DIFFRACTION AT CORNERS FOR THE WAVE EQUATION ON DIFFERENTIAL FORMS

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Abstract. In this paper we prove the propagation of singularities for the wave equation on differential forms with natural (i.e. relative or absolute) boundary conditions on Lorentzian manifolds with corners, which in particular includes a formulation of Maxwell's equations. These results are analogous to those obtained by the author for the scalar wave equation [23] and for the wave equation on systems with Dirichlet or Neumann boundary conditions in [22]. The main novelty is thus the presence of natural boundary conditions, which effectively make the problem non-scalar, even 'to leading order', at corners of codimension ≥ 2.

1. Introduction

Let X be a C^∞ manifold with corners of dimension n (a notion we recall below), and suppose that h is a Lorentz metric on X of signature $(1, n-1)$ with dual metric H. Thus, for $p \in X$, H gives a non-degenerate symmetric bilinear form on $T_p^* X$, which however is not positive definite. Then H induces a symmetric bilinear form on the real form bundle as usual (again, not positive definite), and thus a Hermitian symmetric bilinear form on the complex form bundle which we write as $(.,.)_H$, or simply $(.,.)$. Moreover, h gives rise to a non-vanishing density $|dh|$ on X; non-vanishing is the consequence of nondegeneracy. In particular, for smooth forms, one of which has compact support, one has a pairing

$$\langle \alpha, \beta \rangle_H = \int_X (\alpha, \beta)_H \, |dh|.$$

We recall here that a *tied (or t-) manifold with corners* X of dimension n is a paracompact Hausdorff topological space with a C^∞ structure with corners, i.e. such that the local coordinate charts map into $[0,\infty)^k \times \mathbb{R}^{n-k}$ rather than into $\mathbb{R}^n$. Here k varies with the coordinate chart. We write $\partial_\ell X$ for the set of points $p \in X$ such that in any local coordinates $\phi = (\phi_1, \ldots, \phi_k, \phi_{k+1}, \ldots, \phi_n)$ near p, with k as above, precisely ℓ of the first k coordinate functions vanish at $\phi(p)$. We usually write such local coordinates as $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$. A *boundary face* F of codimension ℓ is the closure of a connected component of $\partial_\ell X$. We write F_{reg} for the interior of F; note that F is a manifold with corners, while F_{reg} is a manifold without boundary. A boundary face of codimension 1 is called a *boundary hypersurface*. A *manifold with corners* is a tied manifold with corners such that all

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boundary hypersurfaces are embedded submanifolds – as our results are local, this distinction is inessential.

Throughout this paper we assume that every proper boundary face F of X (i.e. all boundary faces but X itself) is time-like, in the sense that H restricted to the conormal bundle N^*F of F is negative definite.

For C^∞ differential forms on X, i.e. elements of $C^\infty(X;\Lambda X)$, natural boundary conditions at the boundary hypersurfaces $\mathcal{S} \in \partial_1(X)$ with non-vanishing conormal $\nu_{\mathcal{S}}$ are $\nu_{\mathcal{S}} \wedge u = 0$ (relative boundary condition) and $\iota_{\nu_{\mathcal{S}}} u = 0$ (absolute boundary condition), and we write

$$C^\infty_R(X;\Lambda X) = \{u \in C^\infty(X;\Lambda X) :\ \forall \mathcal{S} \in \partial_1(X),\ \nu_{\mathcal{S}} \wedge u|_{\mathcal{S}} = 0\},$$
$$C^\infty_A(X;\Lambda X) = \{u \in C^\infty(X;\Lambda X) :\ \forall \mathcal{S} \in \partial_1(X),\ \iota_{\nu_{\mathcal{S}}} u|_{\mathcal{S}} = 0\}.$$

For $s \geq 0$ integer we let $H^s(X;\Lambda X)$ be the completion of $C^\infty(X;\Lambda X)$ in the $H^s(X;\Lambda X)$ norm (defined up to equivalence of norms on compact sets). The restriction map $C^\infty(X;\Lambda X) \to C^\infty(\mathcal{S};\Lambda_{\mathcal{S}} X)$ extends by continuity to $H^1(X;\Lambda X) \to H^{1/2}(\mathcal{S};\Lambda_{\mathcal{S}} X)$, and we let

$$H^1_R(X;\Lambda X) = \{u \in H^1(X;\Lambda X) :\ \forall \mathcal{S} \in \partial_1(X),\ \nu_{\mathcal{S}} \wedge u = 0\},$$
$$H^1_A(X;\Lambda X) = \{u \in H^1(X;\Lambda X) :\ \forall \mathcal{S} \in \partial_1(X),\ \iota_{\nu_{\mathcal{S}}} u = 0\};$$

these Sobolev spaces are also the closure of $C^\infty_R(X;\Lambda X)$, resp. $C^\infty_A(X;\Lambda X)$, in $H^1(X;\Lambda X)$.

We consider the wave equation $\Box u = f$ on X, where $\Box = \Box_h$ is the d'Alembertian of h, with natural boundary conditions. That is, for relative boundary conditions,

$$\Box : H^1_{R,\mathrm{loc}}(X;\Lambda X) \to (H^1_{R,\mathrm{comp}}(X;\Lambda X))^*, \tag{1.1}$$

and for $u \in H^1_{R,\mathrm{loc}}(X;\Lambda X)$, $\Box u$ is given by

$$\langle \Box u, v\rangle = \langle du, dv\rangle + \langle d^* u, d^* v\rangle,\ v \in H^1_{R,\mathrm{comp}}(X;\Lambda X). \tag{1.2}$$

That this is indeed the 'right' boundary condition, e.g. in the sense that if $X = M \times \mathbb{R}_t$, M Riemannian with metric g, $h = dt^2 - g$, and one considers the Laplacian on M with relative boundary conditions, and then the functional analytic solutions of $\Box u = f$ with this boundary condition, follows from [18], namely that the quadratic form domain of Δ is $H^1_R(M;\Lambda M)$ in this case – see [22, Section 2] for more details.

The Hodge star $*$ maps $H^1_R(X;\Lambda X)$ to $H^1_A(X;\Lambda X)$ and conversely, and it intertwines solutions of $\Box u = f$ with relative and absolute boundary conditions (of course, one has to take $*f$ as well). Thus, it suffices to study relative boundary conditions, which is how we proceed throughout this paper.

We recall that the analysis of singularities of solutions of the wave equation takes place on the b-cotangent bundle ${}^{\mathrm{b}}T^*X$ or on the b-cosphere bundle ${}^{\mathrm{b}}S^*X$. Smooth sections of ${}^{\mathrm{b}}T^*X$ have the form

$$\sum_{j=1}^{k} \sigma_j(x,y)\,\frac{dx_j}{x_j} + \sum_{j=1}^{n-k} \zeta_j(x,y)\,dy_j, \tag{1.3}$$

with σ_j and ζ_j being C^∞, and correspondingly (x,y,σ,ζ) are local coordinates on ${}^{\mathrm{b}}T^*X$. Let o denote the zero section of ${}^{\mathrm{b}}T^*X$ (as well as other related vector bundles below). Then ${}^{\mathrm{b}}T^*X \setminus o$ is equipped with an $\mathbb{R}^+$-action (fiberwise multiplication) which has no fixed points. It is often natural to take the quotient with the $\mathbb{R}^+$-action, and work on the b-cosphere bundle, ${}^{\mathrm{b}}S^*X$. In a region where, say,

$$|\sigma_j| < C|\zeta_{n-k}|,\ j = 1, \dots, k,\ |\zeta_j| < C|\zeta_{n-k}|, j = 1, \dots, n-k-1, \tag{1.4}$$

$C > 0$ fixed, we can take

$$x_1, \dots, x_k, y_1, \dots, y_{n-k}, \hat\sigma_1, \dots, \hat\sigma_k, \hat\zeta_1, \dots, \hat\zeta_{n-k-1}, |\zeta_{n-k}|,$$
$$\hat\sigma_j = \frac{\sigma_j}{|\zeta_{n-k}|},\ \hat\zeta_j = \frac{\zeta_j}{|\zeta_{n-k}|},$$

as (projective) local coordinates on ${}^{\mathrm{b}}T^*X \setminus o$, hence

$$x_1, \dots, x_k, y_1, \dots, y_{n-k}, \hat\sigma_1, \dots, \hat\sigma_k, \hat\zeta_1, \dots, \hat\zeta_{n-k-1}$$

as local coordinates on the image of this region under the quotient map in ${}^{\mathrm{b}}S^*X$.

A somewhat different perspective is gained by considering the dual bundle, ${}^{\mathrm{b}}TX$, of ${}^{\mathrm{b}}T^*X$. Locally its smooth sections have the form

$$\sum_{j=1}^{k} a_j(x_j\partial_{x_j}) + \sum_{j=1}^{n-k} b_j\partial_{y_j}, \tag{1.5}$$

with $a_j, b_j \in C^\infty(X)$, corresponding to (1.3). Thus, these are exactly the C^∞ vector fields on X which are tangent to every boundary face: they annihilate x_j at $x_j = 0$. The space of these vector fields is denoted $\mathcal{V}_{\mathrm{b}}(X)$, and the corresponding differential operator algebra (locally finite sum of finite products of elements of $\mathcal{V}_{\mathrm{b}}(X)$) is $\mathrm{Diff}_{\mathrm{b}}(X)$.

The principal symbol of $\Box \in \mathrm{Diff}^2(X; \Lambda X)$ is $p\,\mathrm{Id}$, where p is the dual metric function of H on T^*X (so $p(\alpha) = H(\alpha, \alpha)$), and we denote the characteristic set of $\Box$ by

$$\Sigma = p^{-1}(\{0\}) = \{q \in T^*X \setminus o :\ p(q) = 0\}.$$

We denote the Hamilton vector field of p (on T^*X) by H_p. There is a natural map $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ induced by the corresponding map between sections

$$\sum_{j=1}^{k} \xi_j\, dx_j + \sum_{j=1}^{n-k} \zeta_j\, dy_j = \sum_{j=1}^{k} (x_j\xi_j)\, \frac{dx_j}{x_j} + \sum_{j=1}^{n-k} \zeta_j\, dy_j,$$

thus

$$\pi(x, y, \xi, \zeta) = (x, y, x\xi, \zeta),\ x\xi = (x_1\xi_1, \dots, x_k\xi_k). \tag{1.6}$$

We denote the image of Σ under π by

$$\dot\Sigma = \pi(\Sigma),$$

called the compressed characteristic set. As we show below in Section 2, our assumptions on the time-like nature of every boundary face F imply that a neighborhood of $\dot\Sigma$ is covered by coordinate charts as in (1.4), if the ζ_j are appropriately numbered. We next define generalized broken bicharacteristics.

Definition 1.1. *Generalized broken bicharacteristics,* or GBB, are continuous maps $\gamma : I \to \dot\Sigma$, where I is an interval, satisfying

(1) for all $f \in C^\infty({}^{\mathrm{b}}T^*X)$ real valued,

$$\liminf_{s\to s_0} \frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0} \geq \inf\{\mathsf{H}_p(\pi^* f)(q):\ q\in\pi^{-1}(\gamma(s_0))\cap\Sigma(P)\},$$

(2) and if $q_0=\gamma(s_0)\in {}^{\mathrm{b}}T^*_{p_0}X$, and p_0 lies in the interior of a boundary hypersurface (i.e. a boundary face which has codimension 1, so near p_0, ∂X is smooth), then in a neighborhood of s_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand [8], see also [3, Definition 24.3.7].

We mention that there is a different (more concrete) but equivalent version of this definition, due to Lebeau [7], which was used in [23]; we describe this in the next section and refer to [20] for a discussion of the relationship.

We next need to recall the definition of the b-wave front set, $\mathrm{WF}_{\mathrm{b}}(u)$, which was introduced by Melrose originally in order to study propagation of singularities on manifolds with smooth boundaries [12]. More precisely, what we need is the wave front set relative to $H^1(X;\Lambda X)$ rather than the more usual $L^2(X;\Lambda X)$, although they are equivalent (with an appropriate shift of orders) for solutions of the wave equation by Lemma 4.2, Proposition 4.5 and the argument of [23, Lemma 6.1] – see [23] and [22] for a discussion.

As a first step we recall the space of the b-pseudodifferential operators (b-ps.d.o's) which perform the required microlocalization. There are two closely related pseudodifferential algebras, corresponding to the classical and symbolic algebras, $\Psi_{\mathrm{cl}}(X)$ and $\Psi(X)$, in the boundaryless case. These are denoted by $\Psi_{\mathrm{b}}(X)$ and $\Psi_{\mathrm{bc}}(X)$, respectively. There is also a principal symbol on $\Psi_{\mathrm{b}}^m(X)$; this is now a homogeneous degree m function on ${}^{\mathrm{b}}T^*X\setminus o$. $\Psi_{\mathrm{b}}(X)$ has the algebraic properties analogous to $\Psi(X)$ on manifolds without boundary. $\Psi_{\mathrm{b}}(X)$ can be described quite explicitly; this was done for instance in [14, 23, 22] in the corners setting, and in [3, Section 18.3] for smooth boundaries. In particular, a *subset* of $\Psi_{\mathrm{bc}}(X)$ (which would morally suffice for our purposes here) consists of operators with Schwartz kernels supported in $U\times U$, $U\subset X$ a coordinate chart with coordinates x,y as above, with Schwartz kernels of the form

(1.7)
$$q(a)u(x,y) = (2\pi)^{-n}\int e^{i((x-x')\cdot\xi+(y-y')\cdot\zeta)}\phi(\frac{x-x'}{x})a(x,y,x\xi,\zeta)u(x',y')\,dx'\,dy'\,d\xi\,d\zeta,$$

understood as an oscillatory integral, where $a\in S^m(\mathbb{R}^n_{x,y};\mathbb{R}^n_{\sigma,\zeta})$ (with $\sigma=x\xi$, cf. (1.6)), $\phi\in C^\infty_{\mathrm{comp}}((-1/2,1/2)^k)$ is identically 1 near 0, $\frac{x-x'}{x}=(\frac{x_1-x_1'}{x_1},\ldots,\frac{x_k-x_k'}{x_k})$, and the integral in x' is over $[0,\infty)^k$. This formula is similar to the standard quantization formula, but ξ is replaced by $x\xi$ here in the argument of a, and there is a localizating factor ϕ which being identically 1 near the diagonal, does not play an important role. A subset of $\Psi_{\mathrm{b}}(X)$ is similarly obtained if we require that a is a classical (i.e. one-step polyhomogeneous) symbol. Thus, if a is a polynomial in its third and fourth slots, i.e. in $x\xi$ and ζ, depending smoothly on x,y, i.e.

$$a(x,y,\xi,\zeta)=\sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}(x,y)(x\xi)^\alpha\zeta^\beta,$$

then

$$q(a) = \sum_{|\alpha|+|\beta|\le m} a_{\alpha\beta}(x,y)(xD_x)^\alpha D_y^\beta,$$

thus connecting $\mathcal{V}_{\mathrm{b}}(X)$ and $\mathrm{Diff}_{\mathrm{b}}(X)$ to $\Psi_{\mathrm{b}}(X)$ in view of (1.5). For vector bundles E, F over X, one can also construct $\Psi_{\mathrm{b}}(X; E, F)$ via trivializations – acting between distributional sections of vector bundles E and F over X. Elements of $\Psi^m_{\mathrm{bc}}(X)$ have the important property that they map $C^\infty(X) \to C^\infty(X)$, and more generally they map $x_j C^\infty(X) \to x_j C^\infty(X)$, so if $A \in \Psi^m_{\mathrm{bc}}(X)$, then $(Au)|_{\mathcal{S}_j}$ depends only on $u|_{\mathcal{S}_j}$ for $u \in C^\infty(X)$. In particular, Dirichlet boundary conditions are automatically preserved by such A, which makes $\Psi_{\mathrm{b}}(X)$ easy to use in the analysis of the Dirichlet problem in [23]. We will need more care for natural boundary conditions, which is a point we address in the next section, see also [22].

The space of 'very nice' functions corresponding to $\mathcal{V}_{\mathrm{b}}(X)$ and $\mathrm{Diff}_{\mathrm{b}}(X)$, replacing $C^\infty(X)$, is the space of conormal functions to the boundary relative to a fixed space of functions, in this case $H^1(X; \Lambda X)$, i.e. functions $v \in H^1_{\mathrm{loc}}(X; \Lambda X)$ such that $Qv \in H^1_{\mathrm{loc}}(X; \Lambda X)$ for every $Q \in \mathrm{Diff}_{\mathrm{b}}(X; \Lambda X)$ (of any order). Then $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}_{\mathrm{b}}(u)$ if there is an $A \in \Psi^0_{\mathrm{b}}(X; \Lambda X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and Au is H^1-conormal to the boundary. Spelling out the latter explicitly, and also defining the wave front set with finite regularity:

Definition 1.2. Suppose $u \in H^1_{\mathrm{loc}}(X; \Lambda X)$. Then $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}^{1,\infty}_{\mathrm{b}}(u)$ if there is an $A \in \Psi^0_{\mathrm{b}}(X; \Lambda X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $QAu \in H^1_{\mathrm{loc}}(X; \Lambda X)$ for all $Q \in \mathrm{Diff}_{\mathrm{b}}(X; \Lambda X)$.

Moreover, $q \in {}^{\mathrm{b}}T^*X \setminus o$ is *not* in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there is an $A \in \Psi^m_{\mathrm{b}}(X; \Lambda X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $Au \in H^1_{\mathrm{loc}}(X; \Lambda X)$.

The wave front set relative to the dual space, $\dot{H}^{-1}_R(X; \Lambda X)$, is defined similarly.

We recall that the definition of WF could be stated in a completely parallel manner: we would require (for X without boundary) $QAu \in L^2(X)$ for all $Q \in \mathrm{Diff}(X)$ – this is equivalent to $Au \in C^\infty(X)$ by the Sobolev embedding theorem. Here $L^2(X)$ can be replaced by $H^m(X)$ instead, with m arbitrary. Similarly, WF^m can also be defined analogously: we require $Au \in L^2(X)$ for $A \in \Psi^m(X)$ elliptic at q.

The usefulness of the definition relies on the fact that any $A \in \Psi^0_{\mathrm{bc}}(X; \Lambda X)$ with compact support defines a continuous linear maps $A : H^1(X; \Lambda X) \to H^1(X; \Lambda X)$ with norms bounded by a seminorm of A in $\Psi^0_{\mathrm{bc}}(X; \Lambda X)$, see [23, Lemma 3.2] in the scalar case and the discussion after [22, Definition 6] for the vector-valued case.

Our main result is the following:

Theorem 1.3 (See [23] for the scalar equation if $X = M \times \mathbb{R}$ with a product metric, and [22] for the vector-valued equation with Dirichlet or Neumann boundary conditions, and see Theorem 7.1 for a strengthened restatement.)**.** *Suppose that X is a manifold with corners with a C^∞ Lorentz metric h with respect to which every boundary face is timelike. Suppose $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$ and $\Box u = f$ in the sense of* (1.1)-(1.2) *holding for all $v \in H^1_{R,\mathrm{comp}}(X; \Lambda X)$. Then*

$$(\mathrm{WF}^{1,m}_{\mathrm{b}}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}^{-1,m+1}_{\mathrm{b}}(f)$$

is a union of maximally extended generalized broken bicharacteristics of $\Box$ in

$$\dot{\Sigma} \setminus \mathrm{WF}^{-1,m+1}_{\mathrm{b}}(f).$$

In particular, if $\Box u = 0$ *then* $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \subset \dot\Sigma$ *is a union of maximally extended generalized broken bicharacteristics of* P.

The same results hold with the relative boundary condition (R) replaced by the absolute boundary condition (A) throughout.

On manifolds with C^∞ boundaries (and no corners), in the scalar setting this result is due to Melrose, Sjöstrand and Taylor [8, 9, 17, 15]. Still for C^∞ boundaries and singularities, but for systems, including differential forms with natural boundary conditions, Taylor, and Melrose and Taylor have shown the theorem at diffractive points via a parametrix construction [19, 10], see also Yingst's work [24]. In addition, Ivrii [5] has obtained propagation results for systems. Thus, the theorem is known for C^∞ boundaries and C^∞ singularities.

The analogue of this theorem for the scalar equation but for analytic singularities and spaces with an analytically stratified boundary was proved by Lebeau [7], following the work of Sjöstrand [16] when the boundary is analytic. As far as the author is aware, there is no known analogue of this result in the analytic setting for systems with natural boundary conditions, including gliding rays.

A special case of the scalar equation with codimension 2 corners in $\mathbb{R}^2$ had been considered by P. Gérard and Lebeau [2] in the real analytic setting, and by Ivrii [4] in the smooth setting. It should also be mentioned that due to its relevance, this problem has a long history, and has been studied extensively by Keller in the 1940s and 1950s in various special settings, see e.g. [1, 6].

The structure of this paper is the following. In Section 2 we describe the geometric background in more detail, and use this to explain the idea of the proof. In Section 3 we recall some commutator calculation preliminaries and prove the main 'commutator' lemma that we use later in the paper. In Section 4 we recall the elliptic results from [23] and [22]. Then in Section 5 we prove the normal propagation estimate, and in Section 6 we prove glancing propagation. Finally, in Section 7 we put together these results, and also extend the results to a larger class of solutions, possessing a negative order of b-regularity relative to $H^1_{R,\mathrm{loc}}(X;\Lambda X)$.

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2. Setup and idea of the proof

In order to explain how the theorem is proved, we describe the geometry more precisely.

First, we choose local coordinates more carefully. In arbitrary local coordinates

$$(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$$

on a neighborhood $\mathcal{U}$ of a point in the interior of a codimension k corner F given by $x_1 = \ldots = x_k = 0$ inside $x_1 \geq 0, \ldots, x_k \geq 0$, any symmetric bilinear form on T^*X can be written as

$$H(x,y) = \sum_{i,j} A_{ij}(x,y)\,\partial_{x_i}\,\partial_{x_j} + \sum_{i,j} 2C_{ij}(x,y)\,\partial_{x_i}\,\partial_{y_j} + \sum_{i,j} B_{ij}(x,y)\,\partial_{y_i}\,\partial_{y_j} \tag{2.1}$$

with A, B, C smooth. Below we write covectors as

$$\alpha = \sum_{i=1}^{k} \xi_i\,dx_i + \sum_{i=1}^{n-k} \zeta_i\,dy_i. \tag{2.2}$$

Since we assume that every boundary face, in particular F, is time-like in the sense that the restriction of H to N^*F is negative definite, we deduce that A is negative definite, for locally the conormal bundle N^*F is given by

$$\{(x, y, \xi, \zeta) :\ x = 0,\ \zeta = 0\}.$$

Then H is Lorentzian on the H-orthocomplement $(N^*F)^\perp$ of N^*F. In fact, note that for $p_0 \in F$,

$$T^*_{p_0}X = N^*_{p_0}X \oplus (N^*_{p_0}X)^\perp, \tag{2.3}$$

for if V is in the intersection of the two summands, then $H(V, V) = 0$ and $V \in N^*_{p_0}F$, so the definiteness of the inner product on N^*F shows that $V = 0$, hence (2.3) follows as the dimension of the summands sums up to the dimension of $T^*_{p_0}X$. Choosing an orthogonal basis of $(N^*F)^\perp$ consisting of vectors of length ± 1 at a given point $p_0 \in F^\circ$, and then coordinates y_j with differentials equal to these basis vectors, we have in the new basis that $C_{ij}(0, 0) = 0$ and

$$\sum B_{ij}(0, 0)\partial_{y_i}\partial_{y_j} = \partial^2_{y_{n-k}} - \sum_{i<n-k} \partial^2_{y_i}, \tag{2.4}$$

and we write coordinates on T^*X as

$$x,\ t = y_{n-k},\ \tilde{y} = (y_1, \ldots, y_{n-k-1}),\ \xi,\ \tau = \zeta_{n-k},\ \tilde{\zeta} = (\zeta_1, \ldots, \zeta_{n-k-1}),$$

cf. (2.2). Thus B is non-degenerate, Lorentzian, near p_0, and a simple calculation shows that the coordinates on X can be chosen (i.e. the y_j can be adjusted) so that $C(0, y) = 0$. Then

$$H|_{x=0} = \sum_{i,j} A_{ij}(0, y)\, \partial_{x_i}\, \partial_{x_j} + \sum_{i,j} B_{ij}(0, y)\, \partial_{y_i}\, \partial_{y_j}, \tag{2.5}$$

and hence the metric function is

$$p|_{x=0} = \xi \cdot A(y)\xi + \zeta \cdot B(y)\zeta. \tag{2.6}$$

This gives that

$$\dot{\Sigma} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_F X = \{(0, y, 0, \zeta) :\ 0 \le \zeta \cdot B(y)\zeta,\ \zeta \neq 0\}. \tag{2.7}$$

In particular, in view of (2.4), $\dot{\Sigma} \cap \mathcal{U}$ lies in the region (1.4), at least after we possibly shrink $\mathcal{U}$.

In order to better understand the generalized broken bicharacteristics for $\Box$, we divide $\dot{\Sigma}$ into two subsets. We thus define the *glancing set* $\mathcal{G}$ as the set of points in $\dot{\Sigma}$ whose preimage under $\hat{\pi} = \pi|_\Sigma$ consists of a single point, and define the *hyperbolic set* $\mathcal{H}$ as its complement in $\dot{\Sigma}$. Thus, $q \in \dot{\Sigma}$ lies in $\mathcal{G}$ if and only if on $\hat{\pi}^{-1}(\{q\})$, $\xi_j = 0$ for all j. More explicitly, with the notation of (2.7),

$$\begin{aligned} \mathcal{G} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_F X &= \{(0, y, 0, \zeta) :\ \zeta \cdot B(y)\zeta = 0,\ \zeta \neq 0\}, \\ \mathcal{H} \cap \mathcal{U} \cap {}^{\mathrm{b}}T^*_F X &= \{(0, y, 0, \zeta) :\ \zeta \cdot B(y)\zeta > 0,\ \zeta \neq 0\}. \end{aligned} \tag{2.8}$$

Thus, $\mathcal{G}$ corresponds to generalized broken bicharacteristics which are tangent to F in view of the vanishing of ξ_j, while $\mathcal{H}$ corresponds to generalized broken bicharacteristics which are normal to F. Note that if F is one-dimensional, which is the lowest dimension it can be in view of the time-like restriction, then $\zeta \cdot B(y)\zeta$ necessarily implies $\zeta = 0$, so in fact $\mathcal{G} \cap {}^{\mathrm{b}}T^*_F X = \emptyset$.

We next make the role of $\mathcal{G}$ and $\mathcal{H}$ more explicit, which explains the relevant phenomena better. An equivalent characterization of GBB is

Lemma 2.1. *(See the discussion in* [20, Section 1] *after the statement of Definition 1.1.) A continuous map $\gamma : I \to \dot{\Sigma}$, where $I \subset \mathbb{R}$ is an interval, is a* GBB *if and only if it satisfies the following requirements:*

(1) *If $q_0 = \gamma(s_0) \in \mathcal{G}$ then for all $f \in C^\infty({}^{\mathrm{b}}T^*X)$,*

$$\frac{d}{ds}(f \circ \gamma)(s_0) = \mathsf{H}_p(\pi^* f)(\tilde{q}_0),\ \tilde{q}_0 = \hat{\pi}^{-1}(q_0). \tag{2.9}$$

(2) *If $q_0 = \gamma(s_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$ then there exists $\epsilon > 0$ such that*

$$s \in I,\ 0 < |s - s_0| < \epsilon \Rightarrow \gamma(t) \notin {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X. \tag{2.10}$$

(3) *If $q_0 = \gamma(s_0) \in \mathcal{G} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$, and F is a boundary hypersurface (i.e. has codimension 1), then in a neighborhood of s_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand* [8]*, see also* [3, Definition 24.3.7].

The general strategy of the proof of the main theorem is to prove propagation estimates at $\mathcal{G}$ and $\mathcal{H}$ separately. The estimates at $\mathcal{H}$ can be weaker: as the GBB through these points are normal, one only needs to prove that singularities leave ${}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$ immediately, for then an inductive argument, using that locally F is the most singular stratum, allows one to deduce the desired propagation.

However, at $\mathcal{G}$ one has to prove a more precise result. Namely, if $q_0 \in \mathcal{G}$, there is a unique point $\alpha_0 \in \hat{\pi}^{-1}(\{q_0\})$, and we need to prove that roughly speaking singularities propagate in the direction of $(\pi_*)_{\alpha_0}\mathsf{H}_p$. More precisely (although we actually use a vector field on T^*F in Section 6, and a product decomposition of X near a point in F_{reg}), let W be a vector field on ${}^{\mathrm{b}}T^*X$ with $W(q_0) = (\pi_*)_{\alpha_0}\mathsf{H}_p$. Then we need to prove that for small $\delta > 0$, there is an $o(\delta)$-sized ball around $\exp(\delta W)\alpha_0$ such that if this ball contains no singularities of u, then $\alpha_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,m}(u)$ either. We show that indeed there is an $O(\delta^2)$-sized ball with this property, just like for the scalar or the vector-valued equation on manifolds with corners with Dirichlet or Neumann boundary condition.

One basic difficulty is that we need to use operators which preserve the boundary conditions in order to microlocalize. As we want to use principally scalar operators, at the principal symbol level this is automatic, but we need operators fully (not merely symbolically) preserving the boundary conditions. To achieve this, we locally trivialize the form bundle (and use microlocalizers supported in such a coordinate chart) in such a way that the boundary conditions state the vanishing of various components of the trivialization. More concretely, a local trivialization over an open set $\mathcal{U}$ of $\Lambda^p X$ is a map $\Lambda^p_{\mathcal{U}}X \to \mathcal{U} \times \mathbb{R}^N$, $N = \dim \Lambda^p_q X$, $q \in X$, being given by the binomial coefficient; we want this such that there is an index set $J_j \subset \{1, \ldots, N\}$ for $j = 1, \ldots, k$, such that for each j and at each $q \in \mathcal{U} \cap \mathcal{S}_j = \{x_j = 0\}$, for a form α to satisfy $dx_j \wedge u = 0$ requires precisely that $\alpha_m = 0$ for $m \in J_j$, where $\alpha = (\alpha_1, \ldots, \alpha_N)$ with repect to the trivialization. The construction of such a trivialization is straightforward, however, using

$$dx_{i_1} \wedge \ldots \wedge dx_{i_s} \wedge dy_{\ell_1} \wedge \ldots \wedge dy_{\ell_{p-s}},\ i_1 < \ldots < i_s,\ \ell_1 < \ldots < \ell_{p-s}, \tag{2.11}$$

as the basis of $\Lambda^p_q X$, $dx_j \wedge u = 0$ amounts to saying that all components of α in which j is not one the i_r's vanish. Similarly, using the Hodge star operator, there is such a good trivialization for the absolute boundary condition as well, namely $*$ applied to the basis of (2.11).

Now the mere existence of such a trivialization, and hence of 'scalar' (namely, diagonal with respect to the trivialization) b-pseudodifferential operators guarantees that the elliptic regularity arguments go through since in these arguments commutators are lower order hence negligible. Thus, microlocal elliptic regularity was proved by the author in [22]. However, the matters are much more complicated for hyperbolic and glancing points, as at these points the propagation estimates are positive commutator estimates. In positive commutator estimates it is convenient to have formally self-adjoint commutants; however, if one has a scalar operator, its adjoint is usually not scalar – and indeed, usually does not preserve even the relevant subbundles, hence the boundary conditions[1]. One can also work with non-self-adjoint commutants, in which case one has in the boundaryless setting

$$\langle \Box u, Au\rangle - \langle Au, \Box u\rangle = \langle (A^* - A)\Box u, u\rangle + \langle [A, \Box]u, u\rangle;$$

we refer to Proposition 3.10 for the correct statement in the presence of boundaries. Roughly speaking, the difficulty here is that $A^* - A$ does not preserve the natural boundary conditions, and $(A^* - A)\Box$ is the same order as $[A, \Box]$ (if A has real scalar principal symbol) – this is a problem since if $A^* - A$ preserved boundary conditions, $\langle \Box u, (A^* - A)u\rangle$ would be controlled by the PDE, but this is not so otherwise[2]. However, it turns out that modulo terms one can easily control (because normal derivatives of u are small at glancing points), one can replace $\Box$ by its tangential part, and instead of using the PDE, use the positivity of the commutator to control this term.

Our results then combine to prove the main theorem, using the argument of Melrose and Sjöstrand [8, 9], as modified by Lebeau [7, Proposition VII.1].

3. Commutator constructions

We start by recalling[3] that for any vector bundle E over X,

$$\Psi_{\mathrm{bc}}(X;E) = \cup_s \Psi^s_{\mathrm{bc}}(X;E),\ \ \Psi_{\mathrm{b}}(X;E) = \cup_s \Psi^s_{\mathrm{b}}(X;E) \subset \Psi_{\mathrm{bc}}(X;E),$$

are sets of operators

$$\begin{aligned} &\Psi_{\mathrm{bc}}(X;E) \ni A : \dot C^\infty(X;E) \to \dot C^\infty(X;E),\\ &\Psi_{\mathrm{bc}}(X;E) \ni A : C^\infty(X;E) \to C^\infty(X;E), \end{aligned} \tag{3.1}$$

where $\dot C^\infty(X;E)$ denotes the subspace of $C^\infty(X;E)$ (C^∞ sections of E) which vanish with all derivatives at ∂X. Then $\Psi_{\mathrm{bc}}(X;E)$ is a filtered algebra of operators, $\Psi_{\mathrm{b}}(X;E)$ is closed under composition, as well as under addition under a compatibility condition on orders, with a principal symbol map

$$\sigma_{\mathrm{b},s} : \Psi^s_{\mathrm{b}}(X;E) \to S^s_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o; \pi^*\mathrm{Hom}(E,E)),$$

where $\pi : {}^{\mathrm{b}}T^*X \to X$ is the bundle projection, and S^s_{hom} denotes homogeneous degree s, C^∞ functions on ${}^{\mathrm{b}}T^*X \setminus o$, while

$$\sigma_{\mathrm{b},s} : \Psi^s_{\mathrm{bc}}(X;E) \to S^s({}^{\mathrm{b}}T^*X; \pi^*\mathrm{Hom}(E,E))/S^{s-1}({}^{\mathrm{b}}T^*X; \pi^*\mathrm{Hom}(E,E)).$$

[1] This is the difference with codimension one boundaries, where one can simply take an *orthogonal* decomposition of the cotangent bundle dx_1 being orthogonal to the span of $dy_1, \ldots, dy_{n-1}$, which implies that adjoints of block-diagonal operators, i.e. operators respecting this decomposition, are also block-diagonal at $\mathcal{S} = \mathcal{S}_1$, which is what matters.

[2] Indeed, even the principal symbol of $A^* - A$ does not preserve boundary conditions typically.

[3] See [13], which mostly deals with the C^∞ boundary case, and especially [14] as a background reference; [23] has a brief discussion as well on $\Psi_{\mathrm{bc}}(X)$.

Thus, if $B_j \in \Psi_{\mathrm{b}}^{s_j}(X;E)$, $j=1,2$, then

$$B_1 B_2 \in \Psi_{\mathrm{b}}^{s_1+s_2}(X;E),$$

with

$$\sigma_{\mathrm{b},s_1+s_2}(B_1B_2)(q) = \sigma_{\mathrm{b},s_1}(B_1)(q)\sigma_{\mathrm{b},s_2}(B_2)(q),\ q \in {}^{\mathrm{b}}T^*X \setminus o,$$

where the product of the right is composition of endomorphisms of the fiber of E at the point $\pi(q)$, and similarly for $\Psi_{\mathrm{bc}}^s(X;E)$.

If $B_j \in \Psi_{\mathrm{b}}^{s_j}(X;E)$ and $\sigma_{\mathrm{b},s_j}(B_j)$ is *scalar*, $j=1,2$, i.e. is a multiple of the identity homomorphism:

$$\sigma_{\mathrm{b},s_j}(B_j) = b_j \,\mathrm{Id},\ b_j \in S^{s_j}_{\mathrm{hom}}({}^{\mathrm{b}}T^*X),$$

then their commutator is

$$[B_1,B_2] = B_1B_2 - B_2B_1 \in \Psi_{\mathrm{b}}^{s_1+s_2-1}(X;E),$$

with

$$\sigma_{\mathrm{b},s_1+s_2-1}([B_1,B_2]) = \imath(\mathsf{H}_{\mathrm{b},b_1}b_2)\,\mathrm{Id};$$

the analogous result also holds for $\Psi_{\mathrm{bc}}(X;E)$. Here $\mathsf{H}_{\mathrm{b},a}$ is the b-Hamilton vector field of $a \in C^\infty({}^{\mathrm{b}}T^*X)$, i.e. it is the unique C^∞ vector field on ${}^{\mathrm{b}}T^*X$ which agrees with the standard Hamilton vector field H_a on T^*X° under the natural identification of T^*X° with ${}^{\mathrm{b}}T^*_{X^\circ}X$. Thus, in the notation of (1.6), $\mathsf{H}_{\mathrm{b},a} = \pi_*\mathsf{H}_{\pi^*a}$. In local coordinates,

$$\mathsf{H}_{\mathrm{b},a} = \sum_j (\partial_{\sigma_j}a)\,x_j\partial_{x_j} + \sum_j (\partial_{\zeta_j}a)\,\partial_{y_j} - \sum_j (x_j\partial_{x_j}a)\,\partial_{\sigma_j} - \sum_j (\partial_{y_j}a)\,\partial_{\zeta_j},$$

see e.g. [23, Proof of Lemma 2.8]. In particular, it is a vector field tangent to all boundary faces of ${}^{\mathrm{b}}T^*X$, and has vanishing ∂_{σ_j} component at $\{x_j=0\}$. Note also that $\sigma_{\mathrm{b},s_1+s_2-1}([B_1,B_2])$ depends only on b_1 and b_2.

On the other hand, suppose now that $B_j \in \Psi_{\mathrm{b}}^{s_j}(X;E)$, $j=1,2$, and $\sigma_{\mathrm{b},s_1}(B_1)$ is scalar. Then $\sigma_{\mathrm{b},s_1}(B_1)$ and $\sigma_{\mathrm{b},s_2}(B_2)$ commute, hence

$$\sigma_{\mathrm{b},s_1+s_2}([B_1,B_2]) = 0,$$

so

$$[B_1,B_2] \in \Psi_{\mathrm{b}}^{s_1+s_2-1}(X;E).$$

However, the principal symbol of the commutator now depends on B_1 via more than its principal symbol – see also Remark 3.9.

We also recall that if we equip E with a Hermitian inner product and put a C^∞ density ν on X, thus obtaining an inner product on $L^2(X;E)$, then $A \in \Psi^0_{\mathrm{bc}}(X;E)$ with compactly supported Schwartz kernel is bounded, with norm bounded by a seminorm of A in $\Psi^0_{\mathrm{bc}}(X;E)$ – see [14, Equation (2.16)]. Moreover, if A has scalar principal symbol $\sigma_{\mathrm{b},0}(A) = a\,\mathrm{Id}$, then there exists $A' \in \Psi_{\mathrm{b}}^{-1}(X;E)$ such that for all $v \in L^2(X;E)$,

$$\|Av\| \le 2\sup|a|\,\|v\| + \|A'v\|;$$

see [23, Section 2]. In addition, $\Psi^0_{\mathrm{bc}}(X;E)$ and $\Psi^0_{\mathrm{b}}(X;E)$ are closed under L^2-adjoints. Thus, dually – with respect to the L^2-inner product – to (3.1), with $C^{-\infty}(X;E)$ the dual of $\dot C^\infty(X;E)$, $\dot C^{-\infty}(X;E)$ dual to $C^\infty(X;E)$,

$$\begin{aligned}&\Psi_{\mathrm{bc}}(X;E) \ni A : C^{-\infty}(X;E) \to C^{-\infty}(X;E),\ \text{resp.}\\ &\Psi_{\mathrm{bc}}(X;E) \ni A : \dot C^{-\infty}(X;E) \to \dot C^{-\infty}(X;E),\end{aligned} \tag{3.2}$$

defined by

$$Au(\phi) = u(A^*\phi),\ \phi \in \dot{C}^\infty(X;E),\ \text{resp. } \phi \in C^\infty(X;E). \tag{3.3}$$

Next, the definition of $\mathrm{Diff}\Psi_{\mathrm{b}}(X)$ from [23, Definition 2.3]:

Definition 3.1. $\mathrm{Diff}^k\Psi_{\mathrm{b}}^s(X)$ is the vector space of operators of the form

$$\sum_j P_j A_j,\ P_j \in \mathrm{Diff}^k(X),\ A_j \in \Psi_{\mathrm{b}}^s(X), \tag{3.4}$$

where the sum is locally finite in X.

Equivalently, the order of the factors can be reversed, i.e. these operators can be written as

$$\sum_j A'_j P'_j,\ P'_j \in \mathrm{Diff}^k(X),\ A'_j \in \Psi_{\mathrm{b}}^s(X).$$

The key point (in local coordinates) is that while $D_{x_j} \notin \mathcal{V}_{\mathrm{b}}(X)$, for any $A \in \Psi_{\mathrm{b}}^m(X)$ there is an operator $\tilde{A} \in \Psi_{\mathrm{b}}^m(X)$ such that

$$D_{x_j} A - \tilde{A} D_{x_j} \in \Psi_{\mathrm{b}}^m(X), \tag{3.5}$$

and analogously for $\Psi_{\mathrm{b}}^m(X)$ replaced by $\Psi_{\mathrm{bc}}^m(X)$, see [23, Equation (2.3)]. Indeed, one may write

$$D_{x_j} A = \tilde{A} D_{x_j} + \tilde{B},\ \tilde{A} = x_j^{-1} A x_j,\ \tilde{B} = x_j^{-1}[x_j D_{x_j}, A], \tag{3.6}$$

and thus we even have $\sigma_{\mathrm{b},m}(\tilde{A}) = \sigma_{\mathrm{b},m}(A)$.

Indeed, recall from [23, Lemma 2.5] that $\mathrm{Diff}^k\Psi_{\mathrm{b}}^s(X)$ is a filtered algebra with respect to operator composition, with $B_j \in \mathrm{Diff}^{k_j}\Psi_{\mathrm{b}}^{s_j}(X)$, $j = 1, 2$, implying $B_1 B_2 \in \mathrm{Diff}^{k_1+k_2}\Psi_{\mathrm{b}}^{s_1+s_2}(X)$. Moreover, with B_1, B_2 as above,

$$[B_1, B_2] \in \mathrm{Diff}^{k_1+k_2}\Psi_{\mathrm{b}}^{s_1+s_2-1}(X).$$

We also recall the following lemma that computes the principal symbol of a commutator:

Lemma 3.2. *(*[23, Lemma 2.8]*) Let ∂_{x_j}, ∂_{σ_j} denote local coordinate vector fields on ${}^{\mathrm{b}}T^*X$ in the coordinates (x, y, σ, ζ). For $A \in \Psi_{\mathrm{b}}^m(X)$ with Schwartz kernel supported in the coordinate patch, $a = \sigma_{\mathrm{b},m}(A) \in C^\infty({}^{\mathrm{b}}T^*X \setminus o)$, we have $[D_{x_j}, A] = A_1 D_{x_j} + A_0 \in \mathrm{Diff}^1\Psi_{\mathrm{b}}^{m-1}(X)$ with $A_0 \in \Psi_{\mathrm{b}}^m(X)$, $A_1 \in \Psi_{\mathrm{b}}^{m-1}(X)$ and*

$$\sigma_{\mathrm{b},m-1}(A_1) = \frac{1}{\imath}\partial_{\sigma_j} a,\ \sigma_{\mathrm{b},m}(A_0) = \frac{1}{\imath}\partial_{x_j} a. \tag{3.7}$$

This result also holds with $\Psi_{\mathrm{b}}(X)$ replaced by $\Psi_{\mathrm{bc}}(X)$ everywhere.

These results extend immediately to operators acting on sections of a vector bundle E, provided that in the case of (3.7) we require that A has scalar principal symbol, and provided that we replace D_{x_j} by any $Q \in \mathrm{Diff}^1(X;E)$ with scalar principal symbol $\xi_j \,\mathrm{Id}_{\Lambda X}$.

Adjoints play a major role in positive commutator estimates, with a prominent role due to the boundary conditions. We consider operators acting on functions first before turning to operators acting on forms. For the Dirichlet problem the boundary condition can be handled in a number of ways (essentially because of the density of $\dot{C}^\infty(X)$ in $H_0^1(X)$), but for other boundary conditions more care is required. In [23], for the Neumann problem, pairings were considered, and one factor of a differential operator was always left on each slot of the pairing. Here we use the approach of [11] to enlarge $\mathrm{Diff}\,\Psi_{\mathrm{b}}(X)$ somewhat by adjoints of differential operators, so that one need not write down quadratic forms at every point. However, this is mostly only a stylistic issue.

We first recall the basic function spaces. For $k \geq 0$ integer, we let $H^k(X)$ be the completion of $C^\infty_{\mathrm{comp}}(X)$ with respect to the $H^k(X)$ norm. Then we define $H^k_0(X)$ as the closure of $\dot{C}^\infty_{\mathrm{comp}}(X)$ inside $H^k(X)$. If $\tilde{X}$ is a manifold without boundary, and X is embedded into it, one can also extend elements of $H^k(X)$ to elements $H^k_{\mathrm{loc}}(\tilde{X})$. With Hörmander's notation [3, Appendix B.2], $H^k_{\mathrm{loc}}(X) = \bar{H}^k_{\mathrm{loc}}(X^\circ)$ – [3, Appendix B.2] discusses the case of a smooth boundary only, but the general case is similar, see [23, Section 3]. As is clear from the completion definition, $H^k_{0,\mathrm{loc}}(X)$ can be identified with the subset of $H^k_{\mathrm{loc}}(\tilde{X})$ consisting of functions supported in X. Thus, $H^k_{0,\mathrm{loc}}(X) = \dot{H}^k_{\mathrm{loc}}(X)$ with the notation of [3, Appendix B.2].

We let $H^{-k}(X)$ be the dual of $H^k_0(X)$ and $\dot{H}^{-k}(X)$ be the dual of $H^k(X)$, with respect to an extension of the sesquilinear form $\langle u, v\rangle = \int_X u\,\overline{v}\,d\tilde{g}$, i.e. the L^2 inner product. As $H^k_0(X)$ is a closed subspace of $H^k(X)$, $H^{-k}(X)$ is the quotient of $\dot{H}^{-k}(X)$ by the annihilator of $H^k_0(X)$, hence there is a canonical map

$$\rho : \dot{H}^{-k}(X) \to H^{-k}(X).$$

In terms of the identification of the H^k spaces above, $H^{-k}_{\mathrm{loc}}(X) = \bar{H}^{-k}_{\mathrm{loc}}(X^\circ)$ in the notation of [3, Appendix B.2], i.e. its elements are the restrictions to X° of elements of $H^{-k}_{\mathrm{loc}}(\tilde{X})$. Analogously, $\dot{H}^{-k}_{\mathrm{loc}}(X)$ consists of those elements of $H^{-k}_{\mathrm{loc}}(\tilde{X})$ which are supported in X.

If $P \in \mathrm{Diff}^k(X)$, then it defines a continuous linear map

$$P : H^k(X) \to L^2(X).$$

Thus, its Banach space adjoint (with respect to the *sesquilinear* dual pairing) is a map

$$\begin{aligned} P^* : (L^2(X))^* = L^2(X) \to (H^k(X))^* = \dot{H}^{-k}(X),\\ \langle P^*u, v\rangle = \langle u, Pv\rangle,\ u \in L^2(X),\ v \in H^k(X). \end{aligned} \tag{3.8}$$

There is an important distinction here between considering P^* as stated, or as composed with the quotient map, $\rho \circ P^*$.

Lemma 3.3. *(cf.* [11, Lemma 5.18]*) Suppose that $P \in \mathrm{Diff}^k(X)$. Then there exists a unique $Q \in \mathrm{Diff}^k(X)$ such that $\rho \circ P^* = Q$. However, in general, acting on $C^\infty(X)$, $P^* \neq Q$.*

If, on the other hand, $P \in \mathrm{Diff}^k_{\mathrm{b}}(X)$, then there exists a unique $Q \in \mathrm{Diff}^k_{\mathrm{b}}(X)$ such that $P^* = Q$.

As the proof is a simple modification of that of [11, Lemma 5.18] (which actually deals with a somewhat more complicated case), we omit it here. Indeed, one can simply regard our setting as a special case of that of [11, Lemma 5.18], namely when one is working away from the edge that is blown up there – see [11, Section 2] for a discussion of the relationship.

Following [11] we now define an extension of $\mathrm{Diff}(X)$ as follows.

Definition 3.4. Let $\mathrm{Diff}^k_\dagger(X)$ denote the set of Banach space adjoints of elements of $\mathrm{Diff}^k(X)$ in the sense of (3.8).

Also let $\mathrm{Diff}^{2k}_{\sharp}(X)$ denote operators of the form

$$\sum_{j=1}^{N} Q_j P_j,\ P_j \in \mathrm{Diff}^k(X),\ Q_j \in \mathrm{Diff}^k_{\dagger}(X).$$

For X non-compact, the sum is taken to be locally finite.

Thus, if $P \in \mathrm{Diff}^{2k}_{\sharp}(X)$, P_j, Q_j as above, and $Q_j = R_j^*$, $R_j \in \mathrm{Diff}^k(X)$, then

$$\langle Pu, v\rangle = \sum_{j=1}^{N} \langle P_j u, R_j v\rangle.$$

Remark 3.5. While for $R \in \mathrm{Diff}^k(X)$, $R^* \in \mathrm{Diff}^k_{\dagger}(X)$ depends on the inner product on $L^2(X)$, i.e. on the C^∞ density inducing it, the set of adjoints is independent of the choice of inner product.

We now turn to differential operators acting on vector bundles. For this purpose it is often useful (but is sometimes not absolutely necessary) to have a positive definite inner product on ΛX (unlike the pairing induced by the Lorentz metric h). We will only consider such inner products induced by a Riemannian metric $\tilde{h}$. Let H, resp. $\tilde{H}$ denoting the dual metrics, as well as the induced metrics on forms; these can be thought of as maps $\Lambda_p X \to (\Lambda_p X)^*$, hence

$$J = H^{-1}\tilde{H} \tag{3.9}$$

is an isomorphism of $\Lambda_p X$. The inner products then satisfy

$$(u, Jv)_H = (\tilde{H}v)(u) = (u, v)_{\tilde{H}} = (Ju, v)_H, \tag{3.10}$$

and the inner product $(.,.)_{\tilde{H}}$ is positive definite. In particular

$$J^*_H = J,\ (J^{-1})^*_{\tilde{H}} = J^{-1},$$

where $(.)^*_H$ denotes the adjoint of an endomorphism with respect to the H inner product. Thus, with the last equation being a definition,

$$\langle u, Jv\rangle_H = \int (u, Jv)_H\,|dh| = \int (u, v)_{\tilde{H}}\,|dh| = \langle u, v\rangle_{L^2(X;\Lambda X;|dh|\otimes\tilde{H})} \equiv \langle u, v\rangle, \tag{3.11}$$

where the inner product on the right is thus with respect to $\tilde{H}$ on the fibers, but using the density $|dh|$, and is positive definite.

As we only consider natural boundary conditions, and indeed relative boundary conditions only, we discuss adjoints only in this setting to avoid overburdening the notation. Namely, proceeding as above, if $P \in \mathrm{Diff}^k(X;\Lambda X)$, then it defines a continuous linear map

$$P : H^k_R(X;\Lambda X) \to L^2(X;\Lambda X).$$

Thus, its Banach space adjoint is the map

$$\begin{gathered} P^* : (L^2(X;\Lambda X))^* = L^2(X;\Lambda X) \to (H^k_R(X;\Lambda X))^* \equiv \dot{H}^{-k}_R(X;\Lambda X), \\ \langle P^* u, v\rangle = \langle u, Pv\rangle,\ u \in L^2(X;\Lambda X),\ v \in H^k_R(X;\Lambda X). \end{gathered} \tag{3.12}$$

We extend the preceeding definitions:

Definition 3.6. Let $\mathrm{Diff}^k_\dagger(X;\Lambda X)$ denote the set of Banach space adjoints of elements of $\mathrm{Diff}^k(X;\Lambda X)$ in the sense of (3.12).

Also let $\mathrm{Diff}^{2k}_\sharp(X;\Lambda X)$ denote operators of the form

$$\sum_{j=1}^N Q_j P_j,\ \ P_j \in \mathrm{Diff}^k(X;\Lambda X),\ \ Q_j \in \mathrm{Diff}^k_\dagger(X;\Lambda X).$$

For X non-compact, the sum is taken to be locally finite.

Again, $\mathrm{Diff}^{2k}_\sharp(X;\Lambda X)$ is independent of the choice of the metrics h and $\tilde{h}$.

However, for calculations below we need to be somewhat careful in our choice of $\tilde{h}$. One convenient choice is a Riemannian metric $\tilde{H}$ which satisfies

$$\tilde{H} = -H + 2\partial^2_{y_{n-k}} \tag{3.13}$$

in a neighborhood of $p_0 \in F$ with local coordinates nearby as in (2.4). Note that $-H + 2\partial^2_{y_{n-k}}$ is indeed Riemannian in a sufficiently small neighborhood of the point p_0 in view of (2.4), so the desired $\tilde{H}$ exists. Notice that on this neighborhood $\tilde{H}(dx_j,.) = H(dx_j,.)$, thus relative boundary conditions are preserved by J. (Note that the definition of J given in [22] after Equation (10) is not correct in all situations; the present definition should be used instead.)

We can now describe the form of

$$\Box \in \mathrm{Diff}^2_\sharp(X;\Lambda X) : H^1_{R,\mathrm{loc}}(X;\Lambda X) \to (H^1_{R,\mathrm{comp}}(X;\Lambda X))^*. \tag{3.14}$$

Lemma 3.7. *Let $\mathcal{U}$ be a coordinate chart with coordinates $(x_1,\ldots,x_k,y_1,\ldots,y_{n-k})$ such that* (2.5) *holds,* (3.13) *is valid, and trivialize ΛX using the coordinate differentials. With $Q_i = D_{x_i} \otimes \mathrm{Id}_{\Lambda X}$, the wave operator with relative boundary conditions, i.e. as a map* (3.14)*, satisfies*

$$\Box = \sum_{i,j} Q_i^* A_{ij}(x,y) Q_j + \sum_i (M_i Q_i + Q_i^* M_i') + \tilde{P}\ \text{ on } \mathcal{U} \tag{3.15}$$

with

$$\begin{aligned} &M_i, M_i' \in \mathrm{Diff}^1_{\mathrm{b}}(X;\Lambda X),\ \ \tilde{P} \in \mathrm{Diff}^2_{\mathrm{b}}(X;\Lambda X)\\ &\sigma_{\mathrm{b},1}(M_i) = m_i\,\mathrm{Id} = \sigma_{\mathrm{b},1}(M_i'),\ \ \sigma_{\mathrm{b},2}(\tilde{P}) = \tilde{p}\,\mathrm{Id},\\ &m_i|_F = 0,\ \ \tilde{p} = \sum_{i,j=1}^{n-k} B_{ij}(x,y)\zeta_i\zeta_j. \end{aligned} \tag{3.16}$$

Proof. As shown in [22, Section 5] (which in turn follows [18, Section 4]), using the trivialization to define ∇, for $u \in C^\infty_R(X;\Lambda X)$, $v \in C^\infty_{R,\mathrm{comp}}(X;\Lambda X)$:

$$\langle du, dv\rangle_H + \langle \delta u, \delta v\rangle_H = \langle \nabla u, \nabla v\rangle_H + \langle Ru, v\rangle_H + \int_{\partial X} (\tilde{R}u, v)_H\, dS_{\partial X}$$

for some smooth bundle endomorphism $\tilde{R}$ and a first order differential operator R; by continuity and density this holds whenever $u \in H^1_{R,\mathrm{loc}}(X;\Lambda X)$, $v \in H^1_{R,\mathrm{comp}}(X;\Lambda X)$. Thus, the wave equation $\Box u = f$ becomes

$$\langle f, v\rangle_H = \langle \nabla u, \nabla v\rangle_H + \langle Ru, v\rangle_H + \int_{\partial X} (\tilde{R}u, v)_H\, dS_{\partial X} \tag{3.17}$$

for all $v \in H^1_{R,\mathrm{comp}}(X;\Lambda X)$. Rewriting this, replacing v by $Jv \in H^1_{R,\mathrm{comp}}(X;\Lambda X)$ (where we use that J preserves the boundary condition),

$$\begin{aligned}\langle f, v\rangle = \langle f, Jv\rangle_H &= \langle \nabla u, \nabla Jv\rangle_H + \langle Ru, Jv\rangle_H + \int_{\partial X} (\tilde{R}u, Jv)_H \, dS_{\partial X}\\ &= \langle \nabla u, \nabla Jv\rangle_H + \langle Ru, v\rangle + \int_{\partial X} (\tilde{R}u, v)_{\tilde{H}} \, dS_{\partial X}\end{aligned} \tag{3.18}$$

for all $v \in H^1_{R,\mathrm{comp}}(X;\Lambda X)$.

Now $R \in \mathrm{Diff}^1(X;\Lambda X)$ means that it can be absorbed in the $M_i Q_i$ and $\tilde{P}$ terms, as subprincipal terms (i.e. the contribution to the principal symbol of M_i and $\tilde{P}$ vanishes). Similarly, $\tilde{R}$ can be rewritten as the boundary term arising by taking the adjoint of a first order differential operator, i.e. is of the form $(R')^* - R''$ for $R', R'' \in \mathrm{Diff}^1(X;\Lambda X)$, hence again can be absorbed in the subprincipal terms of the $M_i Q_i$, $Q_i^* M_i'$ and $\tilde{P}$ terms. Thus, it suffices to check the form of $\langle \nabla u, \nabla Jv\rangle_{L^2(X;\Lambda X)}$.

But writing the coordinates (x, y) as w and using (3.10) pointwise,

$$\begin{aligned}\langle \nabla u, \nabla Jv\rangle_H &= \sum_{ij} \int (H_{ij} D_{w_i} u, D_{w_j} Jv)_H \, |dh|\\ &= \sum_{ij} \int (J^{-1} H_{ij} D_{w_i} u, D_{w_j} Jv)_{\tilde{H}} \, |dh|\\ &= \sum_{ij} \int (H_{ij} D_{w_i} u, J^{-1} D_{w_j} Jv)_{\tilde{H}} \, |dh| = \sum_{ij} \langle (J^{-1} D_{w_j} J)^* H_{ij} D_{w_i} u, v\rangle\end{aligned}$$

where the remaining pairing is the L^2-pairing on functions, i.e. is the integral of the product (with a complex conjugation). Now, $J^{-1} D_{w_j} J = D_{w_j} + T_j$, $T_j \in C^\infty(X; \mathrm{End}(\Lambda X))$. Rewriting w as (x, y), and using that the crossterm $C_{ij}(x, y)$ vanishes at $x = 0$, we obtain (3.15) by noting that the terms with T_i and T_j can be incorporated in the subprincipal terms of the M_i, M_i' and $\tilde{P}$ terms of (3.15). □

The operators whose solutions we consider below are perturbations of $\Box$ by first order operators, i.e. we assume that

$$\begin{aligned}&P = \Box + P_1 : H^1_{R,\mathrm{loc}}(X;\Lambda X) \to \dot{H}^{-1}_{R,\mathrm{loc}}(X;\Lambda X),\\ &P_1 \in \mathrm{Diff}^1(X;\Lambda X) + \mathrm{Diff}^1_\dagger(X;\Lambda X).\end{aligned} \tag{3.19}$$

We now consider adjoints; these are the main cause of difficulty for commutator constructions. For simplicity (as this is what we need below), we assume that ΛX is trivialized as above, and $A \in \Psi^m_{\mathrm{b}}(X;\Lambda X)$ is scalar with respect to this trivialization, i.e. is of the form $A_0 \otimes \mathrm{Id}$, $A_0 \in \Psi^m_{\mathrm{b}}(X)$, $\sigma_{\mathrm{b},m}(A_0) = a$. For $u = (u_\alpha)$, $v = (v_\beta)$ with respect to this trivialization, the fiber inner product takes the form

$$(u, v) = \sum_{\alpha\beta} \tilde{H}_{\alpha\beta} u_\alpha \overline{v_\beta},$$

hence the inner product on sections of ΛX supported in the coordinate chart takes the form

$$\langle u, v\rangle_{L^2(X,\Lambda X;|dh|\otimes \tilde{H})} = \int_X \sum_{\alpha\beta} \tilde{H}_{\alpha\beta} u_\alpha \overline{v_\beta} \, |dh| = \sum_{\alpha,\beta} \langle \tilde{H}_{\alpha\beta} u_\alpha, v_\beta\rangle_{L^2(X)}.$$

In particular, for A_0 formally self-adjoint with respect to the $L^2(X)$-inner product,

$$\begin{aligned}\langle u, Av\rangle_{L^2(X,\Lambda X;|dh|\otimes\tilde H)} &= \sum_{\alpha,\beta}\langle \tilde H_{\alpha\beta}u_\alpha, A_0 v_\beta\rangle_{L^2(X)} = \sum_{\alpha,\beta}\langle A_0\tilde H_{\alpha\beta}u_\alpha, v_\beta\rangle_{L^2(X)}\\ &= \sum_{\alpha,\beta}\langle \tilde H_{\alpha\beta}A_0u_\alpha, v_\beta\rangle_{L^2(X)} + \langle [A_0,\tilde H_{\alpha\beta}]u_\alpha, v_\beta\rangle_{L^2(X)}\\ &= \langle (A+C)u,v\rangle_{L^2(X,\Lambda X;|dh|\otimes\tilde H)},\end{aligned}$$

where $C=(C_{\alpha\nu})$ is the matrix form of $C\in\Psi_{\mathrm{b}}^{m-1}(X;\Lambda X)$ with respect to the trivialization, and

$$\begin{aligned}C_{\alpha\nu} &= \sum_\mu \tilde h_{\mu\nu}[A_0,\tilde H_{\alpha\mu}],\\ \sigma_{\mathrm{b},m-1}(C_{\alpha\nu}) &= \imath\sum_\mu \tilde h_{\mu\nu}\mathsf{H}_{\mathrm{b},\tilde H_{\alpha\mu}}a,\end{aligned}$$

In particular, notice that $\mathsf{H}_{\mathrm{b},\tilde H_{\alpha\mu}}$ is a vertical vector field on the vector bundle ${}^{\mathrm{b}}T^*X$, and at $F=\{x_1=\ldots=x_k=0\}$, it is a linear combination of the vector fields ∂_{ζ_j}:

$$\mathsf{H}_{\mathrm{b},\tilde H_{\alpha\mu}} = -\sum_j(\partial_{y_j}\tilde H_{\alpha\mu})\partial_{\zeta_j} - \sum_j x_j(\partial_{x_j}\tilde H_{\alpha\mu})\partial_{\sigma_j}. \tag{3.20}$$

Note that even the principal symbol C does *not* usually preserve boundary conditions, which means that we cannot estimate an expression like $\langle Cu,\Box u\rangle$ by using the PDE.

To summarize, we have proved the following result.

Proposition 3.8. *With ΛX trivialized as above, suppose that $A\in\Psi_{\mathrm{b}}^m(X;\Lambda X)$ is scalar, i.e. is of the form $A_0\otimes\mathrm{Id}$, $A_0\in\Psi_{\mathrm{b}}^m(X)$, A_0 is self-adjoint with respect to $|dh|$, $\sigma_{\mathrm{b},m}(A_0)=a$. Then there are smooth vector fields $V_{\alpha\beta}$ on ${}^{\mathrm{b}}T^*X$ such that*

$$\begin{aligned}&A^*-A=C,\ \sigma_{\mathrm{b},m-1}(C)_{\alpha\beta}=(V_{\alpha\beta}a),\\ &(\pi_{{}^{\mathrm{b}}T^*X\to X})_*(V_{\alpha\beta})=0,\ V_{\alpha\beta}|_{x_j=0}\sigma_j=0,\end{aligned} \tag{3.21}$$

where $\pi_{{}^{\mathrm{b}}T^*X\to X}:{}^{\mathrm{b}}T^*X\to X$ is the bundle projection.

Remark 3.9. It is important that A was not merely principally scalar. Indeed, if we replace A by $A+A_1$, $A_1\in\Psi_{\mathrm{b}}^{m-1}(X;\Lambda X)$, then A^*-A is replaced by $C_1=(A^*-A)+(A_1^*-A_1)$, with $A_1^*-A_1\in\Psi_{\mathrm{b}}^{m-1}(X;\Lambda X)$, so the principal symbol of C_1 is not determined by a.

In order to have the A_0 self-adjoint as above, it is convenient to introduce the following notation. For $A_0\in\Psi_{\mathrm{b}}(X)$, let $A_0^\dagger$ denote the $L^2(X)$-adjoint of A_0, and let

$$A^\dagger = A_0^\dagger\otimes\mathrm{Id}. \tag{3.22}$$

Thus, $A^\dagger$ is the $L^2(X;\Lambda X)$ adjoint of A if we put the Euclidean inner product on the fibers of ΛX using the trivialization, and use $|dh|$ as the density to integrate with respect to.

First, however, we need to discuss the action of $A\in\Psi_{\mathrm{bc}}^s(X;\Lambda X)$, $s\in\mathbb{R}$, on $C_R^\infty(X;\Lambda X)$ and $H_R^1(X;\Lambda X)$. Again, for simplicity assume that A is supported in a coordinate chart $\mathcal{U}$ as above. Then A has a normal family

$$\hat{N}_{\mathcal{S}_j}(A)(\sigma_j) : C^\infty(\mathcal{S}_j; \Lambda_{\mathcal{S}_j} X) \to C^\infty(\mathcal{S}_j; \Lambda_{\mathcal{S}_j} X),\ \sigma_j \in \mathbb{R},$$

at each boundary hypersurface $\mathcal{S}_j$, $j = 1, \dots, k$, of X intersecting $\mathcal{U}$, defined by

$$\hat{N}_{\mathcal{S}_j}(A)(\sigma_j) f = (x_j^{-i\sigma_j} A x_j^{i\sigma_j} u)|_{\mathcal{S}_j},\ u|_{\mathcal{S}_j} = f,$$

where $x_j^{-i\sigma_j} A x_j^{i\sigma_j} \in \Psi^s_{\mathrm{bc}}(X)$, hence $x_j^{-i\sigma_j} A x_j^{i\sigma_j} u \in C^\infty(X; \Lambda X)$, and the right hand side does not depend on the choice of u. This captures the behavior of A at $\mathcal{S}_j$ in that

$$(\forall \sigma_j \in \mathbb{R})\ \hat{N}_{\mathcal{S}_j}(A)(\sigma_j) = 0 \Rightarrow A \in x_j \Psi^s_{\mathrm{bc}}(X; \Lambda X).$$

We refer to [13] and [14] for more details.

In general, $A \in \Psi^s_{\mathrm{bc}}(X; \Lambda X)$ does not preserve $C^\infty_R(X; \Lambda X)$; for this to happen we need that for all j and all σ_j,

$$\hat{N}_{\mathcal{S}_j}(A)(\sigma_j) : C^\infty(\mathcal{S}_j; \Lambda_{\mathcal{S}_j,N} X) \to C^\infty(\mathcal{S}_j; \Lambda_{\mathcal{S}_j,N} X), \tag{3.23}$$

where $\Lambda_{\mathcal{S}_j,N} X$ denotes the bundle of normal forms at $\mathcal{S}_j$. If $s \geq 0$, then in addition $A \in \mathcal{L}(H^1(X; \Lambda X))$, and thus (recalling that $C^\infty_R(X; \Lambda X)$ is dense in $H^1_R(X; \Lambda X)$ in the H^1-norm) $A \in \mathcal{L}(H^1_R(X; \Lambda X))$. In particular, if A is scalar with respect to the coordinate trivialization, then $\hat{N}_{\mathcal{S}_j}(A)$ satisfies (3.23) for each j, and it follows that $A : C^\infty_R(X; \Lambda X) \to C^\infty_R(X; \Lambda X)$, and the corresponding mapping property on $H^1_R(X; \Lambda X)$ also holds for $s \geq 0$.

We are now ready to state our main commutator result. Recall that the topology on $\Psi^s_{\mathrm{bc}}(X; \Lambda X)$ is given by conormal (Besov) seminorms on the Schwartz kernel, or equivalently by symbol seminorms (which capture the near diagonal behavior) combined with C^∞ seminorms.

Proposition 3.10. *Let $A_0 \in \Psi^0_{\mathrm{b}}(X)$ with $\sigma_{\mathrm{b},0}(A_0) = a$, supported in the coordinate chart as in Lemma 3.7, and let $A = A_0 \otimes \mathrm{Id}$ with respect to the trivialization. Also let $s \in \mathbb{R}$, Λ_r be scalar, with symbol*

$$w_r = |\zeta_{n-k}|^{s+1/2} (1 + r|\zeta_{n-k}|^2)^{-s}\, \mathrm{Id}, \quad r \in [0, 1), \tag{3.24}$$

so $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}(X; \Lambda X)$ for $r > 0$ and the family $\mathcal{A} = \{A_r : r \in (0, 1)\}$ is uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X; \Lambda X)$. Then, for P as in (3.19), and with $A_r^\dagger$ as in (3.22), we have

(3.25)

$$\begin{aligned}
&\imath (A_r^\dagger A_r)^* P - \imath P A_r^\dagger A_r = Q_i^* C_{r,ij} Q_j + Q_i^* C'_{r,i} + C''_{r,j} Q_j + C_{r,0} + F_r,\\
&\sigma_{\mathrm{b},2s}(C_{r,ij}) = 2 w_r^2 (a V_{ij} a\, \mathrm{Id}_{\Lambda X} + a A_{ij} \tilde{V} a + a^2 \tilde{c}_{r,ij}),\\
&\sigma_{\mathrm{b},2s+1}(C'_{r,i}) = \sigma_{\mathrm{b},2s}(C''_{r,i}) = 2 w_r^2 (a V_i a\, \mathrm{Id}_{\Lambda X} + a m_i \tilde{V} a + a^2 \tilde{c}_{r,i}),\\
&\sigma_{\mathrm{b},2s+2}(C_{r,0}) = 2 w_r^2 (a V_0 a\, \mathrm{Id}_{\Lambda X} + a \tilde{p} \tilde{V} a + a^2 \tilde{c}_{r,0}),\\
&\tilde{c}_{r,ij} \in L^\infty((0,1]_r; S^{-1}(T^*X \setminus o; \mathrm{End}(\Lambda X))\\
&\tilde{c}_{r,i} \in L^\infty((0,1]_r; S^0(T^*X \setminus o; \mathrm{End}(\Lambda X)),\ \tilde{c}_{r,0} \in L^\infty((0,1]_r; S^1(T^*X \setminus o; \mathrm{End}(\Lambda X))\\
&V_{ij} \in \mathcal{V}(T^*X \setminus o),\ V_i \in \mathcal{V}(T^*X \setminus o),\\
&V_0 \in \mathcal{V}(T^*X \setminus o),\ \tilde{V} \in \mathcal{V}(T^*X \setminus o; \mathrm{End}(\Lambda X)),\\
&F_r \in L^\infty((0,1]_r; \mathrm{Diff}^2_\sharp \Psi^{2s-1}_{\mathrm{b}}(X; \Lambda X)),
\end{aligned}$$

$\tilde V, V_{ij}, V_i, V_0$ smooth homogeneous of degree $-1,-1,0,1$ respectively, $\tilde V$ is vertical and annihilates σ_i at $x_i=0$. Moreover,

$$
\begin{aligned}
V_{ij}|_F &= -A_{ij}(\partial_{\sigma_i}+\partial_{\sigma_j})+\sum_k(\partial_{y_k}A_{ij})\partial_{\zeta_k},\\
V_i|_F &= -A_{ij}\partial_{x_j},\ V_0|_F=-\mathsf{H}_{\mathrm{b},\tilde p}.
\end{aligned}
\tag{3.26}
$$

Proof. We use (3.15) and $[B^*,Q_i^*]=B^*Q_i^*-Q_i^*B^*=[Q_i,B]^*$, etc. For instance,

$$
\begin{aligned}
&(A_r^\dagger A_r)^*Q_i^*A_{ij}Q_j-Q_i^*A_{ij}Q_jA_r^\dagger A_r\\
&=[Q_i,A_r^\dagger A_r]^*A_{ij}Q_j\\
&\quad -Q_i^*A_{ij}[Q_j,A_r^\dagger A_r]+Q_i^*\Big((A_r^\dagger A_r)^*A_{ij}-A_{ij}A^\dagger A\Big)Q_j,
\end{aligned}
\tag{3.27}
$$

and

$$(A_r^\dagger A_r)^*A_{ij}-A_{ij}A_r^\dagger A_r=((A_r^\dagger A_r)^*-(A_r^\dagger A_r))A_{ij}+[A_r^\dagger A_r,A_{ij}].$$

We have already calculated $(A_r^\dagger A_r)^*-(A_r^\dagger A_r)$, including its principal symbol, in Proposition 3.8, while the principal symbol of $[A_r^\dagger A_r,A_{ij}]\in\Psi^{2s}_{\mathrm{bc}}(X)$ can be computed in $\Psi^{2s}_{\mathrm{bc}}(X)$. Further, $[Q_j,A_r^\dagger A_r]$ can be computed using Lemma 3.2. Note that in all these terms the principal symbol is given by applying a vector field to $a|\zeta_{n-k}|^{s+1/2}(1+r|\zeta_{n-k}|^2)^{-s}$. When $|\zeta_{n-k}|^{s+1/2}(1+r|\zeta_{n-k}|^2)^{-s}$ is differentiated, a is not differentiated, hence it contributes to the term $a^2\tilde q_r$. Thus, we need to collect the terms in which a is differentiated. Apart from the contribution of $(A_r^\dagger A_r)^*-(A_r^\dagger A_r)$, these are all principally scalar. The contribution of $(A_r^\dagger A_r)^*-(A_r^\dagger A_r)$ gives rise to the $\tilde V$ terms; the verticality and the property of annihilating σ_i follow from Proposition 3.8. The other terms in which a is differentiated give V_{ij}, V_i and V_0. In particular, the V_{ij} arises from the A_1 term in Lemma 3.2 when either Q_i^* or Q_j in $Q_i^*A_{ij}Q_j$ is commuted with $A^\dagger A$ as well as when A_{ij} is commuted with $A^\dagger A$; these give rise to the three terms for $V_{ij}|_F$ in (3.26) respectively. The V_i term arises both from the A_0 term in Lemma 3.2 when either Q_i^* or Q_j in $Q_i^*A_{ij}Q_j$ is commuted with $A^\dagger A$ (which gives $V_i|_F$ in (3.26)), as well as the A_1 term in Lemma 3.2 when Q_i^* in $Q_i^*M_i'$ or Q_i in M_iQ_i is commuted with $A^\dagger A$, as well as when M_i or M_i' is commuted with $A^\dagger A$: note that all but the first of these have vanishing principal symbol at F as $m_i|_F=0$. Finally, the V_0 term arises from the A_0 term in Lemma 3.2 when Q_i^* in $Q_i^*M_i'$ or Q_i in M_iQ_i is commuted with $A^\dagger A$, as well as when $\tilde P$ is commuted with $A^\dagger A$: all but the last of these have vanishing principal symbol at F as $m_i|_F=0$. □

4. Elliptic estimates

We collect here the elliptic estimates from [23], whose validity for forms with natural boundary conditions was discussed in [22], though they were not all stated as explicit lemmas there. Recall that

$$\langle .,.\rangle=\langle .,.\rangle_{L^2(X,\Lambda X;|dh|\otimes\tilde H)}$$

is a positive definite inner product, and $\langle .,.\rangle_H$ is the metric inner product with respect to H. We also let

$$\langle .,.\rangle_{\tilde H\otimes H}=\langle .,.\rangle_{L^2(X,\Lambda X\otimes T^*X;|dh|\otimes\tilde H\otimes H)}=\int(u,v)_{\tilde H\otimes H}\,|dh|,$$

where $(u, v)_{\tilde{H}\otimes H}$ is the inner product on $\Lambda X \otimes T^*X$ with the T^*X inner product given by H and the ΛX inner product given by $\tilde{H}$ – cf. the twisted Dirichlet form in [22, Equation (28)].

We recall the convention for 'local norms' from [23, Remark 4.1]:

Remark 4.1. Since X is non-compact and our results are microlocal, we may always fix a compact set $\tilde{K} \subset X$ and assume that all ps.d.o's have Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. We also let $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X such that $\tilde{U}$ has compact closure, and use the $H^1(\tilde{U})$ norm in place of the $H^1(X)$ norm to accommodate $u \in H^1_{\mathrm{loc}}(X)$. (We may instead take $\phi \in C^\infty_{\mathrm{comp}}(\tilde{U})$ identically 1 in a neighborhood of $\tilde{K}$, and use $\|\phi u\|_{H^1(X)}$.) Here we use the notation $\|.\|_{H^1_{\mathrm{loc}}(X;\Lambda X)}$ for $\|.\|_{H^1(\tilde{U};\Lambda X)}$ to avoid having to specify $\tilde{U}$; indeed we usually drop $(X; \Lambda X)$ and the subscript R as well. We also use $\|v\|_{\dot{H}^{-1}_{R,\mathrm{loc}}(X;\Lambda X)}$ for $\|\phi v\|_{\dot{H}^{-1}_R(X;\Lambda X)}$.

For all the estimates in this section, namely Lemmas 4.2, 4.4 and 4.6, we fix a coordinate chart, the corresponding trivialization of ΛX, and let ∇ be the connection on ΛX given by the trivialization, so

$$\nabla \in \mathrm{Diff}^1(X; \Lambda X; \Lambda X \otimes T^*X),$$

and

$$\sigma_1(\nabla)(w, \tilde{\xi}) = \imath \mathrm{Id} \otimes \tilde{\xi} \in \mathrm{End}(\Lambda_w X; \Lambda_w X \otimes T^*_w X),\ (w, \tilde{\xi}) \in T^*X \setminus o. \tag{4.1}$$

First the basic estimate on the Dirichlet form is:

Lemma 4.2. *(cf.* [23, Lemma 4.2]*; see* [22, Section 5] *for how the proof of* [23, Lemma 4.2] *needs to be modified in this case.) Suppose that P is as in* (3.19)*. Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, $U \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ is open, $K \subset U$, $\overline{\mathcal{U}} \subset \mathcal{U}_0$. Suppose that $\mathcal{A} = \{A_r :\ r \in (0, 1]\}$ is a bounded family of scalar ps.d.o's in $\Psi^s_{\mathrm{bc}}(X; \Lambda X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \subset K$, and with $A_r \in \Psi^{s-1}_{\mathrm{b}}(X; \Lambda X)$ for $r \in (0, 1]$. Then there are $G \in \Psi^{s-1/2}_{\mathrm{b}}(X; \Lambda X)$, $\tilde{G} \in \Psi^{s+1/2}_{\mathrm{b}}(X; \Lambda X)$ scalar with $\mathrm{WF}'_{\mathrm{b}}(G), \mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$ and $C_0 > 0$ such that for $r \in (0, 1]$, $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$ with $\mathrm{WF}^{1,s-1/2}_{\mathrm{b}}(u) \cap U = \emptyset$, $\mathrm{WF}^{-1,s+1/2}_{\mathrm{b}}(Pu) \cap U = \emptyset$, we have*

$$\begin{aligned} &|\langle \nabla A_r u, \nabla A_r u\rangle_{\tilde{H}\otimes H}| \\ &\qquad \leq C_0\big(\|u\|^2_{H^1_{\mathrm{loc}}} + \|Gu\|^2_{H^1} + \|Pu\|^2_{\dot{H}^{-1}_{R,\mathrm{loc}}} + \|\tilde{G}Pu\|^2_{\dot{H}^{-1}_R}\big). \end{aligned} \tag{4.2}$$

Remark 4.3. It is straightforward to modify this lemma so that we do not need to assume $U \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ is open, $K \subset U$, $\overline{\mathcal{U}} \subset \mathcal{U}_0$, rather simply $U \subset {}^{\mathrm{b}}S^*X$ is open, and also A_r needs to be merely principally scalar rather than scalar, and ∇ can be replaced by any first order differential operator with principal symbol (4.1). Indeed, we merely need to use a partition of unity and observe that any new terms introduced by the partition of unity and the other changes can be absorbed into $C_0(\|u\|^2_{H^1_{\mathrm{loc}}} + \|Gu\|^2_{H^1})$, by possibly adjusting C_0 and G (but keeping its properties). However, as the setting relevant to our estimates is local, and we choose A_r (which we choose to be scalar), this is not needed here.

A slightly strengthened version in terms of the order of $\tilde{G}$ (corresponding to the right hand side of the equation $Pu = f$) is:

Lemma 4.4. *(cf.* [23, Lemma 4.4]*; see* [22, Section 5] *for how the proof of* [23, Lemma 4.2] *needs to be modified in this case.) Suppose that P is as in* (3.19)*. Suppose that $K \subset {}^{\mathrm{b}}S^*X$ is compact, $U \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ is open, $K \subset U$, $\overline{\mathcal{U}} \subset \mathcal{U}_0$. Suppose that $\mathcal{A} = \{A_r : r \in (0,1]\}$ is a bounded family of scalar ps.d.o's in $\Psi^s_{\mathrm{bc}}(X;\Lambda X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \subset K$, and with $A_r \in \Psi^{s-1}_{\mathrm{b}}(X;\Lambda X)$ for $r \in (0,1]$. Then there are $G \in \Psi^{s-1/2}_{\mathrm{b}}(X;\Lambda X)$, $\tilde{G} \in \Psi^s_{\mathrm{b}}(X;\Lambda X)$ scalar with $\mathrm{WF}'_{\mathrm{b}}(G), \mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$ and $C_0 > 0$ such that for $\epsilon > 0$, $r \in (0,1]$, $u \in H^1_{R,\mathrm{loc}}(X;\Lambda X)$ with $\mathrm{WF}^{1,s-1/2}_{\mathrm{b}}(u) \cap U = \emptyset$, $\mathrm{WF}^{-1,s}_{\mathrm{b}}(Pu) \cap U = \emptyset$, we have*

$$|\langle \nabla A_r u, \nabla A_r u\rangle_{\tilde{H}\otimes H}| \leq \epsilon \|(D_{y_{n-k}} \otimes \mathrm{Id}) A_r u\|^2_{L^2} + C_0\big(\|u\|^2_{H^1_{\mathrm{loc}}} + \|Gu\|^2_{H^1} + \epsilon^{-1}\|Pu\|^2_{\dot{H}^{-1}_{R,\mathrm{loc}}} + \epsilon^{-1}\|\tilde{G}Pu\|^2_{\dot{H}^{-1}_R}\big).$$

We then recall the statement of microlocal elliptic regularity from [22]:

Proposition 4.5. *(Microlocal elliptic regularity, see* [22, Theorem 9]*.) Suppose that P is as in* (3.19)*, $m \in \mathbb{R}$ or $m = \infty$. Suppose $u \in H^1_{R,\mathrm{loc}}(X;\Lambda X)$. Then*

$$\mathrm{WF}^{1,m}_{\mathrm{b}}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}^{-1,m}_{\mathrm{b}}(Pu).$$

We also need a result giving more precise control of $\|Q_i A_r u\|$. Given Lemma 4.2, the proof proceeds exactly as in [23, Lemma 7.1], namely Equation (7.2) follows as there, and from that point the argument is a b-ps.d.o. argument (rather than a $\mathrm{Diff}_\sharp \Psi_{\mathrm{b}}$ argument), and is unaffected by the boundary conditions.

Lemma 4.6. *(cf.* [23, Lemma 7.1]*) Suppose that P is as in* (3.19)*. Suppose $u \in H^1_{R,\mathrm{loc}}(X;\Lambda X)$, and suppose that we are given $K \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ compact, $\overline{\mathcal{U}} \subset \mathcal{U}_0$, satisfying*

$$K \subset \mathcal{G} \cap {}^{\mathrm{b}}S^*_{F_{\mathrm{reg}}}X \setminus \mathrm{WF}^{-1,s+1/2}_{\mathrm{b}}(Pu).$$

Then there exist $\delta_0 > 0$ and $C_{\mathcal{G},K} > 0$ with the following property. Let $\delta < \delta_0$, $U \subset {}^{\mathrm{b}}S^*X$ open in a δ-neighborhood of K, and $\mathcal{A} = \{A_r : r \in (0,1]\}$ be a bounded family of scalar ps.d.o's in $\Psi^s_{\mathrm{bc}}(X;\Lambda X)$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \subset U$, and with $A_r \in \Psi^{s-1}_{\mathrm{b}}(X;\Lambda X)$ for $r \in (0,1]$.

Then there exist

$$G \in \Psi^{s-1/2}_{\mathrm{b}}(X;\Lambda X),\ \tilde{G} \in \Psi^{s+1/2}_{\mathrm{b}}(X;\Lambda X)$$

scalar with $\mathrm{WF}'_{\mathrm{b}}(G), \mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$ and $\tilde{C}_0 = \tilde{C}_0(\delta) > 0$ such that for all $r > 0$,

$$\sum_i \|Q_i A_r u\|^2 \leq C_{\mathcal{G},K}\delta\|(D_{y_{n-k}} \otimes \mathrm{Id})A_r u\|^2 + \tilde{C}_0\big(\|u\|^2_{H^1_{\mathrm{loc}}} + \|Gu\|^2_{H^1} + \|Pu\|^2_{\dot{H}^{-1}_{R,\mathrm{loc}}} + \|\tilde{G}Pu\|^2_{\dot{H}^{-1}_R}\big).$$

Here $Q_i = D_{x_i} \otimes \mathrm{Id}$ as in Lemma 3.7, and $D_{y_{n-k}} \otimes \mathrm{Id}$ is defined with respect to the same trivialization.

5. Normal propagation

We now turn to propagation of singularities at hyperbolic points. Recall from (1.3) that σ_j is the b-dual variable of x_j, $\hat{\sigma}_j = \sigma_j/|\zeta_{n-k}|$.

Proposition 5.1. *(Normal propagation.) Suppose that P is as in (3.19), i.e. consider P with relative boundary conditions. Let $q_0 = (0, y_0, 0, \zeta_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$, $F \cap U = U \cap \{x = 0\}$, and let*

$$\eta = -\sum_j \hat{\sigma}_j$$

be the function defined in the local coordinates discussed above, and suppose that $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$, $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(f)$, $f = Pu$. If there exists a conic neighborhood U of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ such that

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \tag{5.1}$$

then $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$.

In fact, if the wave front set assumptions are relaxed to $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(f)$ ($f = Pu$) and the existence of a conic neighborhood U of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ such that

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u), \tag{5.2}$$

then we can still conclude that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$.

Remark 5.2. The analogous result also holds for absolute boundary condition, either by a simple modification of the proof given below, or simply using the Hodge star operator to move between the boundary conditions.

As follows immediately from the proof given below, in (5.1) and (5.2), one can replace $\eta(q) < 0$ by $\eta(q) > 0$, i.e. one has the conclusion for either direction (backward or forward) of propagation.

Moreover, every neighborhood U of $q_0 = (y_0, \zeta_0) \in \mathcal{H} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X$ in $\dot{\Sigma}$ contains an open set of the form

$$\{q : \ |x(q)|^2 + |y(q) - y_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2 < \delta\}, \tag{5.3}$$

see [23, Equation (5.1)]. Note also that (5.1) implies the same statement with U replaced by any smaller neighborhood of q_0; in particular, for the set (5.3), provided that δ is sufficiently small. We can also assume by the same observation that $\mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu) \cap U = \emptyset$. Furthermore, with $\tilde{p} = \sigma_{\mathrm{b},2}(\tilde{P})$, we can also arrange that $\tilde{p}(x, y, \sigma, \zeta) > |(\sigma, \zeta)|^2 |\zeta_0|^{-2} \tilde{p}(q_0)/2$ on U since $\zeta_0 \cdot B(y_0)\zeta_0 = \tilde{p}(0, y_0, 0, \zeta_0) > 0$.

Proof. We first construct a commutant by defining its scalar principal symbol, a. This completely follows the scalar case, see [23, Proof of Proposition 6.2]. Next we show how to obtain the desired estimate.

So, as in [23, Proof of Proposition 6.2], let

$$\omega(q) = |x(q)|^2 + |y(q) - y_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2, \tag{5.4}$$

with $|.|$ denoting the Euclidean norm. For $\epsilon > 0$, $\delta > 0$, with other restrictions to be imposed later on, let

$$\phi = \eta + \frac{1}{\epsilon^2 \delta}\omega, \tag{5.5}$$

Let $\chi_0 \in C^\infty(\mathbb{R})$ be equal to 0 on $(-\infty, 0]$ and $\chi_0(t) = \exp(-1/t)$ for $t > 0$. Thus, $t^2 \chi_0'(t) = \chi_0(t)$ for $t \in \mathbb{R}$. Let $\chi_1 \in C^\infty(\mathbb{R})$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$, with $\chi_1' \geq 0$ satisfying $\chi_1' \in C^\infty_{\mathrm{comp}}((0, 1))$. Finally, let $\chi_2 \in C^\infty_{\mathrm{comp}}(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$, identically 1 on $[-c_1, c_1]$, where c_1 is such that if $|\hat{\sigma}|^2 < c_1/2$ in $\dot{\Sigma} \cap U_0$. Thus, $\chi_2(|\hat{\sigma}|^2)$ is a cutoff in $|\hat{\sigma}|$, with its support properties ensuring that $d\chi_2(|\hat{\sigma}|^2)$ is supported in $|\hat{\sigma}|^2 \in [c_1, 2c_1]$ hence outside $\dot{\Sigma}$ – it should be thought of as a factor that microlocalizes near the characteristic set but effectively commutes with P. Then, for $\digamma > 0$ large, to be determined, let

$$a = \chi_0(\digamma^{-1}(2 - \phi/\delta))\chi_1(\eta/\delta + 2)\chi_2(|\hat{\sigma}|^2); \tag{5.6}$$

so a is a homogeneous degree zero C^∞ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$. Indeed, as we see momentarily, for any $\epsilon > 0$, a has compact support inside this neighborhood (regarded as a subset of ${}^{\mathrm{b}}S^*X$, i.e. quotienting out by the $\mathbb{R}^+$-action) for δ sufficiently small, so in fact it is globally well-defined. In fact, on $\operatorname{supp} a$ we have $\phi \leq 2\delta$ and $\eta \geq -2\delta$. Since $\omega \geq 0$, the first of these inequalities implies that $\eta \leq 2\delta$, so on $\operatorname{supp} a$

$$|\eta| \leq 2\delta. \tag{5.7}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \eta) \leq 4\delta^2\epsilon^2. \tag{5.8}$$

In view of (5.4) and (5.3), this shows that given any $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$, a is supported in U. The role that $\digamma$ large plays (in the definition of a) is that it increases the size of the first derivatives of a relative to the size of a, hence it allows us to give a bound for a in terms of a small multiple of its derivative along the Hamilton vector field.

Now let $A_0 \in \Psi^0_{\mathrm{b}}(X)$ with $\sigma_{\mathrm{b},0}(A_0) = a$, supported in the coordinate chart, and let $A = A_0 \otimes \mathrm{Id}$ with respect to the trivialization. Also let Λ_r be scalar, have symbol

$$|\zeta_{n-k}|^{s+1/2}(1 + r|\zeta_{n-k}|^2)^{-s}\,\mathrm{Id}, \quad r \in [0, 1), \tag{5.9}$$

so $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}(X; \Lambda X)$ for $r > 0$ and it is uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X; \Lambda X)$. Then, for $r > 0$,

$$\langle \imath Pu, A_r^\dagger A_r u\rangle - \langle \imath A_r^\dagger A_r u, Pu\rangle = \langle \imath (A_r^\dagger A_r)^* Pu, u\rangle - \langle \imath P A_r^\dagger A_r u, u\rangle. \tag{5.10}$$

We can compute this using Proposition 3.10. We arrange the terms of the proposition so that the terms in which a vector field differentiates χ_1 are included in E_r, the terms in which a vector fields differentiates χ_2 are included in E'_r. Thus, we have

$$\imath(A_r^\dagger A_r)^*P - \imath P A_r^\dagger A_r = Q_i^* C_{r,ij} Q_j + Q_i^* C'_{r,i} + C''_{r,j} Q_j + C_{r,0} + E_r + E'_r + F_r, \tag{5.11}$$

with

(5.12)

$$\begin{aligned}
\sigma_{\mathrm{b},2s}(C_{r,ij}) &= w_r^2\Big(4\digamma^{-1}\delta^{-1}a|\zeta_{n-k}|^{-1}(-A_{ij} + \hat{f}_{ij} + \epsilon^{-2}\delta^{-1}f_{ij})\chi_0'\chi_1\chi_2 + a^2\tilde{c}_{r,ij}\Big),\\
\sigma_{\mathrm{b},2s+1}(C'_{r,i}) &= w_r^2\Big(\digamma^{-1}\delta^{-1}a(\hat{f}'_i + \delta^{-1}\epsilon^{-2}f'_i)\chi_0'\chi_1\chi_2 + a^2\tilde{c}'_{r,i}\Big),\\
\sigma_{\mathrm{b},2s+1}(C''_{r,i}) &= w_r^2\Big(\digamma^{-1}\delta^{-1}a(\hat{f}''_i + \delta^{-1}\epsilon^{-2}f''_i)\chi_0'\chi_1\chi_2 + a^2\tilde{c}''_{r,i}\Big),\\
\sigma_{\mathrm{b},2s+2}(C_{r,0}) &= w_r^2\Big(\digamma^{-1}\delta^{-1}|\zeta_{n-k}|a(\hat{f}_0 + \delta^{-1}\epsilon^{-2}f_0)\chi_0'\chi_1\chi_2 + a^2\tilde{c}_{r,0}\Big),
\end{aligned}$$

where f_{ij}, f'_i, f''_i and f_0 as well as $\hat{f}_{ij}, \hat{f}'_i, \hat{f}''_i$ and $\hat{f}_0$ are all smooth $\mathrm{End}(\Lambda X)$-valued functions on ${}^{\mathrm{b}}T^*X\setminus o$, homogeneous of degree 0 (independent of ϵ and δ). Moreover, f_{ij}, f'_i, f''_i, f_0 arise from when ω is differentiated in $\chi(\digamma^{-1}(2-\phi/\delta))$, and thus vanish when $\omega = 0$, while $\hat{f}_{ij}, \hat{f}'_i, \hat{f}''_i$ and $\hat{f}_0$ arise when η is differentiated in $\chi(\digamma^{-1}(2-\phi/\delta))$, and comprise all such terms with the exception of those arising from the ∂_{σ_i} and ∂_{σ_j} components of $V_{ij}|_F$ (which give A_{ij} on the first line above) hence are the sums of functions vanishing at $x = 0$ (corresponding to us only specifying the restrictions of the vector fields in (3.26) at F) and functions vanishing at $\hat\sigma = 0$ (when $|\zeta_{n-k}|^{-1}$ in $\eta = -\sum_j \sigma_j |\zeta_{n-k}|^{-1}$ is differentiated)[4].

In this formula we think of

$$-4\digamma^{-1}\delta^{-1}w_r^2 a|\zeta_{n-k}|^{-1}A_{ij}\chi_0'\chi_1\chi_2 \tag{5.13}$$

as the main term; note that $-A_{ij}$ is positive definite. Compared to this, the terms with a^2 are negligible, for they can all be bounded by

$$c\digamma^{-1}(\digamma^{-1}\delta^{-1}w_r^2 a|\zeta_{n-k}|^{-1}\chi_0'\chi_1\chi_2)$$

(cf. (5.13)), i.e. by a small multiple of $\digamma^{-1}\delta^{-1}w_r^2 a|\zeta_{n-k}|^{-1}\chi_0'\chi_1\chi_2$ when $\digamma$ is taken large, using that $2 - \phi/\delta \leq 4$ on $\operatorname{supp} a$ and

$$\chi_0(\digamma^{-1}t) = (\digamma^{-1}t)^2\chi_0'(\digamma^{-1}t) \leq 16\digamma^{-2}\chi_0'(\digamma^{-1}t),\ t \leq 4; \tag{5.14}$$

see the discussion in [22, Section 6] and [23] following Equation (6.19).

The vanishing condition on the f_{ij} and f_i ensures that, with $|.|$ denoting norms in $\operatorname{End}(\Lambda X)$, on $\operatorname{supp} a$,

$$|f_{ij}|, |f_i'|, |f_i''|, |f_0| \leq C\omega^{1/2} \leq 2C\epsilon\delta, \tag{5.15}$$

so the corresponding terms can thus be estimated using $w_r^2\digamma^{-1}\delta^{-1}a|\zeta_{n-k}|^{-1}\chi_0'\chi_1\chi_2$ provided ϵ^{-1} is not too large, i.e. there exists $\tilde\epsilon_0 > 0$ such that if $\epsilon > \tilde\epsilon_0$, the terms with f_{ij} can be treated as error terms.

On the other hand, we have

$$|\hat f_{ij}|, |\hat f_i'|, |\hat f_i''|, |\hat f_0| \leq C|x| + C|\hat\sigma| \leq C\omega^{1/2} + C|\hat\sigma| \leq 2C\epsilon\delta + C|\hat\sigma|. \tag{5.16}$$

Now, on $\dot\Sigma$, $|\hat\sigma| \leq 2|x|$ (for $|\sigma_j| = |x_j||\xi_j| \leq 2|x_j||\zeta_{n-k}|$ with U sufficiently small). Thus we can write $\hat f_{ij} = \hat f_{ij}^\sharp + \hat f_{ij}^\flat$ with $\hat f_{ij}^\flat$ supported away from $\dot\Sigma$ and $\hat f_{ij}^\sharp$ satisfying

$$|\hat f_{ij}^\sharp| \leq C|x| + C|\hat\sigma| \leq C'|x| \leq C'\omega^{1/2} \leq 2C'\epsilon\delta; \tag{5.17}$$

we can also obtain a similar decomposition for $\hat f_i', \hat f_i'', \hat f_0$.

Indeed, using (5.14) it is useful to rewrite (5.12) as

(5.18)

$$\begin{aligned}
\sigma_{\mathrm{b},2s}(C_{r,ij}) &= 4w_r^2\digamma^{-1}\delta^{-1}a|\zeta_{n-k}|^{-1}(-A_{ij} + \hat f_{ij} + \epsilon^{-2}\delta^{-1}f_{ij} + \digamma^{-1}\delta\hat c_{r,ij})\chi_0'\chi_1\chi_2,\\
\sigma_{\mathrm{b},2s+1}(C_{r,i}') &= w_r^2\delta^{-1}\digamma^{-1}a(\hat f_i' + \delta^{-1}\epsilon^{-2}f_i' + \digamma^{-1}\delta\hat c_{r,i}')\chi_0'\chi_1\chi_2,\\
\sigma_{\mathrm{b},2s+1}(C_{r,i}'') &= w_r^2\delta^{-1}\digamma^{-1}a(\hat f_i'' + \delta^{-1}\epsilon^{-2}f_i'' + \digamma^{-1}\delta\hat c_{r,i}'')\chi_0'\chi_1\chi_2,\\
\sigma_{\mathrm{b},2s+2}(C_{r,0}) &= w_r^2\delta^{-1}\digamma^{-1}a|\zeta_{n-k}|(\hat f_0 + \delta^{-1}\epsilon^{-2}f_0 + \digamma^{-1}\hat c_{r,0})\chi_0'\chi_1\chi_2,
\end{aligned}$$

with

- f_{ij}, f_i', f_i'' and f_0 are all smooth $\operatorname{End}(\Lambda X)$-valued functions on ${}^{\mathrm{b}}T^*X\setminus o$, homogeneous of degree 0, satisfying (5.15) (and are independent of $\digamma, \epsilon, \delta, r$),
- $\hat f_{ij}$, $\hat f_i'$, $\hat f_i''$ and $\hat f_0$ are all smooth $\operatorname{End}(\Lambda X)$-valued functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0, with $\hat f_{ij} = \hat f_{ij}^\sharp + \hat f_{ij}^\flat$, $\hat f_{ij}^\sharp, (\hat f_i')^\sharp, (\hat f_i'')^\sharp, \hat f_0^\sharp$ satisfying (5.17) (and are independent of $\digamma, \epsilon, \delta, r$), while $\hat f_{ij}^\flat, (\hat f_i')^\flat, (\hat f_i'')^\flat, \hat f_0^\flat$ is supported away from $\dot\Sigma$,

[4]Terms of the latter kind did not occur in [23] as time-translation invariance was assumed, but it does occur in [22], where the Lorentzian scalar setting is considered.

- and $\hat{c}_{r,ij}$, $\hat{c}'_{r,i}$, $\hat{c}''_{r,i}$ and $\hat{c}_{r,0}$ are all smooth $\mathrm{End}(\Lambda X)$-valued functions on ${}^{\mathrm{b}}T^*X \setminus o$, homogeneous of degree 0, uniformly bounded in $\epsilon, \delta, r, \digamma$.

In fact, it is useful to rewrite the leading term in $Q_i^* C_{r,ij} Q_j$, namely the term that contributes to $C_{r,ij}$ with symbol $-4w_r^2 \digamma^{-1}\delta^{-1} A_{ij} a |\zeta_{n-k}|^{-1}\chi_0'\chi_1\chi_2$ (cf. (5.13)), as a b-operator using the PDE, modulo lower order terms. Thus, let

$$b_r = 2w_r|\zeta_{n-k}|^{1/2}(\digamma\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

and let $\tilde{B}_r \in \Psi_{\mathrm{b}}^{s+1}(X;\Lambda X)$ with principal symbol $b_r \,\mathrm{Id}_{\Lambda X}$. Then let

$$C \in \Psi_{\mathrm{b}}^0(X;\Lambda X),\ \sigma_{\mathrm{b},0}(C) = |\zeta_{n-k}|^{-1}\tilde{p}^{1/2}\psi \,\mathrm{Id}_{\Lambda X}$$

where $\psi \in S^0_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o)$ is identically 1 on U considered as a subset of ${}^{\mathrm{b}}S^*X$; recall from Remark 5.2 that $\tilde{p}$ is bounded below by a positive quantity here.

If $\tilde{C}_r \in \Psi_{\mathrm{b}}^{2s}(X;\Lambda X)$ with principal symbol

$$\sigma_{\mathrm{b},2s}(\tilde{C}_r) = -4w_r^2\digamma^{-1}\delta^{-1}a|\zeta_{n-k}|^{-1}\chi_0'\chi_1\chi_2 \,\mathrm{Id}_{\Lambda X} = -|\zeta_{n-k}|^{-2} b_r^2 \,\mathrm{Id}_{\Lambda X}$$

and with $\tilde{C}_r^*$ preserving boundary conditions[5], then, with $\sim$ denoting operators differing by an element of $\mathrm{Diff}^2_\sharp \Psi_{\mathrm{b}}^{2s-1}(X;\Lambda X)$,

$$\begin{aligned}\sum_{ij} Q_i^* \tilde{C}_r A_{ij} Q_j &\sim \tilde{C}_r \sum_{ij} Q_i^* A_{ij} Q_j = \tilde{C}_r(P - \sum_i (M_i Q_i + Q_i^* M_i') - \tilde{P})\\ &\sim \tilde{C}_r P - B_r^* \tilde{B}_r (\sum_i (M_i Q_i + Q_i^* M_i')) + \tilde{B}_r^* C^* C \tilde{B}_r,\end{aligned}$$

so we deduce from (5.11)-(5.18) that[6]

$$\begin{aligned}&\imath(A_r^\dagger A_r)^* P - \imath P A_r^\dagger A_r\\ &\quad = R'P + \tilde{B}_r^*\big(C^*C + R_0 + \sum_i (Q_i^* R_i + \tilde{R}_i Q_i) + \sum_{ij} Q_i^* R_{ij} Q_j\big)\tilde{B}_r + R'' + E + E'\end{aligned} \tag{5.19}$$

with

$$\begin{aligned}&R_0 \in \Psi_{\mathrm{b}}^0(X;\Lambda X),\ R_i, \tilde{R}_i \in \Psi_{\mathrm{b}}^{-1}(X;\Lambda X),\ R_{ij} \in \Psi_{\mathrm{b}}^{-2}(X;\Lambda X),\\ &R' \in \Psi_{\mathrm{b}}^{-1}(X;\Lambda X),\ R'' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-2}(X;\Lambda X),\ E, E' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-1}(X;\Lambda X),\end{aligned}$$

with $\mathrm{WF}_{\mathrm{b}}'(E) \subset \eta^{-1}((-\infty,-\delta]) \cap U$, $\mathrm{WF}_{\mathrm{b}}'(E') \cap \dot{\Sigma} = \emptyset$, $(R')^*$ preserving the boundary conditions, and with $r_0 = \sigma_{\mathrm{b},0}(R_0)$, $r_i = \sigma_{\mathrm{b},-1}(R_i)$, $\tilde{r}_i = \sigma_{\mathrm{b},-1}(\tilde{R}_i)$, $r_{ij} \in \sigma_{\mathrm{b},-2}(R_{ij})$, $|.|$ denoting endomorphism norms,

$$\begin{aligned}&|r_0| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}),\ |\zeta_{n-k} r_i| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}),\\ &|\zeta_{n-k}\tilde{r}_i| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}),\ |\zeta_{n-k}^2 r_{ij}| \le C_2(\delta\epsilon + \epsilon^{-1} + \delta\digamma^{-1}).\end{aligned}$$

This is exactly the form-valued version of [23, Equation (6.18)], except the presence of the $\delta\digamma^{-1}$ term which however is treated like the $\epsilon\delta$ term for $\digamma$ sufficiently large, hence the rest of the proof proceeds exactly as in that paper – the only point where one needs to use the boundary conditions from here on is that $(R')^*$ preserves these,

[5]But not necessarily C_r; we can construct C_r by first constructing an operator preserving the boundary conditions and with symbol equal the the adjoint of the desired symbol of C_r, and then taking its adjoint.

[6]The $f_{ij}^\sharp$ terms are included in R_{ij}, while the $f_{ij}^\flat$ terms are included in E', and similarly for the other analogous terms in f_i', f_i'', f_0.

so $\langle Pu, (R')^* u\rangle = \langle f, (R')^* u\rangle$. In order to eliminate duplication of the rest of the argument of [23, Proof of Proposition 6.2], we do not repeat it here. $\square$

6. Tangential propagation

We now consider tangential propagation.

Proposition 6.1. *(Tangential propagation.) Suppose that P is as in (3.19), i.e. consider P with relative boundary conditions. Let $\mathcal{U}_0$ be a coordinate chart in X, $\mathcal{U}$ open with $\overline{\mathcal{U}} \subset \mathcal{U}_0$. Let $u \in H^1_{R,\mathrm{loc}}(X; \Lambda X)$, and let $\tilde\pi : T^*X \to T^*F$ be the coordinate projection*

$$\tilde\pi : (x, y, \xi, \zeta) \mapsto (y, \zeta).$$

Given $K \subset {}^{\mathrm{b}}S^*_{\mathcal{U}}X$ compact with

$$K \subset (\mathcal{G} \cap {}^{\mathrm{b}}T^*_{F_{\mathrm{reg}}}X) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(f),\ f = Pu, \tag{6.1}$$

there exist constants $C_0 > 0$, $\delta_0 > 0$ such that the following holds. If $q_0 = (y_0, \zeta_0) \in K$, $\alpha_0 = \hat\pi^{-1}(q_0)$, $W_0 = \tilde\pi_*|_{\alpha_0}\mathsf{H}_p$ considered as a constant vector field in local coordinates, and for some $0 < \delta < \delta_0$, $C_0\delta \le \epsilon < 1$ and for all $\alpha = (x, y, \xi, \zeta) \in \Sigma$

$$\begin{aligned} &\alpha \in T^*X \text{ and } |\tilde\pi(\alpha - \alpha_0 - \delta W_0)| \le \epsilon\delta \text{ and } |x(\alpha)| \le \epsilon\delta \\ &\qquad \Rightarrow \pi(\alpha) \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u), \end{aligned} \tag{6.2}$$

then $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$.

Proof. Again, we first construct the symbol a of our commutator following [23, Proof of Proposition 7.3] as corrected in [21]. Note that

$$W_0(q_0) = \mathsf{H}_{\tilde p}(q_0),$$

and let

$$W = |\zeta_{n-k}|^{-1} W_0,$$

so W is homogeneous of degree zero (with respect to the $\mathbb{R}^+$-action on the fibers of $T^*F \setminus o$). We can use

$$\tilde\eta = (\mathrm{sgn}(\zeta_{n-k})_0)(y_{n-k} - (y_{n-k})_0)$$

now to measure propagation, since $\zeta_{n-k}^{-1}\mathsf{H}_{\tilde p}(y_{n-k}) = 2 > 0$ at q_0, so $\mathsf{H}_{\tilde p}\tilde\eta$ is $2|\zeta_{n-k}| > 0$ at q_0.

First, we require

$$\rho_1 = \tilde p(y, \hat\zeta) = |\zeta_{n-k}|^{-2}\tilde p(y, \zeta);$$

note that $d\rho_1 \neq 0$ at q_0 for $\zeta \neq 0$ there, but $\mathsf{H}_{\tilde p}\tilde p \equiv 0$, so

$$W\rho_1(q_0) = 0.$$

Next, since $\dim F = n - k$, $\dim T^*F = 2n - 2k$, hence $\dim S^*F = 2n - 2k - 1$. With a slight abuse of notation, we also regard q_0 as a point in S^*F – recall that $S^*F = (T^*F \setminus o)/\mathbb{R}^+$. We can also regard W as a vector field on S^*F in view of its homogeneity. As W does not vanish as a vector in $T_{q_0}S^*F$ in view of $W\tilde\eta(q_0) \neq 0$, $\tilde\eta$ being homogeneous degree zero, hence a function on S^*F, the kernel of W in $T^*_{q_0}S^*F$ has dimension $2n - 2k - 2$. Thus there exist ρ_j, $j = 2, \dots, 2n - 2k - 2$ be homogeneous degree zero functions on T^*F (hence functions on S^*F) such that

$$\begin{aligned} &\rho_j(q_0) = 0,\ j = 2, \dots, 2n - 2k - 2, \\ &W\rho_j(q_0) = 0,\ j = 2, \dots, 2n - 2k - 2, \\ &d\rho_j(q_0),\ j = 1, \dots, 2n - 2k - 2 \text{ are linearly independent at } q_0. \end{aligned} \tag{6.3}$$

By dimensional considerations, $d\rho_j(q_0)$, $j = 1, \ldots, 2n-2k-2$, together with $d\tilde{\eta}$ span the cotangent space of S^*F at q_0, i.e. of the quotient of T^*F by the $\mathbb{R}^+$-action.

Hence,

$$|\zeta_{n-k}|^{-1} W_0 \rho_j = \sum_{i=1}^{2n-2k-2} \tilde{F}_{ij}\rho_j + \tilde{F}_{2n-2k-1}\tilde{\eta},\ j = 2, \ldots, 2n-2k-2,$$

with $\tilde{F}_{ij}$ smooth, $i = 1, \ldots, 2n-2k-2$, $j = 2, \ldots, 2n-2k-2$. Then we extend ρ_j to a function on ${}^{\mathrm{b}}T^*X \setminus o$ (using the coordinates (x, y, σ, ζ)), and conclude that (6.4)

$$|\zeta_{n-k}|^{-1}\mathsf{H}_{\tilde{p}}\rho_j = \sum_{l=1}^{2n-2k-2} \tilde{F}_{jl}\rho_l + \tilde{F}_{2n-2k-1}\tilde{\eta} + \sum_l \tilde{F}_{0,jl}x_l,\ j = 2, \ldots, 2n-2k-2,$$

with $\tilde{F}_{jl}$, $\tilde{F}_{0,jl}$ smooth, $\partial_{x_l}\rho_j$ vanishes. Similarly,

$$|\zeta_{n-k}|^{-1}\mathsf{H}_{\tilde{p}}\tilde{\eta} = 2 + \sum_{l=1}^{2n-2k-2} \hat{F}_l\rho_l + \hat{F}_{2n-2k-1}\tilde{\eta} + \sum_l \hat{F}_{0,l}x_l, \tag{6.5}$$

with $\hat{F}_l$, $\hat{F}_{0,l}$ smooth, $\partial_{x_l}\tilde{\eta}$ vanishes, as do vertical derivatives of $\tilde{\eta}$.

Let

$$\omega = |x|^2 + \sum_{j=1}^{2n-2k-2} \rho_j^2. \tag{6.6}$$

Finally, we let

$$\phi = \tilde{\eta} + \frac{1}{\epsilon^2\delta}\omega, \tag{6.7}$$

and define a by

$$a = \chi_0(F^{-1}(2 - \phi/\delta))\chi_1((\tilde{\eta}\delta)/\epsilon\delta + 1)\chi_2(|\sigma|^2/\zeta_{n-k}^2), \tag{6.8}$$

with χ_0, χ_1 and χ_2 as in the case of the normal propagation estimate, stated after (5.5). We always assume $\epsilon < 1$, so on $\operatorname{supp} a$ we have

$$\phi \leq 2\delta \text{ and } \tilde{\eta} \geq -\epsilon\delta - \delta \geq -2\delta.$$

Since $\omega \geq 0$, the first of these inequalities implies that $\tilde{\eta} \leq 2\delta$, so on $\operatorname{supp} a$

$$|\tilde{\eta}| \leq 2\delta. \tag{6.9}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \tilde{\eta}) \leq 4\delta^2\epsilon^2. \tag{6.10}$$

Moreover, on $\operatorname{supp} d\chi_1$,

$$\tilde{\eta} \in [-\delta - \epsilon\delta, -\delta],\ \omega^{1/2} \leq 2\epsilon\delta, \tag{6.11}$$

so this region lies in (6.2) after ϵ and δ are both replaced by appropriate constant multiples, namely the present δ should be replaced by $\delta/(2|(\zeta_{n-k})_0|)$.

We proceed as in the case of hyperbolic points, letting $A_0 \in \Psi_{\mathrm{b}}^0(X)$ with $\sigma_{\mathrm{b},0}(A_0) = a$, supported in the coordinate chart, and letting $A = A_0 \otimes \mathrm{Id}$ with respect to the trivialization. Also let Λ_r be scalar with symbol

$$|\zeta_{n-k}|^{s+1/2}(1 + r|\zeta_{n-k}|^2)^{-s}\,\mathrm{Id}, \quad r \in [0, 1), \tag{6.12}$$

so $A_r = A\Lambda_r \in \Psi^0_{\mathrm{b}}(X;\Lambda X)$ for $r>0$ and it is uniformly bounded in $\Psi^{s+1/2}_{\mathrm{bc}}(X;\Lambda X)$. Then, for $r>0$,

$$\langle \imath Pu, A_r^\dagger A_r u\rangle - \langle \imath A_r^\dagger A_r u, Pu\rangle = \langle \imath (A_r^\dagger A_r)^* Pu, u\rangle - \langle \imath P A_r^\dagger A_r u, u\rangle. \tag{6.13}$$

We can compute this using Proposition 3.10, noting that the vector fields $\tilde V_{ij}$, $\tilde V_i$ and $\tilde V_0$ satisfy

$$\tilde V_\bullet \chi(\digamma^{-1}(2-\phi/\delta)) = \digamma^{-1}\epsilon^{-2}\delta^{-2}(\tilde V_\bullet \omega)\cdot \chi_0'(\digamma^{-1}(2-\phi/\delta)),$$

since $\tilde\eta$ is the pull-back of a function on the base and $\tilde V_\bullet$ is vertical.

We arrange the terms of the proposition so that the terms in which a vector field differentiates χ_1 are included in E_r, the terms in which a vector fields differentiates χ_2 are included in E_r'. Thus, we have

$$\imath(A_r^\dagger A_r)^*P - \imath P A_r^\dagger A_r = Q_i^* C_{r,ij} Q_j + Q_i^* C'_{r,i} + C''_{r,j} Q_j + C_{r,0} + E_r + E_r' + F_r, \tag{6.14}$$

with

(6.15)

$$\begin{aligned}
\sigma_{\mathrm{b},2s}(C_{r,ij}) &= w_r^2\Big(\digamma^{-1}\delta^{-1}a|\zeta_{n-k}|^{-1}\big(f_{1,ij}+\epsilon^{-2}\delta^{-1}f_{0,ij}\big)\chi_0'\chi_1\chi_2 + a^2\tilde c_{r,ij}\Big),\\
\sigma_{\mathrm{b},2s+1}(C^\bullet_{r,i}) &= w_r^2\Big(\digamma^{-1}\delta^{-1}a\big(f^\bullet_{1,i}+\delta^{-1}\epsilon^{-2}f^\bullet_{0,i}\big)\chi_0'\chi_1\chi_2 + a^2\tilde c^\bullet_{r,i}\Big),\ \bullet = ',''\\
\sigma_{\mathrm{b},2s+2}(C_{r,0}) &= w_r^2\Big(|\zeta_{n-k}|\digamma^{-1}\delta^{-1}a\big(4+f_{1,0}+\delta^{-1}\epsilon^{-2}f_{0,0}\big)\chi_0'\chi_1\chi_2 + a^2\tilde c_{r,0}\Big),
\end{aligned}$$

where $f_{k,ij}$, $f_{k,i}$ and $f_{k,0}$ are all smooth $\mathrm{End}(\Lambda X)$-valued functions on ${}^{\mathrm{b}}T^*X\setminus o$, homogeneous of degree 0 (independent of $\epsilon\in[0,1)$, $k=0,1$, and $\delta\in[0,1]$), arising when ω is differentiated in $\chi_0(\digamma^{-1}(2-\phi/\delta))$ for $k=0$ and when $\tilde\eta$ is differentiated for $k=1$, and all such terms are included in these except $f_{1,0}$ is missing $|\zeta_{n-k}|^{-1}\mathsf{H}_{\tilde p}\tilde\eta(q_0)=2>0$ extended as the constant function near q_0. Moreover, as $V_\bullet\rho^2 = 2\rho V_\bullet\rho$ for any function ρ, the terms with $k=0$ have vanishing factors of ρ_l, resp. x_l, with the structure of the remaining factor dictated by the form of $V_\bullet\rho_l$, resp. $V_\bullet x_l$, and similarly for $\tilde V$ (which only affects the $k=0$ terms as discussed earlier). Thus, using (6.4) to compute $f_{0,0}$, (6.5) to compute $f_{1,0}$, and recalling that $\tilde V$ appears in the form $\tilde p\tilde V = \rho_1\tilde V$ in $f_{0,0}$,

$$\begin{aligned}
f_{0,ij} &= \sum_k \rho_k\tilde f_{0,ijk} + \sum_k x_k\hat f_{0,ijk},\\
f^\bullet_{0,i} &= \sum_l \rho_k\tilde f^\bullet_{0,ik} + \sum_k x_k\hat f^\bullet_{0,ik},\ \bullet = ','',\\
f_{0,0} &= \sum_{kl}\rho_k\rho_l\tilde f_{0,0kl} + \sum_{kl}\rho_k x_l\hat f_{0,0kl} + \sum_{kl}x_k x_l\check f_{0,0kl}\sum_k\rho_k\tilde\eta f^\sharp_{0,0k} + \sum_k x_k\tilde\eta f^\flat_{0,0k},\\
f_{1,0} &= \sum_k x_k\tilde f_{1,0k} + \sum_k \rho_k\hat f_{1,0k} + \tilde\eta f^\flat_{1,0},
\end{aligned}$$

with $\tilde f_{0,ijk}$, etc., smooth. With $|.|$ denoting norms in $\mathrm{End}(\Lambda X)$, we deduce that

$$\epsilon^{-2}\delta^{-1}|f_{0,ij}| \le C\epsilon^{-1},\ |f_{1,ij}|\le C, \tag{6.16}$$

while

$$\epsilon^{-2}\delta^{-1}|f^\bullet_{0,i}| \le C\epsilon^{-1},\ |f^\bullet_{1,i}|\le C, \tag{6.17}$$

$\bullet = ', ''$, and

$$\epsilon^{-2}\delta^{-1}|f_{0,0}| \leq C\epsilon^{-1}\delta,\ |f_{1,0}| \leq C\delta. \tag{6.18}$$

We remark that although thus far we worked with a single $q_0 \in K$, the same construction works with q_0 in a neighborhood $U_{q_0'}$ of a fixed $q_0' \in K$, with a *uniform* constant C. In view of the compactness of K, this suffices (by the rest of the argument we present below) to give the uniform estimate of the proposition.

For a small constant $c_0 > 0$ to be determined, which we may assume to be less than C, we demand below that the expressions on the right hand sides of (6.16) are bounded by $c_0(\epsilon\delta)^{-1}$, those on the right hand sides of (6.17) are bounded by $c_0(\epsilon\delta)^{-1/2}$, while those on the right hand sides of (6.18) are bounded by c_0. This demand is due to the appearance of two, resp. one, resp. zero, factors of Q_i in (6.14) for the terms whose principal symbols are affected by these, taking into account that in view of Lemma 4.6 we can estimate $\|Q_i v\|$ by $C_{\mathcal{G},K}(\epsilon\delta)^{1/2}\|(D_{y_{n-k}}\otimes \mathrm{Id})v\|$ if v is microlocalized to a $\epsilon\delta$-neighborhood of $\mathcal{G}$, which is the case for us with $v = A_r u$ in terms of support properties of a.

Thus, we need[7]

$$\begin{aligned} &C\epsilon^{-1} \leq c_0\delta^{-1}\epsilon^{-1},\ C \leq c_0\delta^{-1}\epsilon^{-1};\\ &C\epsilon^{-1} \leq c_0\delta^{-1/2}\epsilon^{-1/2},\ C \leq c_0\delta^{-1/2}\epsilon^{-1/2};\\ &C\epsilon^{-1}\delta \leq c_0,\ C\delta \leq c_0; \end{aligned} \tag{6.19}$$

here the semicolons correspond to the breakup corresponding to the various lines of (6.16)-(6.18). By the first inequality on the last line, we must have $\epsilon \geq (C/c_0)\delta$, by the first equation on the second line $\epsilon \geq (C/c_0)^2\delta$, which implies the preceeding equation as $c_0 < C$. Assuming

$$(C/c_0)^2\delta \leq \epsilon \leq 1, \tag{6.20}$$

all the preceeding equations hold for sufficiently small δ, namely

$$\delta < (c_0/C)^2, \tag{6.21}$$

as we check below.

Thus, with ϵ, δ satisfying (6.20) and (6.21), hence $\delta^{-1} > (C/c_0)^2 > C/c_0$, (6.16)-(6.18) give that

$$\epsilon^{-2}\delta^{-1}|f_{0,ij}| \leq C\epsilon^{-1} \leq c_0\delta^{-1}\epsilon^{-1},\ |f_{1,ij}| \leq C \leq c_0\delta^{-1} \leq c_0\delta^{-1}\epsilon^{-1}, \tag{6.22}$$

while

$$\epsilon^{-2}\delta^{-1}|f^{\bullet}_{0,i}| \leq C\epsilon^{-1} \leq c_0\delta^{-1/2}\epsilon^{-1/2},\ |f^{\bullet}_{1,i}| \leq C \leq c_0\delta^{-1/2} \leq c_0\delta^{-1/2}\epsilon^{-1/2}, \tag{6.23}$$

$\bullet = ', ''$, and

$$\epsilon^{-2}\delta^{-1}|f_{0,0}| \leq C\epsilon^{-1}\delta \leq c_0,\ |f_{1,0}| \leq C\delta \leq c_0. \tag{6.24}$$

Again, the terms with a^2 in (6.14) are negligible, for they can all be rewritten using (5.14).

Let $\tilde{B}_r \in \Psi_{\mathrm{b}}^{s+1}(X;\Lambda X)$ with $\sigma_{\mathrm{b},0}(\tilde{B}_r) = \tilde{b}_r\,\mathrm{Id}$,

$$\tilde{b}_r = 2w_r|\zeta_{n-k}|^{1/2}(\digamma\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2 \in C^\infty({}^{\mathrm{b}}T^*X \setminus o).$$

[7] In the sense that if these hold, the right hand side of (6.16)-(6.18) satisfy the desired estimates.

Again, χ_0 stands for $\chi_0(F^{-1}(2-\frac{\phi}{\delta}))$, etc. Also, let $C \in \Psi_{\mathrm{b}}^0(X;\Lambda X)$ have symbol $\sigma_{\mathrm{b},0}(C) = \psi \operatorname{Id}$ where $\psi \in S^0_{\mathrm{hom}}({}^{\mathrm{b}}T^*X \setminus o)$ is identically 1 on U considered as a subset of ${}^{\mathrm{b}}S^*X$. Then

$$\begin{aligned}&\imath(A_r^\dagger A_r)^*P - \imath P A_r^\dagger A_r\\ &\quad= \tilde{B}_r^*(C^*C + R_0 + \sum_i (Q_i^* R_i + \tilde{R}_i Q_i) + \sum_{ij} Q_i^* R_{ij} Q_j)\tilde{B}_r + R'' + E + E'\end{aligned}$$

with

$$\begin{gathered}R_0 \in \Psi_{\mathrm{b}}^0(X;\Lambda X),\ R_i, \tilde{R}_i \in \Psi_{\mathrm{b}}^{-1}(X;\Lambda X),\ R_{ij} \in \Psi_{\mathrm{b}}^{-2}(X;\Lambda X),\\ R'' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-2}(X;\Lambda X),\ E, E' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-1}(X;\Lambda X),\end{gathered}$$

with $\mathrm{WF}_{\mathrm{b}}'(E) \subset \tilde{\eta}^{-1}((-\infty,-\delta]) \cap U$, $\mathrm{WF}_{\mathrm{b}}'(E') \cap \dot{\Sigma} = \emptyset$, and with $r_0 = \sigma_{\mathrm{b},0}(R_0)$, $r_i = \sigma_{\mathrm{b},-1}(R_i)$, $\tilde{r}_i = \sigma_{\mathrm{b},-1}(\tilde{R}_i)$, $r_{ij} \in \sigma_{\mathrm{b},-2}(R_{ij})$, satisfying

$$\begin{aligned}&|r_0| \le 2c_0 + C_2\delta F^{-1},\\ &|\zeta_{n-k} r_i|, |\zeta_{n-k}\tilde{r}_i| \le 2c_0\delta^{-1/2}\epsilon^{-1/2} + C_2\delta F^{-1},\\ &|\zeta_{n-k}^2 r_{ij}| \le 2c_0\delta^{-1}\epsilon^{-1} + C_2\delta F^{-1}.\end{aligned} \tag{6.25}$$

Here the $C_2\delta F^{-1}$ terms arise by incorporating the a^2-terms of (6.15), using (5.14), as in the normal case.

These are exactly the formed-valued versions of the result of the second displayed equation after [23, Equation (7.16)], as corrected in [21], with the small (at this point arbitrary) constant c_0 replacing some constants given there in terms of ϵ and δ: in [21]. Thus, the rest of the argument thus proceeds as in [23, Proof of Proposition 7.3], taking into account [21]. For example, the estimate on $\tilde{R}_i$ takes the following form.

We let, as in [23], $T \in \Psi_{\mathrm{b}}^{-1}(X;\Lambda X)$ be elliptic with principal symbol $|\zeta_{n-k}|^{-1}$ on a neighborhood of $\operatorname{supp} a$, $T^- \in \Psi_{\mathrm{b}}^1(X;\Lambda X)$ a parametrix, so $T^-T = \operatorname{Id} + F$, $F \in \Psi_{\mathrm{b}}^{-\infty}(X;\Lambda X)$. In view of (6.25) there exists $R_i' \in \Psi_{\mathrm{b}}^{-1}(X;\Lambda X)$ such that for any $\gamma > 0$,

$$\begin{aligned}\|\tilde{R}_i w\| &= \|\tilde{R}_i(T^-T - F)w\| \le \|(\tilde{R}_i T^-)(Tw)\| + \|\tilde{R}_i F w\|\\ &\le 2(2c_0\delta^{-1/2}\epsilon^{-1/2} + C_2\delta F^{-1})\|Tw\| + \|R_i' T w\| + \|\tilde{R}_i F w\|\end{aligned}$$

for all w with $Tw \in L^2(X;\Lambda X)$, hence

$$\begin{aligned}|\langle \tilde{R}_i Q_i v, v\rangle| \le& 2(2c_0\delta^{-1/2}\epsilon^{-1/2} + C_2\delta F^{-1})\|TQ_i v\|\,\|v\|\\ &+ 2\gamma\|v\|^2 + \gamma^{-1}\|R_i' T Q_i v\|^2 + \gamma^{-1}\|F_i Q_i v\|^2,\end{aligned}$$

with $F_i \in \Psi_{\mathrm{b}}^{-\infty}(X;\Lambda X)$. Now we use that a is microlocalized in an $\epsilon\delta$-neighborhood of $\mathcal{G}$, hence the same can be arranged for T: $\mathcal{G}$ is given by $\rho_1 = 0$, $x = 0$, and we are microlocalized to the region where $|\rho_1| \le 2\epsilon\delta$, $|x| \le 2\epsilon\delta$. For $v = \tilde{B}_r u$, Lemma 4.6 thus gives (taking into account that we need to estimate $\|TQ_i v\|$ rather than its square)

$$\begin{aligned}|\langle \tilde{R}_i Q_i v, v\rangle| \le\ & 4C_{\mathcal{G},K}(2c_0\delta^{-1/2}\epsilon^{-1/2} + C_2\delta F^{-1})(\epsilon\delta)^{1/2}\|\tilde{B}_r u\|^2\\ &+ C_0\gamma^{-1}\big(\|G\tilde{B}_r u\|^2_{H^1} + \|\tilde{B}_r u\|^2_{H^1_{\mathrm{loc}}} + \|\tilde{G}Pu\|^2_{\dot{H}^{-1}} + \|Pu\|^2_{H^1_{\mathrm{loc}}}\big)\\ &+ 3\gamma\|\tilde{B}_r u\|^2 + \gamma^{-1}\|R_i' T D_{x_i}\tilde{B}_r u\|^2 + \gamma^{-1}\|F_i D_{x_i}\tilde{B}_r u\|^2.\end{aligned}$$

The first term is the main term of interest, and its coefficient satisfies

$$4C_{\mathcal{G},K}(2c_0\delta^{-1/2}\epsilon^{-1/2}+C_2\delta F^{-1})(\epsilon\delta)^{1/2}\leq C_3(c_0+\delta F^{-1})$$

with some C_3 depending only on $C_{\mathcal{G},K},C,C_2$. We can proceed as in [23] if given a prescribed quantity $c_1>0$ we can find $\delta_0>0$ such that this coefficient is less than c_1 for $\delta\in(0,\delta_0)$ (and can do the same with analogous terms for R_{ij}, R_i and R_0, which however are easily handled by a similar argument). But we can indeed achieve this by first choosing c_0 sufficiently small, then δ_0 sufficiently small according (6.21) (i.e. take $\delta_0<(c_0/C)^2$), finally F sufficiently large. The proof is thus finished as in in [23, Proof of Proposition 7.3], thus completing the proof. □

7. Propagation of singularities

Recall from (3.19) that we assume that

$$\begin{aligned}&P=\Box+P_1:H^1_{R,\mathrm{loc}}(X;\Lambda X)\to\dot H^{-1}_{R,\mathrm{loc}}(X;\Lambda X),\\&P_1\in\mathrm{Diff}^1(X;\Lambda X)+\mathrm{Diff}^1_\dagger(X;\Lambda X).\end{aligned}\tag{7.1}$$

The theorem on propagation of singularities is the following.

Theorem 7.1 (Slightly strengthened restatement of Theorem 1.3)**.** *Suppose that P is as in* (3.19)*, i.e. consider P with relative boundary conditions, $Pu=f$. If $u\in H^1_{R,\mathrm{loc}}(X;\Lambda X)$, then for all $s\in\mathbb{R}^+\cup\{\infty\}$* (*with the convention $\infty+1=\infty$*)*,*

$$\mathrm{WF}^{1,s}_{\mathrm{b}}(u)\setminus\mathrm{WF}^{-1,s+1}_{\mathrm{b}}(f)\subset\dot\Sigma,$$

and it is a union of maximally extended generalized broken bicharacteristics of P in $\dot\Sigma\setminus\mathrm{WF}^{-1,s+1}_{\mathrm{b}}(Pu)$.

The same conclusion holds with relative boundary conditions replaced by absolute boundary conditions.

Proof. The proof proceeds as in [23, Proof of Theorem 8.1], since the Propositions 5.1 and 6.1 are complete analogues of [23, Proposition 6.2] and [23, Proposition 7.3]. Given the results of the previous sections, this argument itself is only a slight modification of an argument originally due to Melrose and Sjöstrand [8], as presented by Lebeau [7]. □

One can relax the hypotheses of this Theorem in order to allow solutions of $Pu=f$ with negative order of b-regularity relative to $H^1_{R,\mathrm{loc}}(X;\Lambda X)$. This is of importance because this way one can deal with the (say, forward) fundamental solution of the wave equation directly.

The b-Sobolev spaces with negative order relative to $H^1(X)$ and $\dot H^{-1}(X)$ were defined in [23, Definition 3.15]; we refer to Section 3 of [23] for most details. Here we first state the bundle valued analogue:

Definition 7.2. Let E be a vector bundle over X. Let $m<0$, and $A\in\Psi^{-m}_{\mathrm{b}}(X;E)$ be elliptic on ${}^{\mathrm{b}}S^*X$ with proper support. We let $H^{1,m}_{\mathrm{b,comp}}(X;E)$ be the space of all $u\in\dot C^{-\infty}(X;E)$ of the form $u=u_1+Au_2$ with $u_1,u_2\in H^1_{\mathrm{comp}}(X;E)$. We let

$$\|u\|_{H^{1,m}_{\mathrm{b,comp}}(X;E)}=\inf\{\|u_1\|_{H^1(X;E)}+\|u_2\|_{H^1(X;E)}:\ u=u_1+Au_2\}.$$

We also let $H^{1,m}_{\mathrm{b,loc}}(X;E)$ be the space of all $u\in\dot C^{-\infty}(X;E)$ such that $\phi u\in H^{1,m}_{\mathrm{b,comp}}(X;E)$ for all $\phi\in C^\infty_{\mathrm{comp}}(X)$.

We define $\dot H^{-1,m}_{\mathrm{b,comp}}(X;E)$ and $\dot H^{-1,m}_{\mathrm{b,loc}}(X;E)$ analogously, replacing $H^1(X;E)$ $\dot H^{-1}(X;E)$ throughout the above discussion.

Remark 7.3. For u supported in a coordinate chart in which E is trivialized, without loss of generality we may require that A is scalar in that particular trivialization; as shown in [23, Section 3, following Remark 3.16] all choices of A are equivalent (as long as they are elliptic on a neighborhood of $\operatorname{supp} u$).

Indeed, this – together with the locality of these spaces, i.e. that they are preserved by multiplication by $\phi \in C^\infty(X)$ – shows that $H^{1,m}_{\mathrm{b,comp}}(X;E)$ could also be defined by localization, and requiring that in local coordinates in which E is trivial, the sections are N-tuples of $H^{1,m}_{\mathrm{b,comp}}(X)$ functions with N being the rank of E.

The restriction map to a boundary hypersurface $\mathcal{S}$, $\gamma_{\mathcal{S}} : C^\infty(X;E) \to C^\infty(\mathcal{S};E_{\mathcal{S}})$ extends to a map

$$\gamma_{\mathcal{S}} : H^{1,m}_{\mathrm{b,comp}}(X;E) \to \dot C^{-\infty}_{\mathrm{comp}}(\mathcal{S};E_{\mathcal{S}}),$$
$$\gamma_{\mathcal{S}}(u_1 + Au_2) = \gamma_{\mathcal{S}}(u_1) + \hat N_{\mathcal{S}}(A)(0)\gamma_{\mathcal{S}}(u_2);$$

see [23, Remark 3.16]. In particular, we make the following definition:

Definition 7.4. If E is a vector bundle over X, $m < 0$,

$$\dot H^{1,m}_{\mathrm{b,comp}}(X;E) = \{u \in H^{1,m}_{\mathrm{b,comp}}(X;E) :\ \forall \mathcal{S} \in \partial_1(X),\ \gamma_{\mathcal{S}}(u) = 0\}$$

and

$$H^{1,m}_{\mathrm{b},R,\mathrm{comp}}(X;\Lambda X) = \{u \in H^{1,m}_{\mathrm{b,comp}}(X;\Lambda X) : \\ \forall \mathcal{S} \in \partial_1(X),\ \gamma_{\mathcal{S}}(u) \in \dot C^{-\infty}(\mathcal{S};\Lambda_{\mathcal{S},N}X)\}.$$

The local spaces are defined analogously.

Equivalently, as follows from the corresponding statements for $\dot H^1(X;E) = H^1_0(X;E)$ resp. $H^1_R(X;\Lambda X)$, $\dot H^{1,m}_{\mathrm{b,comp}}(X;E)$ is the closure of $\dot C^\infty_{\mathrm{comp}}(X;E)$ and $H^{1,m}_{\mathrm{b},R,\mathrm{comp}}(X;\Lambda X)$ is the closure of $C^\infty_{R,\mathrm{comp}}(X;\Lambda X)$ in the $H^{1,m}$ topology.

Also, equivalently, $H^{1,m}_{\mathrm{b},R,\mathrm{comp}}(X;\Lambda X)$ is the space of all $u \in \dot C^{-\infty}_{\mathrm{comp}}(X;\Lambda X)$ of the form $u = u_1 + Au_2$ with $u_1, u_2 \in H^1_{R,\mathrm{comp}}(X;\Lambda X)$ if A satisfies (3.23) (this can always be assumed locally by Remark 7.3). This follows by considering a parametrix G for A, $E = GA - \mathrm{Id}, F = AG - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{b}}(X;\Lambda X)$, such that G also satisfies (3.23). Then $u = u_1' + Au_2'$ with $u_k' \in H^1_{\mathrm{comp}}(X;\Lambda X)$, so

$$u = AGu - Fu = A(Gu_1' + GAu_2') - Fu_1' - FAu_2' = u_1 + Au_2,$$
$$u_1 = -Fu = -Fu_1' - FAu_2',\ u_2 = Gu = Gu_1' + GAu_2',$$

where the first expression for u_k shows $\gamma_{\mathcal{S}}(u_k) = 0$, and the second shows $u_k \in H^1_{\mathrm{comp}}(X;\Lambda X)$.

We still need to define the negative b-regularity version of $\dot H^{-1}_R(X;\Lambda X)$. As $\dot H^{-1}_R(X;\Lambda X)$ is a quotient space of $\dot H^{-1}(X;\Lambda X)$, just like $\dot C^{-\infty}_R(X;\Lambda X)$ (the dual of $C^\infty_R(X;\Lambda X)$) is a quotient space of $\dot C^{-\infty}(X;\Lambda X)$, we proceed as follows.

Definition 7.5. We let

$$\dot H^{-1,m}_{\mathrm{b},R,\mathrm{comp}}(X;\Lambda X),\ \text{resp. } \dot H^{-1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X),$$

be the image of $\dot H^{-1,m}_{\mathrm{b,comp}}(X;\Lambda X)$, resp. $\dot H^{-1,m}_{\mathrm{b,loc}}(X;\Lambda X)$, in $\dot C^{-\infty}_R(X;\Lambda X)$.

If $A \in \Psi_{\mathrm{bc}}(X;\Lambda X)$ with normal operators of A^* satisfying (3.23) then

$$A^* : C^\infty_R(X;\Lambda X) \to C^\infty_R(X;\Lambda X),$$

so A actually descends to the quotient space $\dot{C}^{-\infty}_R(X;\Lambda X)$, cf. (3.2)-(3.3). We conclude the following:

Lemma 7.6. *Let $m < 0$, and $A \in \Psi^{-m}_{\mathrm{b}}(X;\Lambda X)$ be elliptic on ${}^{\mathrm{b}}S^*X$ with proper support and with the normal operators $\hat{N}_{\mathcal{S}}(A^*)$ of A^* satisfying* (3.23). *Then $u \in \dot{H}^{-1,m}_{\mathrm{b},R,\mathrm{comp}}(X;\Lambda X)$ if and only if $u = u_1 + Au_2$ with $u_1, u_2 \in \dot{H}^{-1}_{R,\mathrm{comp}}(X;\Lambda X)$.*

Proof. If $u' \in \dot{H}^{-1,m}_{\mathrm{b},\mathrm{comp}}(X;\Lambda X)$ with the image of u' in $\dot{C}^{-\infty}_R(X;\Lambda X)$ being u, then (with this A, see Remark 7.3 on the independence of the definition from the choice of A) $u' = u'_1 + Au'_2$, $u'_k \in \dot{H}^{-1}(X;\Lambda X)$. Let u_k be the image of u'_k in $\dot{H}^{-1}_R(X;\Lambda X) \subset \dot{C}^{-\infty}_R(X;\Lambda X)$; the claim then follows immediately as A acts on $\dot{C}^{-\infty}_R(X;\Lambda X)$. The converse follows by letting $u'_k \in \dot{H}^{-1}_{\mathrm{comp}}(X;\Lambda X)$ have image $u_k \in \dot{H}^{-1}_{R,\mathrm{comp}}(X;\Lambda X)$. □

In view of Definition 3.1 and the following remarks, namely that one can rearrange the order of factors in $\mathrm{Diff}(X)$ and $\Psi_{\mathrm{b}}(X)$ (changing the factors but not their (pseudo)differential orders), without affecting the principal symbol, and without affecting the mapping property (3.23) of normal operators (due to the explicit form of $\tilde{A}$ and $\tilde{B}$ in (3.6); by Lemma 7.6 we presently need that the adjoints satisfy (3.23)), we deduce that any $Q \in \mathrm{Diff}^2(X;\Lambda X)$ gives a map

$$Q : H^{1,m}_{\mathrm{b},\mathrm{loc}}(X;\Lambda X) \to H^{-1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X).$$

In particular, this is the case for P as in (3.19), and thus

$$P : H^{1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X) \to H^{-1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X). \tag{7.2}$$

We also recall from [23, Section 3] the wave front set with negative order of regularity relative to H^1 and $\dot{H}^{-1}$. Indeed, since any $A \in \Psi^m_{\mathrm{b}}(X;\Lambda X)$ defines a map $A : \dot{C}^{-\infty}(X;\Lambda X) \to \dot{C}^{-\infty}(X;\Lambda X)$, our definition of the wave front set makes sense for $m < 0$ as well; it is independent of s if we take $u \in H^{1,s}_{\mathrm{b},\mathrm{loc}}(X;\Lambda X)$ since the action of $\Psi_{\mathrm{b}}(X;\Lambda X)$ is well-defined on the larger spaces $\dot{C}^{-\infty}(X;\Lambda X)$ already.

Definition 7.7. Suppose $u \in H^{1,s}_{\mathrm{b},\mathrm{loc}}(X;\Lambda X)$ for some $s \leq 0$, and suppose that $m \in \mathbb{R}$. We say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^m_{\mathrm{b}}(X;\Lambda X)$ such that $\sigma_{\mathrm{b},m}(A)(q)$ is invertible and $Au \in H^1(X;\Lambda X)$.

For $m = \infty$, we say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^0_{\mathrm{b}}(X;\Lambda X)$ such that $\sigma_{\mathrm{b},0}(A)(q)$ is invertible and $LAu \in H^1(X;\Lambda X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X;\Lambda X)$, i.e. if $Au \in H^{1,\infty}_{\mathrm{b}}(X;\Lambda X)$.

The wave front set $\mathrm{WF}^{-1,m}_{\mathrm{b}}(u)$ relative to $\dot{H}^{-1}(X;\Lambda X)$ is defined similarly for $u \in \dot{H}^{-1,s}_{\mathrm{b},\mathrm{loc}}(X;\Lambda X)$, and the same also holds for $u \in \dot{H}^{-1,s}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X)$ except that we must require A such that A^* satisfies (3.23).

Remark 7.8. When A as in the definition of $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ exists, there also exists $A \in \Psi^m_{\mathrm{b}}(X;\Lambda X)$ which is elliptic at q and which satisfies (3.23). Indeed, one may arrange that A is supported in a local coordinate chart and is scalar in the trivialization of ΛX.

With this background we have the following strengthening of Theorem 7.1.

Theorem 7.9 (Negative order version of Theorem 7.1). *Suppose that P is as in* (3.19) *considered as a map* (7.2)*, i.e. consider P with relative boundary conditions, $Pu = f$.*

If $u \in H^{1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X)$ for some $m \leq 0$, then for all $s \in \mathbb{R} \cup \{\infty\}$,

$$\mathrm{WF}_{\mathrm{b}}^{1,s}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(f) \subset \dot{\Sigma},$$

and it is a union of maximally extended generalized broken bicharacteristics of P in $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$.

The same conclusion holds with relative boundary conditions replaced by absolute boundary conditions.

Proof. As noted in [23, Remark 8.3], the only part of our estimates that need changing is the treatment of the 'background terms', such as $\|u\|_{H^1_{\mathrm{loc}}}$ in Lemma 4.2 (and Lemma 4.6), and $\|Pu\|_{H^{-1}_{R,\mathrm{loc}}(X)}$. Explicitly, we need to replace the

$$H^1_{\mathrm{loc}}(X;\Lambda X), \text{ resp. } H^{-1}_{R,\mathrm{loc}}(X;\Lambda X),$$

norms by the

$$H^{1,m}_{\mathrm{b},\mathrm{loc}}(X;\Lambda X), \text{ resp. } H^{-1,m}_{\mathrm{b},R,\mathrm{loc}}(X;\Lambda X),$$

norms. The microlocal norms, in which we are gaining regularity, such as those of Gu and $\tilde{G}Pu$ in Lemma 4.2 and Lemma 4.6 are *unchanged.* Indeed, now we merely need to apply [23, Lemma 3.18] in place of [23, Lemma 3.13]. □

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