any $\tilde{B}_0 \in \Psi_b^0(M)$ and any $\tilde{B}_3 \in \Psi_b^0(M)$ elliptic on $\text{WF}'_b(\tilde{B}_0)$, $\|\hat{P}_0 \tilde{B}_0 u\|_{H_b^s} \leq C \|\tilde{B}_0 \mathcal{P}u\|_{H_b^{s-m}} + C' \|\tilde{B}_3 u\|_{H_b^{s-1}}$, by simply using that \hat{P}_0 is an elliptic multiple of P modulo $\Psi_b^{-1}(M)$. Since $\|\tilde{B}_0 \mathcal{P}u\|_{H_b^{s-m}} \leq C \|\tilde{B}_0 \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s-m}}$, the \hat{P}_0 contribution to $\|\tilde{B}_0 u\|_{\mathcal{H}_{b,\Gamma}^s}$ in (A.23) is thus automatically controlled.

So let $\chi_0(t) = e^{-F/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, with $F > 0$ (large) to be specified, $\chi \in C_c^\infty([0, \infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi' \chi = -\chi_1^2$, $\chi_1 \geq 0$, $\chi_1 \in C_c^\infty([0, \infty))$, and let $\psi \in C_c^\infty(\mathbb{R})$ be identically 1 near 0. As we use the Weyl quantization,⁷² we write \mathcal{P} as the Weyl quantization of $p = p_0 + \tilde{\rho}p_1$, with p_1 order $m-1$. Let

$$a = \tilde{\rho}^{-s+(m-1)/2} \chi_0(\rho_+ - \phi_-^2 + \kappa) \chi(\rho_+) \psi(\tilde{\rho}^m p), \quad (\text{A.24})$$

$\kappa > 0$ small. Notice that on $\text{supp } a$, if χ is supported in $[0, R]$,

$$\rho_+ \leq R, \quad \phi_-^2 \leq \rho_+ + \kappa = R + \kappa,$$

so a is localized near Γ if R and κ are taken sufficiently small. In particular, the argument of χ_0 is bounded above by $R + \kappa$, so given any $M_0 > 0$ one can take $F > 0$ large so that

$$\chi'_0 \chi_0 - M_0 \chi_0^2 = b^2 \chi'_0 \chi_0,$$

with $b \geq 1/2$, C^∞ , on the range of the argument of χ_0 .

In fact, we also need to regularize, namely introduce

$$a_\epsilon = (1 + \epsilon \tilde{\rho}^{-1})^{-2} a, \quad \epsilon \in [0, 1], \quad (\text{A.25})$$

which is a symbol of order $s - (m-1)/2 - 2$ for $\epsilon > 0$, and is uniformly bounded in symbols of order $s - (m-1)/2$ as ϵ varies in $[0, 1]$. In order to avoid more cumbersome notation below, we ignore the regularizer and work directly with a ; since the regularizer gives the same kind of contributions to the commutator as the weight $\tilde{\rho}^{-s+(m-1)/2}$, these contributions can be dominated in exactly the same way.

Then, with $p = p_0 + \tilde{\rho}p_1$ as above, $W = \tilde{\rho}^{m-2} \mathbf{H}_{\tilde{\rho}p_1}$, which is a smooth vector field near ${}^b S^*M$ as $\tilde{\rho}p_1$ is order $m-1$, noting $W\tilde{\rho} = \tilde{\alpha}_1 \tau \tilde{\rho}$ similarly to (A.18), and $W\tau = \alpha_{\partial,1} \tau$ by the tangency of W to $\tau = 0$,

$$\begin{aligned} \frac{1}{4} \mathbf{H}_p(a^2) &= -(-\hat{\rho}_+/2 + c_-^2 \phi_-^2 + \nu_- \phi_- \hat{p}_0 - \tilde{\rho} \phi_+(W\phi_+) - \tilde{\rho} M \alpha_{\partial,1} \tau + \tilde{\rho} \phi_-(W\phi_-)) \\ &\quad \times \tilde{\rho}^{-2s} (\chi_0 \chi'_0)(\rho_+ - \phi_-^2 + \kappa) \chi(\rho_+)^2 \psi(\tilde{\rho}^m p)^2 \\ &\quad + \frac{1}{4} (-2s + m - 1) \tilde{\rho}^{-2s} (\tilde{\alpha} + \tilde{\rho} \tilde{\alpha}_1) \tau \chi_0(\rho_+ - \phi_-^2 + \kappa)^2 \chi(\rho_+)^2 \psi(\tilde{\rho}^m p)^2 \\ &\quad + \frac{1}{2} \tilde{\rho}^{-2s} (\hat{\rho}_+ + \tilde{\rho} W \rho_+) (\chi' \chi)(\rho_+) \chi_0(\rho_+ - \phi_-^2 + \kappa)^2 \psi(\tilde{\rho}^m p)^2 \\ &\quad + \frac{m}{2} (\tilde{\alpha} + \tilde{\rho} \tilde{\alpha}_1) \tilde{\rho}^{-2s} (\tilde{\rho}^m p) \tau \chi_0(\rho_+ - \phi_-^2 + \kappa)^2 \chi(\rho_+)^2 (\psi \psi')(\tilde{\rho}^m p). \end{aligned} \quad (\text{A.26})$$

A key point is that the second term on the right hand side, given by the weight $\tilde{\rho}^{-2s+m-1}$ being differentiated, can be absorbed into the first by making $F > 0$

⁷²Again, the Weyl quantization is irrelevant: if $A \in \Psi_b^m(X)$ and the principal symbol of A is real then the real part of the subprincipal symbol is defined independently of choices, which suffices below.

large so that $\hat{\rho}_+\chi'_0(\rho_+ - \phi_-^2 + \kappa)$ dominates

$$|-2s + m - 1|\tilde{\alpha}|\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)$$

on $\text{supp } a$, which can be arranged as $|-2s + m - 1|\tilde{\alpha}|\tau$ is bounded by a sufficiently large multiple of $\hat{\rho}_+$ there. Thus,

$$\frac{1}{4}\mathbf{H}_p(a^2) = -c_+^2a_+^2 - c_-^2a_-^2 - a_\partial^2 + 2g_+a_+ + 2g_-a_- + e + \tilde{e} + 2a_+j_+p + 2a_-j_-p \quad (\text{A.27})$$

with

$$\begin{aligned} a_\pm &= \tilde{\rho}^{-s}\phi_\pm\sqrt{(\chi_0\chi'_0)(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde{\rho}^mp), \\ a_\partial &= \tilde{\rho}^{-s}\tau^{1/2}\left((\mathbf{M}(c_\partial^2/2) - \beta_+\phi_+)(\chi_0\chi'_0)(\rho_+ - \phi_-^2 + \kappa) \right. \\ &\quad \left. - \frac{1}{4}(-2s + m - 1)(\tilde{\alpha} + \tilde{\rho}\tilde{\alpha}_1)\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\right)^{1/2}\chi(\rho_+)\psi(\tilde{\rho}^mp), \\ g_\pm &= \pm\frac{1}{2}\tilde{\rho}^{-s+1}((W\phi_\pm) - \nu_\pm\tilde{\rho}^{m-1}p_1)\sqrt{(\chi_0\chi'_0)(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde{\rho}^mp), \\ e &= -\frac{1}{2}\tilde{\rho}^{-2s}\hat{\rho}_+\chi_1(\rho_+)^2\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\psi(\tilde{\rho}^mp)^2, \\ \tilde{e} &= \frac{m}{2}\tilde{\rho}^{-2s}(\tilde{\rho}^mp)(\tilde{\alpha} + \tilde{\rho}\tilde{\alpha}_1)\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2(\psi\psi')(\tilde{\rho}^mp), \\ j_\pm &= \pm\frac{1}{2}\nu_\pm\tilde{\rho}^{-s+m}\sqrt{(\chi_0\chi'_0)(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde{\rho}^mp), \end{aligned}$$

the square root in a_∂ is that of a non-negative quantity and is C^∞ for \mathbf{M} large (so that $\beta_+\phi_+$ can be absorbed into $\mathbf{M}(c_\partial^2/2)$) and F large (so that a small multiple of χ'_0 can be used to dominate χ_0), as discussed earlier, and

$$\begin{aligned} \text{supp } e &\subset \text{supp } a, \quad \text{supp } e \cap \Gamma_+ = \emptyset, \\ \text{supp } \tilde{e} &\subset \text{supp } a, \quad \text{supp } \tilde{e} \cap \Sigma = \emptyset. \end{aligned}$$

This gives, with the various operators being Weyl quantizations of the corresponding lower case symbols,

$$\begin{aligned} \frac{i}{4}[\mathcal{P}, A^*A] &= -(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) - A_\partial^*A_\partial \\ &\quad + G_+^*A_+ + A_+^*G_+ + G_-^*A_- + A_-^*G_- \\ &\quad + E + \tilde{E} + A_+^*J_+\mathcal{P} + \mathcal{P}^*J_+^*A_+ + A_-^*J_-\mathcal{P} + \mathcal{P}^*J_-^*A_- + F \end{aligned} \quad (\text{A.28})$$

where now $A \in \Psi_b^{s-(m-1)/2}(M)$, $A_\pm, A_\partial \in \Psi_b^s(M)$, $G_\pm \in \Psi_b^{s-1}(M)$, $E \in \Psi_b^{2s}(M)$, $\tilde{E} \in \Psi_b^{2s}(M)$, $J_\pm \in \Psi_b^{s-m}(M)$, $F \in \Psi_b^{2s-2}(M)$ with $\text{WF}'_b(F) \subset \text{supp } a$.

After this point the calculations repeat the semiclassical argument: First using $\mathcal{P} - \mathcal{P}^* \in \Psi_b^{m-2}(M)$,

$$\begin{aligned} &\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2 \\ &\leq |\langle Eu, u \rangle| + |\langle A\mathcal{P}u, Au \rangle| + 2\|A_+u\|\|G_+u\| + 2\|A_-u\|\|G_-u\| \\ &\quad + 2|\langle J_+\mathcal{P}u, A_+u \rangle| + 2|\langle J_-\mathcal{P}u, A_-u \rangle| + C_1\|\tilde{F}_1u\|_{H_b^{s-1}}^2 + C_1\|u\|_{H_b^{-N}}^2, \end{aligned} \quad (\text{A.29})$$

where we took $\tilde{F}_1 \in \Psi_b^0(M)$ elliptic on $\text{WF}'_b(F)$ and with $\text{WF}'_b(\tilde{F}_1)$ near Γ . Noting that $\text{WF}'_b(\tilde{E}) \cap \Sigma = \emptyset$, the elliptic estimates give

$$|\langle \tilde{E}u, u \rangle| \leq C \|B_1 \mathcal{P}u\|_{H_b^{s-m}}^2 + C \|u\|_{H_b^{-N}}^2$$

if $B_1 \in \Psi_b^0(M)$ is elliptic on $\text{supp } \tilde{e}$. Let $\Lambda \in \Psi_b^{(m-1)/2}(M)$ be elliptic with real principal symbol λ , and let $\Lambda^- \in \Psi_b^{-(m-1)/2}(M)$ be a parametrrix for it so that $\Lambda \Lambda^- - \text{Id} = R_0 \in \Psi_b^{-\infty}(M)$. Then

$$\begin{aligned} |\langle A \mathcal{P}u, Au \rangle| &\leq |\langle \Lambda^- A \mathcal{P}u, \Lambda^* Au \rangle| + |\langle R_0 A \mathcal{P}u, Au \rangle| \\ &\leq \frac{1}{2\epsilon} \|\Lambda^- A \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s,0}}^2 + \frac{\epsilon}{2} \|\Lambda^* Au\|_{\mathcal{H}_{b,\Gamma}^0}^2 + C' \|u\|_{H_b^{-N}}^2 \end{aligned}$$

As $\Lambda^* A \in \Psi_b^s(M)$, for sufficiently small $\epsilon > 0$, $\frac{\epsilon}{2} \|\Lambda^* Au\|_{\mathcal{H}_{b,\Gamma}^0}^2$ can be absorbed into⁷³ $\|C_+ A_+ u\|^2 + \|C_- A_- u\|^2 + \|A_\partial u\|^2$ plus $\|\tilde{B}_0 \tilde{F}_0 u\|_{H_b^s}^2$, and as discussed above, the latter already has the control required for (A.23). On the other hand, taking $B_1 \in \Psi_b^0(M)$ elliptic on $\text{WF}'_b(A)$, as $\Lambda^- A \in \Psi_b^{s-m+1}(M)$,

$$\|\Lambda^- A \mathcal{P}u\|_{(\mathcal{H}_{b,\Gamma}^0)^*}^2 \leq C'' \|B_1 \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s,-m+1}}^2 + C'' \|u\|_{H_b^{-N}}^2.$$

Similarly, to deal with the J_\pm terms on the right hand side of (A.29), one writes

$$\begin{aligned} |\langle J_\pm \mathcal{P}u, A_\pm u \rangle| &\leq \frac{1}{2\epsilon} \left(\|B_1 \mathcal{P}u\|_{H_b^{s-m}}^2 + C'' \|u\|_{H_b^{-N}}^2 \right) + \frac{\epsilon}{2} \|A_\pm u\|_{L^2}^2 \\ &\leq \frac{1}{2\epsilon} \left(\|B_1 \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s,-m}}^2 + C'' \|u\|_{H_b^{-N}}^2 \right) + \frac{\epsilon}{2} \|A_\pm u\|_{L^2}^2, \end{aligned}$$

while the G_\pm terms can be estimated by

$$\epsilon \|A_+ u\|^2 + \epsilon^{-1} \|G_+ u\|^2 + \epsilon \|A_- u\|^2 + \epsilon^{-1} \|G_- u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_\pm u\|^2$ terms in both cases can be absorbed into the left hand side of (A.29) while the G_\pm into the error term. This gives, with \tilde{F}_2 having properties as \tilde{F}_1 ,

$$\begin{aligned} &\|C_+ A_+ u\|^2 + \|C_- A_- u\|^2 + \|A_\partial u\|^2 \\ &\leq |\langle Eu, u \rangle| + C \|B_1 \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s,-m+1}}^2 + C_2 \|\tilde{F}_2 u\|_{H_b^{s-1}}^2 + C_2 \|u\|_{H_b^{-N}}^2. \end{aligned}$$

By the remark before the statement of the proposition, if $B_0 \in \Psi_b^0(M)$ is such that $\chi_0(\rho_+ - \phi_+^2 + \kappa)\chi(\rho_+)\psi(p) > 0$ on $\text{WF}'_b(B_0)$, $\|B_0 u\|_{H_b^{s-1/2}}^2$ can be added to the left hand side at the cost of changing the constant in front of $\|\tilde{F}_2 u\|_{H_b^{s-1}}^2 + \|u\|_{H_b^{-N}}^2$ on the right hand side. Taking such $B_0 \in \Psi_b^0(M)$, and B_1 elliptic on $\text{WF}'_b(A)$ as before, $B_2 \in \Psi_b^0(M)$ elliptic on $\text{WF}'_b(E)$ but with $\text{WF}'_b(B_2)$ disjoint from Γ_+ , we conclude that

$$\|B_0 u\|_{\mathcal{H}_{b,\Gamma}^s}^2 \leq C \|B_1 \mathcal{P}u\|_{\mathcal{H}_{b,\Gamma}^{s,-m+1}}^2 + C \|B_2 u\|_{H_b^s}^2 + C \|\tilde{F}_2 u\|_{H_b^{s-1}}^2 + C \|u\|_{H_b^{-N}}^2,$$

⁷³The point being that $A_\pm^* C_\pm^* C_\pm A_\pm - \epsilon A^* \Lambda Q_\pm^* Q_\pm \Lambda^* A$ has principal symbol $c_\pm^2 a_\pm^2 - \epsilon a^2 \phi_\pm^2 \lambda^2$ which can be written as the square of a real symbol for $\epsilon > 0$ small in view of the main difference in vanishing factors in the two terms being that χ'_0 in a_\pm^2 is replaced by χ_0 in a , and thus the corresponding operator can be expressed as $\tilde{C}^* \tilde{C}$ for suitable \tilde{C} , modulo an element of $\Psi_b^{2s-2}(M)$, with the latter contributing to the H_b^{s-1} error term on the right hand side of (A.23).

proving (A.23), up to redefining B_j by multiplication by a positive constant. Recall that unless one makes sufficient a priori assumptions on the regularity of u , one actually needs to regularize, but as mentioned after (A.25), the regularizer is handled in exactly the same manner as the weight.

Now in general, with χ as before, but supported in $[0, 1]$ instead of $[0, R]$, writing $\chi_R = \chi(\cdot/R)$, letting $a = a_{R,\kappa}$ to emphasize its dependence on these quantities, when R and κ are decreased, $\text{supp } a_{R,\kappa}$ also decreases in Σ in the strong sense that $0 < R < R'$, $0 < \kappa < \kappa'$ implies that $a_{R',\kappa'}$ is elliptic on $\text{supp } a_{R,\kappa}$ within Σ , and indeed globally if the cutoff ψ is suitably adjusted as well. Thus, if $u \in H_b^{-N}$, say, one uses first (A.23) with $s = -N + 1$, and with B_j given by the proof above, so the B_3u term is a priori bounded, to conclude that $B_0u \in \mathcal{H}_{b,\Gamma}^s$ and the estimate holds, so in particular, u is in $H_b^{-N+1/2}$ microlocally near Γ (concretely, on the elliptic set of B_0). Now one decreases κ and R by an arbitrarily small amount and applies (A.23) with $s = -N + 3/2$; the B_3u term is now a priori bounded by the microlocal membership of u in $H_b^{-N+1/2}$, and one concludes that $B_0u \in \mathcal{H}_{b,\Gamma}^{-N+3/2}$, so in particular u is microlocally in H_b^{-N+1} . Proceeding inductively, one deduces the first statement of the proposition, (A.20).

If one reverses the role of Γ_+ and Γ_- in the statement of the proposition, one simply reverses the roles of $\rho_+ = \phi_+^2 + M\tau$ and ϕ_-^2 in the definition of a in (A.24). This reverses the signs of all terms on the right hand side of (A.26) whose sign mattered below, and thus the signs of the first three terms on the right hand side of (A.28), which then does not affect the rest of the argument.

In order to prove (A.21), one simply adds a factor τ^{-2r} to the definition of a in (A.24). This adds a factor τ^{-2r} to every term on the right hand side of (A.28), as well as an additional term

$$\frac{r}{2}\tau^{-2r}\tilde{\rho}^{-2s}c_{\partial}^2\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2\psi(p)^2,$$

which for $r < 0$ has the same sign as the terms whose sign was used above, and indeed can be written as the negative of a square. Thus (A.27) becomes

$$\begin{aligned} \frac{1}{4}\mathbb{H}_p(a^2) &= -c_+^2a_+^2 - c_-^2a_-^2 - a_{\partial}^2 - a_r^2 \\ &\quad + 2g_+a_+ + 2g_-a_- + e + \bar{e} + 2j_+a_+p + 2j_-a_-p \end{aligned} \quad (\text{A.30})$$

with

$$a_r = \sqrt{\frac{-r}{2}}\tau^{-r}\tilde{\rho}^{-s}c_{\partial}\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(p),$$

and all other terms as above apart from the additional factor of τ^{-r} in the definition of a_{\pm} , etc. Since a_r is actually elliptic at Γ when $r \neq 0$, this proves the desired estimate (and one does not need to use the improved properties given by the Weyl calculus!).

When the role of Γ_+ and Γ_- is reversed, there is an overall sign change, and thus $r > 0$ gives the advantageous sign; the rest of the argument is unchanged. \square

REFERENCES

- [1] D. Baskin. A Strichartz estimate for de Sitter space. *Annales Henri Poincaré*, *arxiv:0909.2696*, 14:221–252, 2013.
- [2] D. Baskin, A. Vasy, and J. Wunsch. Asymptotics of radiation fields in asymptotically Minkowski space. *Preprint*, *arxiv:1212.5141*, 2012.

- [3] Dean Baskin. A parametrix for the fundamental solution of the Klein-Gordon equation on asymptotically de Sitter spaces. *J. Funct. Anal.*, 259(7):1673–1719, 2010.
- [4] Lydia Bieri. Part I: Solutions of the Einstein vacuum equations. In *Extensions of the stability theorem of the Minkowski space in general relativity*, volume 45 of *AMS/IP Stud. Adv. Math.*, pages 1–295. Amer. Math. Soc., Providence, RI, 2009.
- [5] Lydia Bieri and Nina Zipser. *Extensions of the stability theorem of the Minkowski space in general relativity*, volume 45 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
- [6] Jean-François Bony and Dietrich Häfner. Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric. *Comm. Math. Phys.*, 282(3):697–719, 2008.
- [7] Demetrios Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.*, 39(2):267–282, 1986.
- [8] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
- [9] Piotr T. Chruściel and Szymon Łęski. Polyhomogeneous solutions of nonlinear wave equations without corner conditions. *J. Hyperbolic Differ. Equ.*, 3(1):81–141, 2006.
- [10] Mihalis Dafermos and Igor Rodnianski. The wave equation on Schwarzschild-de Sitter space times. *Preprint, arXiv:07092766*, 2007.
- [11] Mihalis Dafermos and Igor Rodnianski. *Lectures on black holes and linear waves*. CMI/AMS Publications, 2013.
- [12] K. Datchev and A. Vasy. Gluing semiclassical resolvent estimates via propagation of singularities. *Int. Math. Res. Notices*, 2012(23):5409–5443, 2012.
- [13] K. Datchev and A. Vasy. Propagation through trapped sets and semiclassical resolvent estimates. *Annales de l’Institut Fourier*, To appear. *arxiv:1010.2190*.
- [14] K. Datchev and A. Vasy. Semiclassical resolvent estimates at trapped sets. *Annales de l’Institut Fourier*, To appear. *arxiv:1206.0768*.
- [15] Semyon Dyatlov. Exponential energy decay for Kerr–de Sitter black holes beyond event horizons. *Math. Res. Lett.*, 18(5):1023–1035, 2011.
- [16] Semyon Dyatlov. Quasi-normal modes and exponential energy decay for the Kerr-de Sitter black hole. *Comm. Math. Phys.*, 306(1):119–163, 2011.
- [17] Semyon Dyatlov. Resonance projectors and asymptotics for r-normally hyperbolic trapped sets. *Preprint, arXiv:1301.5633*, 2013.
- [18] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.
- [19] Colin Guillarmou. Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds. *Duke Math. J.*, 129(1):1–37, 2005.
- [20] N. Haber and A. Vasy. Propagation of singularities around a Lagrangian submanifold of radial points. *Preprint, arxiv:1110.1419*, 2011.
- [21] L. Hörmander. *The analysis of linear partial differential operators*, vol. 1-4. Springer-Verlag, 1983.
- [22] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.
- [23] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Non-linear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.
- [24] Sergiu Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.
- [25] Hans Lindblad. Global solutions of quasilinear wave equations. *Amer. J. Math.*, 130(1):115–157, 2008.
- [26] Hans Lindblad and Igor Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 256(1):43–110, 2005.
- [27] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.

- [28] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.
- [29] R. B. Melrose. *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*. Marcel Dekker, 1994.
- [30] R. B. Melrose, A. Sá Barreto, and A. Vasy. Asymptotics of solutions of the wave equation on de Sitter-Schwarzschild space. *Preprint, arxiv:0811.2229*, 2008.
- [31] R. B. Melrose, A. Vasy, and J. Wunsch. Diffraction of singularities for the wave equation on manifolds with corners. *Astérisque*, to appear; arXiv:0903.3208.
- [32] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993.
- [33] Jason Metcalfe and Daniel Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. *Math. Ann.*, 353(4):1183–1237, 2012.
- [34] Stéphane Nonnenmacher and Maciej Zworski. Quantum decay rates in chaotic scattering. *Acta Math.*, 203(2):149–233, 2009.
- [35] D. Polarski. On the Hawking effect in de Sitter space. *Classical Quantum Gravity*, 6(5):717–722, 1989.
- [36] Antônio Sá Barreto and Maciej Zworski. Distribution of resonances for spherical black holes. *Math. Res. Lett.*, 4(1):103–121, 1997.
- [37] Robert S. Strichartz. A note on Sobolev algebras. *Proc. Amer. Math. Soc.*, 29:205–207, 1971.
- [38] Michael E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.
- [39] A. Vasy. *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates*, volume 60 of *MSRI Publications*. Cambridge University Press, 2012.
- [40] A. Vasy. The relationship of asymptotically hyperbolic and de Sitter analysis with a global problem. *Preprint, arXiv:1303.5653*, 2013.
- [41] A. Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces. *Inventiones Math.*, to appear; arxiv:1012.4391. With an appendix by S. Dyatlov.
- [42] András Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Ann. of Math. (2)*, 168(3):749–812, 2008.
- [43] András Vasy. The wave equation on asymptotically de Sitter-like spaces. *Adv. Math.*, 223(1):49–97, 2010.
- [44] András Vasy. The wave equation on asymptotically anti de Sitter spaces. *Anal. PDE*, 5(1):81–144, 2012.
- [45] F. Wang. *Radiation field for vacuum Einstein equation*. PhD thesis, Massachusetts Institute of Technology, 2010.
- [46] Jared Wunsch and Maciej Zworski. Erratum to ‘resolvent estimates for normally hyperbolic trapped sets’. Posted on http://www.math.northwestern.edu/~jwunsch/erratum_wz.pdf.
- [47] Jared Wunsch and Maciej Zworski. Resolvent estimates for normally hyperbolic trapped sets. *Ann. Henri Poincaré*, 12(7):1349–1385, 2011.
- [48] Karen Yagdjian. The semilinear Klein-Gordon equation in de Sitter spacetime. *Discrete Contin. Dyn. Syst. Ser. S*, 2(3):679–696, 2009.
- [49] Karen Yagdjian. Global existence of the scalar field in de Sitter spacetime. *J. Math. Anal. Appl.*, 396(1):323–344, 2012.
- [50] Karen Yagdjian and Anahit Galstian. Fundamental solutions for the Klein-Gordon equation in de Sitter spacetime. *Comm. Math. Phys.*, 285(1):293–344, 2009.

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