Some recent advances in microlocal analysis

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Abstract. In this talk we describe some recent developments in microlocal analysis that have led to advances in understanding problems such as wave propagation, the Laplacian on asymptotically hyperbolic spaces and the meromorphic continuation of the dynamical zeta function for Anosov flows.

1. Introduction

In this talk we describe some recent developments in microlocal analysis that have led to advances in understanding problems such as wave propagation, the Laplacian on asymptotically hyperbolic spaces and the meromorphic continuation of the dynamical zeta function for Anosov flows. We state some of these results as theorems directly, giving details in the body of the notes.

The first theorem concerns asymptotically hyperbolic spaces, which are n-dimensional manifolds with boundary X_0, with a preferred boundary defining function x, with a complete Riemannian metric g_0 on the interior of X_0 such that $\hat{g}_0 = x^2 g_0$ is Riemannian on X_0 (i.e. up to the boundary) and $|dx|_{\hat{g}_0} = 1$ at ∂X_0. For such metrics the Laplacian is essentially self-adjoint on $\mathcal{C}_c^\infty(X_0^\circ)$, and is positive, and thus the modified resolvent $R(\sigma) = (\Delta_{g_0} - (n-1)^2/4 - \sigma^2)^{-1}$ exists, as a bounded operator on $L^2(dg_0)$ for $\operatorname{Im}\sigma > 0$, $\sigma \notin \imath(0,(n-1)/2]$.

Theorem 1.1. *([74, 73]) Let (X_0, g_0) be an* even *asymptotically hyperbolic space (in the conformally compact sense) of dimension n. Then the (modified) resolvent of the Laplacian on functions, $R(\sigma) = (\Delta_{g_0} - (n-1)^2/4 - \sigma^2)^{-1}$, continues meromorphically from $\operatorname{Im}\sigma > (n-1)/2$ to $\mathbb{C}$ with finite rank Laurent coefficients at the poles (called* resonances*), and if the geodesic flow on (X, g) is non-trapping, i.e. all geodesics escape to infinity, then in strips $\operatorname{Im}\sigma > s$, $R(\sigma)$ satisfies non-trapping estimates $\|R(\sigma)\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \leq C|\sigma|^{-1}$, $\operatorname{Re}\sigma > C_1$, for suitable Hilbert spaces $\mathcal{X}, \mathcal{Y}$.*

Analogous results hold on differential k-forms, with $(n-1)^2/4$ replaced by $(n-2k\pm1)^2/4$, with the sign $+$ corresponding to closed, and $-$ corresponding to coclosed forms.

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The meromorphic extension was proved by Mazzeo and Melrose [48] and Guillarmou [29], using the 0-pseudodifferential algebra of Mazzeo and Melrose. In this algebra the operator is elliptic in the principal symbol sense, but one also needs to invert the *normal operator*. The latter is sufficiently difficult that (suboptimal, but sufficient for some applications) estimates as $\operatorname{Re}\sigma \to \infty$ in strips were only obtained recently by Melrose, Sá Barreto and Vasy [50] by a semiclassical construction in special cases. Recently Vasy [75, 74] gave a new proof, and proved the non-trapping estimates as well, using a new method, extending a renormalized and conjugated version of the spectral family $\Delta_g - (n-1)^2/4 - \sigma^2$ across ∂X to a new operator P_σ which can be thought of as being defined on a manifold without boundary X, so only 'standard' microlocal analysis is needed. The extended operator P_σ is no longer elliptic, but the additional phenomena are well-understood from the point of view of microlocal analysis: real principal type propagation, radial points and complex absorption. This method also allows for a generalization to differential forms; these were previously studied in the context of Hodge theory by Mazzeo [47]. Also, as a byproduct, it gives a new approach for analyzing the wave equation on asymptotically de Sitter spaces, on which wave propagation was described earlier, without the evenness condition, by Vasy [81] and Baskin [6].

In addition to providing a new proof of the meromorphic continuation of the resolvent, as well as the large σ estimates, this approach also allows for microlocalization of the estimates which is crucial in many applications, such as in the gluing work of Datchev and Vasy [16]. A nice application of this theory, in combination with the exotic psedodifferential calculus/second microlocal machinery developed by Sjöstrand and Zworski [63], is the work of Datchev and Dyatlov [15], which gave a proof of fractal upper bounds, in terms of the upper Minkowski dimension of the trapped set, for the resonance counting function on even asymptotically hyperbolic spaces with hyperbolic geodesic flow. This in particular applies for convex cocompact quotients of hyperbolic space and gives analogous upper bounds for the counting function of zeros of the Selberg zeta function then. (These quotients have long been studied; see e.g. [58, 67].)

We also point out that the Euclidean analogue of the theorem has a long history (with stronger restrictions at infinity needed). An effective meromorphic continuation was obtained by complex scaling methods due to Aguilar, Balslev, Combes and Simon, and other authors, including the microlocal perspective of Helffer and Sjöstrand; see [62] and the references therein.

The second theorem concerns wave propagation on Kerr-de Sitter spaces. This is particularly interesting since the asymptotic behavior of waves involves resonances, which are poles of a family $\sigma \mapsto P_\sigma^{-1}$, where P_σ is very similar to the P_σ in the asymptotically hyperbolic case; it is an operator on a manifold without boundary. Concretely, Kerr-de Sitter space has a bordification, or partial compactification, $\overline{M}$, with a boundary defining function τ and P_σ is then an operator on ∂M. The extra complication is that this operator is trapping, but the trapping is of a relatively weak type, called normally hyperbolic trapping, which has been analyzed by Wunsch and Zworski [83] and by Dyatlov [23] recently.

Theorem 1.2. *(See [75, 41], and see [22] for exact Kerr-de Sitter space.) Let*

$(\overline{M}, g)$ be a Kerr-de Sitter type space with normally hyperbolic trapping. Then there is $\kappa > 0$ such that solutions of $(\Box_g - \lambda)u = 0$ have an asymptotic expansion $u \sim \sum_j \sum_{k \leq k_j} \tau^{\imath\sigma_j}(\log|\tau|)^k a_{jk} + \tilde{u}$, where $\tilde{u} \in \tau^\kappa H^s_{\mathrm{b}}(\overline{M})$; here σ_j are resonances *of the associated* normal operator. *For $\lambda = 0$ on Kerr-de Sitter space, the unique σ_j with $\operatorname{Im}\sigma_j \geq 0$ is 0, and the corresponding term is a constant, i.e. waves decay to constants.*

Further, this result is stable under b-perturbations of the metric, with the b-structure understood in the sense of Melrose [54].

In spatially compact parts of Kerr-de Sitter space, $\tau = e^{-t}$ for the usual time function t, i.e. this decay is exponential.

In fact, in a slightly different way, the wave equation for Minkowski-type metrics, more specifically Lorentzian scattering metrics, can also be handled by similar techniques, see [75, 5, 41], for both Cauchy problems and for the Feynman propagator. In fact, Klein-Gordon type equations, even in ultrahyperbolic settings, are also amenable to this type of analysis – in this case in Melrose's scattering framework [49].

A different direction of extending these results is to non-linear equations. In the semilinear setting this was discussed by [41], and then extended to the quasilinear case by Hintz [38]. We briefly discuss this direction at the end of these notes.

The third theorem concerns the dynamical zeta function. It was a conjecture of Smale's, proved by Giulietti, Liverani and Pollicott [26] recently by dynamical systems techniques, but shortly afterwards Dyatlov and Zworski [20] gave a new short proof using microlocal analysis, relying on ideas of Faure and Sjöstrand [24], which are analogous to the setup involved in proving the above theorems, as well as Guillemin's approach to trace formulae [33].

Theorem 1.3. *(See [20].) Let X be a compact manifold and $\phi_t : X \to X$ a $\mathcal{C}^\infty$ Anosov flow with orientable stable and unstable bundles. Let $\{\gamma^\sharp\}$ denote the set of primitive orbits of ϕ_t, and $T^\sharp_\gamma$ their periods. Then the Ruelle zeta function $\zeta_R(\lambda) = \prod_{\gamma^\sharp}(1 - e^{\imath\lambda T^\sharp_\gamma})$, which converges for $\operatorname{Im}\lambda \gg 0$, extends meromorphically to $\mathbb{C}$.*

While of course this lecture cannot cover all of microlocal analysis, at this point we need to point out a particularly glaring omission in the author's opinion: analysis on singular spaces. This has been an extremely active area of study, both in elliptic and non-elliptic settings. The former include for instance N-body scattering, see [72] and references therein for this very extensive topic connected to wave propagation on manifolds with corners, the low energy description of the resolvent of the Laplacian on asymptotically Euclidean spaces [30, 31, 60], scattering theory on geometrically finite hyperbolic manifolds [35, 10, 32], index and K-theory on manifolds with corners and stratified spaces [55, 1], and a general symbolic pseudodifferential theory [2]. The latter include for instance wave propagation on cones, edges, manifolds with corners, see e.g. [46, 52, 79, 80, 51, 7] and references therein.

With these illustrations of the uses of microlocal analysis, we now explain the new developments which facilitated these advances.

2. Pseudodifferential operators

Microlocal, or phase space, analysis in its simplest form concerns itself with the study of functions or distributions on manifolds by means with which one can localize not only in the base manifold, but also conically in the fibers of its cotangent bundle. This corresponds to a description of not only where a distribution lies in, say, a Sobolev space locally, but in which (co)direction this happens. In the most basic setting of $\mathbb{R}^n$, it is closely related to the Fourier transform: one localizes in the base space $\mathbb{R}^n_z$, as well as in conic (i.e. dilation invariant) subsets of $\mathbb{R}^n_\zeta$. For instance, one says that a distribution u on $\mathbb{R}^n$ is *in the Sobolev space H^s microlocally near* $(z_0,\zeta_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})=T^*\mathbb{R}^n\setminus o$ if there exists $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ identically 1 near z_0, and $\psi\in\mathcal{C}^\infty(\mathbb{R}^n)$, homogeneous of degree 0 for $|\zeta|>1$, such that $\psi(t\zeta_0)=1$ for all $t\gg 1$ sufficiently large, and

$$\mathcal{F}^{-1}\psi\mathcal{F}\phi u\in H^s,$$

or equivalently

$$\mathcal{F}^{-1}\langle\zeta\rangle^s\psi\mathcal{F}\phi u\in L^2,$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$, and $\langle\zeta\rangle=(1+|\zeta|^2)^{1/2}$. The operator $A=\mathcal{F}^{-1}\langle\zeta\rangle^s\psi\mathcal{F}\phi$ is a prime example of a *pseudodifferential operator*, a class of operators which includes differential operators. In general, on $\mathbb{R}^n$, a pseudodifferential operator has the form

$$Au(z)=(2\pi)^{-n}\int e^{\imath(z-z')\cdot\zeta}a(z,z',\zeta)\,u(z')\,d\zeta\,dz',$$

with the integral understood as an *oscillatory integral*, where a satisfies appropriate estimates. (For the example above, one can take $a(z,z',\zeta)=\langle\zeta\rangle^s\psi(\zeta)\phi(z')$ independent of z.) A typical class of estimates is

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z,z',\zeta)|\leq C_{\alpha\beta\gamma}\langle\zeta\rangle^{m-|\gamma|}; \tag{2.1}$$

this gives Hörmander's symbol class S^m_∞, and the corresponding operators $A\in\Psi^m_\infty$. Another typical class is $S^{m,\ell}$ given by the estimates

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z,z',\zeta)|\leq C_{\alpha\beta\gamma}\langle z\rangle^{\ell_1-|\alpha|}\langle z'\rangle^{\ell_2-|\beta|}\langle\zeta\rangle^{m-|\gamma|};$$

this gives rise to the operators $A\in\Psi^{m,\ell}$ with $\ell=\ell_1+\ell_2$. Both of these classes form an algebra (with composition giving the ring multiplication) over $\mathbb{C}$. An important property is that composition is symbolic, i.e. one can compute AB modulo *residual* operators, i.e. modulo $\Psi^{-\infty}_\infty$, resp. $\Psi^{-\infty,-\infty}$. Note that $\Psi^{m,0}\subset\Psi^m_\infty$, and elements of Ψ^m_∞ are bounded maps between weighted Sobolev spaces $H^{s,r}\to H^{s-m,r}$, where $H^{s,r}=\langle z\rangle^{-r}H^s$, while those of $\Psi^{m,\ell}$ are bounded $H^{s,r}\to H^{s-m,r-\ell}$. This explains the different properties of these algebras: the residual terms are non-compact on, say, L^2, for the first algebra, but they are compact for the second.

Another class of operators that plays a role below is the *semiclassical family*, $A=(A_h)_{h\in(0,1)}$ of operators:

$$A_hu(z)=(2\pi h)^{-n}\int e^{\imath(z-z')\cdot\zeta/h}a(z,z',\zeta,h)\,u(z')\;d\zeta\,dz',$$

where a satisfies estimates in one of the above cases, uniformly in $h \in [0,1)$; indeed, we typically consider smooth h dependence with h-uniform bounds for all derivatives. This gives rise to the operator algebras $\Psi^m_{\infty,\hbar}$ and $\Psi^{m,\ell}_\hbar$. Here the residual terms (modulo which one can do calculations) are in $h^\infty\Psi^m_{\infty,\hbar}$, resp. $h^\infty\Psi^{m,\ell}_\hbar$, i.e. one can do computations modulo rapidly vanishing, as $h \to 0$, errors, not merely compact (in the case of $\Psi^{m,\ell}_\hbar$) error terms.

Pseudodifferential operators can be transferred to compact manifolds without boundary via local coordinate charts, requiring that the Schwartz kernel is $\mathcal{C}^\infty$ away from the diagonal; one checks easily that for the class stated above on $\mathbb{R}^n$, the Schwartz kernel is indeed $\mathcal{C}^\infty$ away from the diagonal $z = z'$ on $\mathbb{R}^n \times \mathbb{R}^n$. It is worthwhile noticing that differential operators are of the form stated above with polynomial behavior in ζ; the subclass of pseudodifferential operators for which a as above has an asymptotic expansion in terms of homogeneous functions of degree $m-j$, $j \in \mathbb{N}$, is called *classical* or *one-step polyhomogeneous*. In this generality of manifolds, pseudodifferential operators have a well-defined *principal symbol* $[a]$ in $S^m_\infty/S^{m-1}_\infty$; in the case of classical operators, these can be regarded as homogeneous degree m functions on $T^*M \setminus o$.

Since T^*M is not compact even if M is, and as the most interesting objects are homogeneous, it is useful to work instead on the compact (if M is such) space $(T^*M \setminus o)/\mathbb{R}^+ = S^*M$. However, it is even better to compactify the fibers of T^*M as balls to obtain $\overline{T}^*M$; this glues S^*M to T^*M at fiber infinity using 'reciprocal polar coordinates', so $\partial\overline{T}^*M = S^*M$. The principal symbol a of $A \in \Psi^0_{\mathrm{cl}}$, which can be regarded as a homogeneous degree zero function, is then equivalently a function on $\partial\overline{T}^*M$. If $A \in \Psi^m_{\mathrm{cl}}$, then this principal symbol is homogeneous degree m, or a section of a line bundle over $\partial\overline{T}^*M$, but it is also well-defined as a function up to a positive multiplier, so e.g. its zero set, which is also called the *characteristic set* $\mathrm{Char}(A)$ of A, is well-defined as a subset of $\partial\overline{T}^*M$. The complement of the characteristic set is the *elliptic set* $\mathrm{Ell}(A)$; an operator is *elliptic* if it has empty characteristic set, i.e if it is elliptic at every point. It also makes sense to talk of the operator *wave front set* $\mathrm{WF}'(A)$; a point in $\partial\overline{T}^*M$ is in the complement of $\mathrm{WF}'(A)$ if it has a neighborhood on which A is given (say, in local coordinates) by an order $-\infty$, i.e. 'really trivial', symbol, as opposed to one of order $m-1$, which is what would be guaranteed by the vanishing of the principal symbol on such a neighborhood. One thinks of operators A which have wave front set in some open set $U \subset \partial\overline{T}^*M$ as a microlocalizer to U (analogue of the multiplication operator by a compactly supported function on an open set O in M localizing to O); one can also construct microlocal partitions of unity, etc.

In the semiclassical version of this analysis, in addition to the behavior of a at $\partial\overline{T}^*M$ for all h, one is also interested in the behavior of a *on all of* $\overline{T}^*M$ at $h = 0$. This is geometrically encoded by working with the space $[0,1)_h \times \overline{T}^*M$, then the boundary hypersurfaces are $\{0\}_h \times \overline{T}^*M$ and $[0,1)_h \times \partial\overline{T}^*M$. We call the restriction of a to the former the *semiclassical principal symbol.* If a is a classical, h-dependent, order 0 family, then the *joint principal symbol* can be thought of as a function on $\partial([0,1)_h \times \overline{T}^*M)$, consisting of the standard and the semiclassical principal symbol as its two constituents. Correspondingly, the elliptic set, characteristic set and wave front sets are all subsets of $\partial([0,1)_h \times \overline{T}^* M)$.

Much as we compactified the fibers of the cotangent bundle above, we can also compactify the base space $\mathbb{R}^n$ for $\Psi^{m,\ell}$, we again do it as a ball. Then neighborhoods of points at $\partial\overline{\mathbb{R}^n}$ correspond to (cut off to the exterior of a compact subset of $\mathbb{R}^n$) *conic* regions in $\mathbb{R}^n$. Much as transferring the definition of pseudodifferential operators to manifolds is possible via local coordinate charts, we can transfer $\Psi^{m,\ell}$ to *manifolds with boundary* M, requiring smooth and rapidly (order ∞) decaying Schwartz kernel away from the diagonal in $M \times M$. The resulting algebra of operators is Melrose's *scattering algebra* $\Psi_{\mathrm{sc}}^{m,\ell}(M)$, see [49]. The natural place of microlocalization is a replacement for the standard cotangent bundle, called the *scattering cotangent bundle* ${}^{\mathrm{sc}}T^*M$, which is naturally identified with T^*M over M°. Since we transferred our pseudodifferential operators from $\mathbb{R}^n$, the smooth sections of this bundle are locally (near ∂M) of the form $\sum \zeta_j(z)\,dz_j$, where $\zeta_j \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$. It is more convenient to express this in polar coordinates on $\mathbb{R}^n$ and then transfer to M: one then sees that these forms are $\tau\,\frac{dx}{x^2} + \sum \eta_j\,\frac{dy_j}{dx}$, with x a boundary defining function, y_j local coordinates on ∂M. Dually, the vector fields in $\Psi_{\mathrm{sc}}(M)$ are the scattering vector fields, of the form $V = a(x^2\partial_x) + \sum b_j(x\partial_{y_j})$, with a, b_j smooth. More invariantly, they are of the form $\rho\mathcal{V}_{\mathrm{b}}(M)$, ρ now a global boundary defining function, where $\mathcal{V}_{\mathrm{b}}(M)$ is the Lie algebra of Melrose's b-vector fields, namely vector fields tangent to ∂M.

In fact, it is useful to compactify the fibers of ${}^{\mathrm{sc}}T^*M$ to obtain the space ${}^{\mathrm{sc}}\overline{T}^*M$. Now the *joint principal symbol* is an object (section of a line bundle, or simply a function for order $(0,0)$) on $\partial\,{}^{\mathrm{sc}}\overline{T}^*M$, and the elliptic, characteristic and wave front sets are subsets of $\partial\,{}^{\mathrm{sc}}\overline{T}^*M$. There is even a semiclassical version of this, in which case these objects 'live on' $\partial([0,1)_h \times {}^{\mathrm{sc}}\overline{T}^*M)$, which has three boundary hypersurfaces: 'fiber infinity', 'base infinity' and $h = 0$.

Here we shall mostly consider applications of microlocal analysis to partial differential equations. However, this is by no means the only application. For instance, the field of inverse problems, such as determining a Riemannian metric on a manifold with boundary M from its distance function restricted to the boundary, called the *boundary rigidity problem*, has been using microlocal analysis, in particular the theory of Fourier integral operators, extensively. Such problems have linearizations related to the geodesic X-ray transform (on tensors). By introducing an artificial boundary, given by a level set of a convex function, and using Melrose's scattering pseudodifferential algebra let Stefanov, Uhlmann and Vasy [70, 66] show that the local (in M) boundary rigidity problem is well-posed in a conformal class under suitable conditions. Of course, there were many predecessors of this work in inverse problems using microlocal analysis, see for instance works of Greenleaf, Stefanov and Uhlmann [28, 27, 65, 64].

Returning to manifolds without boundary M, T^*M being symplectic, there is a vector field H_a corresponding to the principal symbol a of A, which is homogeneous of degree $m-1$. For $m = 1$, H_a can be regarded as a vector field on ∂T^*M; for other a, we can multiply a by a positive homogeneous degree $1-m$ function b, then H_{ba} is well-defined, and as $H_{ba} = bH_a + aH_b$, so in $\mathrm{Char}(A)$, H_a is well-defined up to a positive multiple, in particular the integral curves of H_a are well-defined.

A bit more precisely, not only is H_a well-defined as a vector field on ∂T^*M, but it is well defined as a vector field in $\mathcal{V}_b(\overline{T}^*M)$ modulo $\rho_{\mathrm{fiber}}\mathcal{V}_b(\overline{T}^*M)$, where on a manifold X, $\mathcal{V}_b(X)$ is the Lie algebra of vector fields tangential to ∂X. Integral curves of H_a inside the characteristic set are called (null)-bicharacteristics.

A useful extension of the setting discussed so far is to allow the operators to have *variable order*. For instance, in the case of $\Psi^m(M)$, one can allow m to be a $\mathcal{C}^\infty$ function on $S^*M = \partial\overline{T}^*M$. For M compact, this is a subset $\Psi^{m_0}(M)$, $m_0 > \sup m$, but it allows for much finer control of regularity. Here one needs to allow symbols which gain less than a full order upon differentiation, so e.g. in the setting of (2.1) one would have, with $\delta \in (0, 1/2)$,

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a(z, z', \zeta)| \leq C_{\alpha\beta\gamma}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|};$$

these are symbols in $S^m_{\infty,1-\delta,\delta}$ in the standard notation, and one can take $\delta > 0$ arbitrarily small. One can still talk about ellipticity (including microlocally) by requiring the invertibility (by symbols of order $-m$) of the principal symbol modulo symbols of order -1. In particular, the Sobolev space $H^m(M)$ is defined, with $\tilde{m} = \inf m$, by

$$H^m(M) = \{u \in H^{\tilde{m}}(M) : \ Au \in L^2\},$$

where A is any elliptic element of $\Psi^m(M)$. Such results appeared in the work of Unterberger [71] (with order depending on the underlying space M) and Duistermaat [18]; a more complete discussion as needed for our purposes is given in [5, Appendix A]. In the semiclassical version the order m can be a function on $\partial([0, 1)_h \times \overline{T}^*M)$. If the only place where it needs to vary is at $h = 0$, it can be thought of as a function, constant outside a compact set, on T^*M. In this case one is considering a fixed Sobolev space, but with an h-dependent norm. Such microlocal norms play a key role in the work of Faure and Sjöstrand [24]; see the Anosov flow discussion below for an application. Similarly, in the scattering setting, one can have an order that is a function on $\partial^{\mathrm{sc}}\overline{T}^*M$.

3. Elliptic and non-elliptic Fredholm theory

3.1. Elliptic theory. The basic results in microlocal analysis concern the structures we have already introduced. In order to explain their significance from the perspective of solving PDE, we remark that the Fredholm property of a (pseudo)differential operator P acting between two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, i.e. the range being closed, the nullspace finite dimensional, the range finite codimensional, is equivalent to estimates

$$\|u\|_{\mathcal{X}} \leq C(\|Pu\|_{\mathcal{Y}} + \|u\|_{\tilde{\mathcal{X}}}), \qquad \|v\|_{\mathcal{Y}^*} \leq C(\|P^*v\|_{\mathcal{X}^*} + \|v\|_{\mathcal{Z}}),$$

where the adjoints are taken relative to a fixed space, such as L^2, and $\tilde{\mathcal{X}}, \mathcal{Z}$ are Hilbert spaces, with compact inclusion maps $\mathcal{X} \to \tilde{\mathcal{X}}$, $\mathcal{Y}^* \to \mathcal{Z}$. The simplest example of such an estimate is an elliptic estimate, in which case one can take all the spaces to be standard Sobolev spaces. For instance if $P \in \Psi^m(M)$ is elliptic and M is compact then $\mathcal{X} = H^s$, $\mathcal{Y} = H^{s-m}$, $\mathcal{X}^* = H^{-s}$, $\mathcal{Y}^* = H^{-s+m}$, $\tilde{\mathcal{X}} = H^{\tilde{N}}$, $\mathcal{Z} = H^{N^*}$ work for any $\tilde{N}, N^*$, which are taken to satisfy $\tilde{N} < s$, $N^* < -s+m$ for the application (the compact inclusion map). Thus, elliptic operators are invertible as maps between these Sobolev spaces, up to a finite dimensional obstruction. The analytic version of the Fredholm theory can also be used to show that for $P \in \Psi^m(M)$ elliptic with $m > 0$, the resolvent family $\mathbb{C} \ni \lambda \mapsto (P - \lambda I)^{-1}$ is meromorphic with finite rank smoothing ($\Psi^{-\infty}(M)$) Laurent coefficients if $P - \lambda_0 I$ is invertible for a single value, λ_0, of λ. All of these have natural extension to operators acting between sections of vector bundles E, F; then the principal symbol $a(z, \zeta)$ has values in bundle endomorphisms $\mathrm{End}(E_z, F_z)$, and ellipticity is the invertibility of these endomorphisms between finite dimensional vector spaces.

The elliptic estimates can be proved by constructing an approximate inverse, or parametrix, for P, which can be done by inverting the principal symbol (i.e. taking its reciprocal in the scalar setting). They can also be microlocalized, namely for any $P \in \Psi^m(M)$ (not necessarily elliptic) one also has estimates of the form

$$\|Q_1 u\|_{H^s} \leq C(\|Q_2 P u\|_{H^{s-m}} + \|u\|_{H^{\tilde{N}}}),$$

$Q_1, Q_2 \in \Psi^0(M)$ with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_2) \cap \mathrm{Ell}(P)$, i.e. on the elliptic set of P, if one microlocalizes (via Q_1 and Q_2), the analogue of the global estimate holds. Note that as the principal symbol of P^* is the complex conjugate of that of P, P^* is elliptic wherever P is. It is also useful to remark here that the elliptic estimates are valid on variable order Sobolev spaces as well.

3.2. Real principal type propagation. While there are many interesting elliptic operators, such as the Laplacian on functions or differential forms, or the Dirac operator, there are even more non-elliptic problems of interest, and it turns out that with a bit of effort they fit into similar Fredholm frameworks. If $P \in \Psi^m(M)$ is non-elliptic, it has a non-empty characteristic set. The non-degenerate version of non-ellipticity is if the principal symbol p vanishes simply there; then the characteristic set is smooth. There is a difference between the real and complex valued cases; we are here most interested in the real valued one, then $\mathrm{Char}(P)$ has codimension one. The symplectic structure turns a non-degenerate dp into a non-vanishing Hamilton vector H_p; however H_p may be *radial*, i.e. tangent to the $\mathbb{R}^+$-orbits. Hörmander's propagation of singularities theorem [42], see also [19], is that one can propagate estimates along the bicharacteristic flow in $\mathrm{Char}(P)$; such a statement is meaningful where H_p is not radial. That is, one has estimates

$$\|Q_1 u\|_{H^s} \leq C(\|Q_2 u\|_{H^s} + \|Q_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}), \tag{3.1}$$

where $Q_1, Q_2, Q_3 \in \Psi^0(M)$, provided the elliptic set of Q_3 contains $\mathrm{WF}'(Q_1)$, and the bicharacteristics through all points in $\mathrm{WF}'(Q_1) \cap \mathrm{Char}(P)$ reach the elliptic set of Q_2 while remaining in $\mathrm{Ell}(Q_3)$. The more usual phrasing of this theorem is that if u is in H^s microlocally at a point in $\mathrm{Char}(P)$, then $u \in H^s$ on the maximal bicharacteristic segment through this point, with 'maximal' being with respect to being contained in the complement of $\mathrm{WF}^{s-m+1}(Pu)$, i.e. in the set where Pu is microlocally H^{s-m+1}. This theorem is proved by *positive commutator estimates*, computing

$$\langle \imath Pu, Au\rangle - \langle \imath Au, Pu\rangle = \langle (\imath[A,P] + \imath(P-P^*)A)u, u\rangle, \tag{3.2}$$

$A = A^* \in \Psi^{m'}(M)$, and the principal symbol of $\imath[A,P]+\imath(P-P^*)A$ in $\Psi^{m+m'-1}(M)$ is $-H_p a - 2\tilde{p}a$ if $\tilde{p}$ is the principal symbol of $\frac{1}{2\imath}(P-P^*) \in \Psi^{m-1}(M)$. One arranges that

$$-H_p a - 2\tilde{p}a = b^2 + e, \tag{3.3}$$

where e has support in the region where the a priori assumptions are imposed (such as $\mathrm{WF}'(Q_2)$ above). Taking B, E with principal symbols b, e, one has $\imath[A,P]+\imath(P-P^*)A = B^*B + E + F$, with F lower order, so substituting into (3.2), one controls Bu in terms of Eu as well as A^*Pu, proving a theorem after a regularization argument.

This theorem, with the proof, is also valid on variable order Sobolev spaces, but only in one direction of flow. Thus, if s is monotone along the Hamilton flow, say s is increasing, then one can propagate H^s estimates in the backward direction, while if s is decreasing, one can propagate H^s estimates in the forward direction. In terms of the sketched proof, the reason for the restriction on the direction is that the Hamilton derivative hitting the weight (giving the order) provides a logarithmically larger term than the other ones, which thus must have a correct sign for the argument to go through; see [71, 5].

We also remark that Hörmander's theorem, with the positive commutator proof, extends easily to systems whose principal symbol is real scalar (a multiple of the identity operator on the vector bundle), and also extends to more general *real principal type systems*, as shown by Dencker [17].

3.3. Complex absorption. Hörmander's theorem, as well as its generalizations, had a key role in understanding propagation phenomena, such as waves. In all these cases estimates propagate, i.e. if one a priori knows that u is well-behaved somewhere (in this case on the wave front set of Q_2) then one can conclude that u is well-behaved somewhere else. From the perspective of Fredholm problems, the problem with this is that the a priori hypothesis need not ever be fulfilled. One way of dealing with this is called *complex absorption*, see [56]. This means that one considers an artificial operator $Q \in \Psi^m(M)$ with non-negative principal symbol, and replaces P by $P - \imath Q$. Then there is still an analogue of Hörmander's theorem, but one can only propagate estimates in the *forward* direction along H_p. Notice that in the elliptic set of Q one has elliptic estimates even in $\mathrm{Char}(P)$, so the point is that one can propagate estimates from and to this elliptic set, in the forward direction, along the H_p flow. Replacing $P - \imath Q$ by $P + \imath Q$, but Q still having non-negative principal symbol, the estimates can be propagated in the *backward* direction. In particular, this works for $(P - \imath Q)^* = P^* + \imath Q^*$, so *estimates for the adjoint can be propagated in the opposite direction as estimates for the operator.* As an example, if all bicharacteristics of P (in $\mathrm{Char}(P)$) reach the elliptic set of Q in both the forward and the backward direction, which we may call the simplest *non-trapping* scenario, one can piece together elliptic and propagation estimates to conclude that

$$\|u\|_{H^s} \leq C(\|(P - \imath Q)u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

and

$$\|v\|_{H^{s'}} \leq C(\|(P^* + \imath Q^*)v\|_{H^{s'-m+1}} + \|v\|_{H^{N'}}), \qquad s' = -s + m - 1.$$

This corresponds to Fredholm estimates, though one has to be a bit careful as $P - \imath Q$ does not map H^s to H^{s-m+1}. So one lets

$$\mathcal{X} = \{u \in H^s : \ (P - \imath Q)u \in H^{s-m+1}\},$$

which is a Hilbert space in the natural norm; this is the simplest example of a coisotropic space, see [51, Appendix A]. One also lets $\mathcal{Y} = H^{s-m+1}$. Then $P - \imath Q : \mathcal{X} \to \mathcal{Y}$ is Fredholm, and indeed, if P depends on a parameter $\sigma \in \mathbb{C}$, with the principal symbol of P independent of σ, then this is an analytic Fredholm family, with a meromorphic inverse if it is invertible at a single point.

3.4. Radial points. While complex absorption is artificial, though very useful in eliminating dynamics in certain regions of ∂T^*M by making the operator elliptic there, it illustrates an important point: in order to have Fredholm problems, we need the bicharacteristics to reach regions in which we have good a priori control, such as $\mathrm{Ell}(Q)$ above. The most natural setting is that of *radial points*, which were already mentioned earlier as the points at which Hörmander's propagation theorem provides no extra information. Unlike in the settings considered above, in which the Sobolev order s was arbitrary, here there are restrictions on it because the positivity, corresponding to b^2 in (3.3), can only be given by the *weight*, $\rho_{\mathrm{fiber}}^{-m'}$. Thus, it is useful to think of H_p, or rather $\rho_{\mathrm{fiber}}^{m-1}H_p$, as an element of $\mathcal{V}_{\mathrm{b}}(\overline{T}^*M)$ modulo $\rho_{\mathrm{fiber}}\mathcal{V}_{\mathrm{b}}(\overline{T}^*M)$, since this encodes $H_p\rho_{\mathrm{fiber}}$ modulo one order additional vanishing. The results in this setting depend on the sign of the Hamilton derivative of the weight relative to the sign of the Hamilton derivative of the microlocalizer: if they have the same sign, one need not make an assumption like $Q_2u \in H^s$ in the propagation estimates (3.1) to get a conclusion at the radial set (set of radial points), but if they have the opposite sign, one does need to do this. Since the sign of $H_p\rho_{\mathrm{fiber}}^{-m'}$ depends on the sign of m', this means that the kind of results one gets in the high regularity (which needs m' bigger than a threshold) versus the low regularity (which needs m' smaller than a threshold) are different. Finally, we need to keep in mind the appearance of $\tilde{p}$ in (3.3), which shifts this threshold value from being $m' = 0$. Thus, the estimate in this setting has two parts. We first make the non-degeneracy assumption that $\rho_{\mathrm{fiber}}^{m-2}H_p\rho_{\mathrm{fiber}} = \mp\beta_0 > 0$ on the radial set $L \subset \mathrm{Char}(P) \subset \partial T^*M$, while $H_p\tilde{p} = \pm\tilde{\beta}\beta_0\tilde{\rho}^{-m+1}$, where we assume for simplicity that $\tilde{\beta}$ is constant on L, which is the case in many applications. Further, assume that L acts as a sink ($-$) or source ($+$) in $\mathrm{Char}P \subset \partial\overline{T}^*M$, in a non-degenerate sense; this basically means that there is a quadratic defining function ρ_1 of L in $\mathrm{Char}(P)$ such that $\rho_{\mathrm{fiber}}^{m-1}H_p\rho_1$ is a positive definite. Then

(i) If $s \geq s_0 > (m-1)/2 - \tilde{\beta}$, then for $u \in H^{s_0}$ one has estimates

$$\|Q_1u\|_{H^s} \leq C(\|Q_3Pu\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1)$ tend to L in either the forward ($-$) or the backward ($+$) direction, while remaining in $\mathrm{Ell}(Q_3)$.

(ii) If $s < (m-1)/2 - \tilde{\beta}$, then one has estimates

$$\|Q_1 u\|_{H^s} \le C(\|Q_2 u\|_{H^s} + \|Q_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{\tilde{N}}}),$$

with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1) \setminus L$ tend to $\mathrm{Ell}(Q_2)$ (which is now typically disjoint from L) in either the forward ($+$) or the backward ($-$) direction, while remaining in $\mathrm{Ell}(Q_3)$.

Note that if P is replaced by P^*, then $P - P^*$ is replaced by its negative, so $\tilde{\beta}$ defined for P^* is the negative of that defined for P, which means that for P^* (keeping $\tilde{\beta}$ as defined for P) one has

(i) If $s' \ge s_0' > (m-1)/2 + \tilde{\beta}$, then for $u \in H^{s_0'}$ one has estimates

$$\|Q_1 u\|_{H^{s'}} \le C(\|Q_3 P^* u\|_{H^{s'-m+1}} + \|u\|_{H^{N'}}),$$

with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics of p from points in $\mathrm{WF}'(Q_1)$ tend to L in either the forward ($-$) or the backward ($+$) direction, while remaining in $\mathrm{Ell}(Q_3)$.

(ii) If $s' < (m-1)/2 + \tilde{\beta}$, then one has estimates

$$\|Q_1 u\|_{H^{s'}} \le C(\|Q_2 u\|_{H^{s'}} + \|Q_3 P^* u\|_{H^{s'-m+1}} + \|u\|_{H^{N'}}),$$

with $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(Q_3)$ and such that all bicharacteristics from points in $\mathrm{WF}'(Q_1) \setminus L$ tend to $\mathrm{Ell}(Q_2)$ in either the forward ($+$) or the backward ($-$) direction, while remaining in $\mathrm{Ell}(Q_3)$.

Substituting in $s' = -s + m - 1$, one sees that the condition in (ii) for P^* is equivalent to that in in condition (i) for P, and similarly with (i) and (ii) interchanged. This means that if one has non-trapping in the sense that both in the forward and in the backward direction the bicharacteristics escape to radial sets, one has Fredholm estimates, provided one can arrange that the Sobolev spaces are such that one can propagate estimates away from a radial set (case (i)) for P from one of the ends of the bicharacteristics, and for P^* from the other end (as this implies case (ii) for P). Often the numerology in (i) and (ii) is such that the Sobolev spaces H^s must have *variable order*. One can also combine such a Fredholm setup with complex absorption; in this case one can often work with constant order Sobolev spaces.

Notice that radial points are in some sense the best thing that can happen to a non-elliptic problem with real principal symbol: if one has a chaotic Hamilton flow, there is no reason to think that one can propagate regularity from *anywhere*; radial points provide just such a location. This being said, note that the requirements above were weaker than L being radial: roughly speaking, there can still be non-trivial Hamilton flow in L, and we care about the Hamilton dynamics normally to

L. This turns out to be important in Kerr-de Sitter black holes, where the conormal bundle of the event horizon at infinity has this kind of structure. (The non-rotating black holes of de Sitter-Schwarzschild spaces give rise to actual radial sets.) Also notice that the estimates we stated were global in a radial set (component) L; one can in fact microlocalize if the set is actually radial, as shown by Haber and Vasy [36], but this becomes impossible when there is a non-trivial Hamilton flow within L, as in Kerr-de Sitter space. We finally remark that the radial point estimates also hold for systems provided the principal symbol is real scalar (a multiple of the identity operator on the fibers of the vector bundle).

3.5. Normally hyperbolic trapping. This lack of ability to microlocalize within a set Γ invariant under the Hamilton flow occurs also in a more degenerate setting, that of *normally hyperbolic trapped sets*. After much earlier work of Gérard and Sjöstrand [25] in the analytic setting, this was analyzed by Wunsch and Zworski [83], to an extent which suffices for the problems we consider here, with refinements by Hintz and Vasy [39], and in more detail by Nonnenmacher and Zworski [57] and by Dyatlov [23, 21]. (The latter is sufficiently precise to locate a sequence of resonances corresponding to Γ, while [57] allows for rather irregular normal dynamics (stable and unstable distributions)!) For us these enter in either the semiclassical, or in the b-settings, with Kerr-de Sitter spaces containing perhaps the prime examples. In the normally hyperbolic setting one drops the non-degeneracy of $\rho_{\mathrm{fiber}}^{m-2} H_p \rho_{\mathrm{fiber}} = \mp\beta_0 > 0$ of the radial setting; in fact, one has a defining function of the boundary hypersurface at which one is doing analysis (so h in the semiclassical setting) which, at L, has vanishing H_p-derivative. (This is automatic in the semiclassical setting!) The subprincipal symbol (in the form of $\tilde{p}$), which shifts the threshold in the radial setting via $\tilde{\beta}$, can still give positivity, but it must have a definite sign to do so. However, one can extend a bit beyond this strict threshold (which cannot be moved by changing the weight, such as h, since H_p annihilates the latter), at the cost of losing powers of the weight relative to the real principal type and radial point settings, provided that the Hamilton dynamics *normally to* Γ is well-behaved. Here, for brevity, we do not discuss details, but the key feature is that, within the characteristic set, there are transversally intersecting smooth codimension 1 manifolds $\Gamma_\pm$ with intersection Γ, with Γ_- and Γ_+ the local stable, resp. unstable, manifolds along the flow. Then one can arrange defining functions $\phi_\pm$ for these such that $H_p\phi_\pm = \mp c_\pm \phi_\pm$, with $c_\pm > 0$ and $H_{\phi_+}\phi_- = \{\phi_+, \phi_-\} > 0$; the latter positivity plays an important role in the control of the subprincipal term in [83, 39]. The functions $c_\pm$ can be chosen in a manner related to the normal hyperbolicity of the flow, namely bounded from below and above, up to an ϵ loss, by the normal minimal and maximal expansion rates. They dictate the size of the 'gap', i.e. the upper bound for the wrong-sign subprincipal symbol, to be (up to an ϵ loss) half of the minimum expansion rate; see [57, 23, 21].

3.6. Semiclassical and scattering settings. These results have natural extensions to the other algebras considered above: $\Psi_\hbar(M)$ and $\Psi_{\mathrm{sc}}(M)$ (as well as its semiclassical version). A straightforward application of the results thus far then is the limiting absorption principle for *scattering metrics*, introduced by Melrose, modelled on the large ends of cones, including non-trapping estimates if the geodesic flow is *non-trapping*, i.e. all geodesics escape to infinity. (Indeed, this setting is where Melrose started studying Lagrangian sets of radial points, see [49]; the earlier work of Guillemin and Schaeffer was on isolated radial points [34], see also [37] and [36].) Concretely, a scattering metric is a Riemannian metric g on M° of the form

$$g = x^{-4}\,dx^2 + x^{-2}h$$

near ∂M, where h is a smooth symmetric 2-cotensor which restricts to a Riemannian metric on ∂M. This generalizes the Euclidean metric, where one would take M to be the radial compactification of $\mathbb{R}^n$, so $\partial M = \mathbb{S}^{n-1}$ with the round metric h. One then has the following result of Melrose, with the semiclassical version due to Vasy and Zworski:

Theorem 3.1. *([49, 78]) The Laplacian Δ_g of a scattering metric has spectrum $[0,\infty)$, and for $\lambda > 0$ the limiting resolvents $R(\lambda^2 \pm \imath 0)$ exist as bounded operators $H^{s,r} \to H^{s+2,r-1}$, provided the weight r satisfies $r > 1/2$ at the incoming radial set, $r < 1/2$ on the outgoing radial set.*

Further, if the manifold is non-trapping then one has non-trapping resolvent estimates

$$\|R(\lambda^2 \pm \imath 0)\|_{\mathcal{L}(H^{s,r}_{|\lambda|^{-1}}, H^{s+2,r-1}_{|\lambda|^{-1}})} \leq C|\lambda|^{-1},\ \ \lambda \gg 1.$$

Here we do not provide further detail, but in fact this scattering framework also works directly for Klein-Gordon equations on *non-trapping Lorentzian scattering metrics* in the sense of Baskin, Vasy and Wunsch [5]; see also [41, Section 5]. Both of these discuss the actual wave equation, which requires b-methods described at the end of these notes, but in fact the Klein-Gordon version is much easier (as far as Fredholm analysis is concerned) as it can be done in the very amenable scattering setting, see [77]. For $\Box_g - \lambda$, $\lambda > 0$, and g of signature $(1, n-1)$, the characteristic set has two components, and within each there are two components of the radial set. One can thus choose the direction of propagation in either component separately. Choosing forward propagation in the base 'time' variable, this is the forward propagator; reversing it one gets the adjoint, the backward propagator. These correspond to the Cauchy problem. However, choosing forward propagation relative to the Hamilton flow, which means propagation in the opposite directions in the two components of the characteristic set relative to the base 'time' variable, gives a Feynman propagator; similarly choosing the backward one relative to the Hamilton flow gives another Feynman propagator. Indeed, even ultrahyperbolic equations are perfectly well-behaved: e.g. if g is a non-degenerate translation invariant metric, then for the corresponding d'Alembertian $\Box_g$, $\Box_g - \lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$, fits into this framework. Here, in general, the radial set has two components, and the Feynman propagator is the only reasonable option – this corresponds to the Cauchy problem being ill-behaved.

4. Applications

4.1. Anosov flows. One of the simplest kinds of differential operator is a vector field. Following earlier work of Faure and Sjöstrand [24], Dyatlov and Zworski [20] adapted a PDE point of view to analyze $\mathcal{C}^\infty$ Anosov flows $\varphi_t : X \to X$ on a compact manifold X, $\phi_t = \exp(tV)$, from the perspective of the generator V. Here the Anosov property means that the tangent space TX has a continuous (in x) decomposition into a stable subspace $E_s(x)$, an unstable subspace $E_u(x)$, and the neutral direction of $E_0(x) = \mathrm{Span}(V(x))$. Then the differential operator one studies is $P = \frac{1}{\imath}\mathcal{L}_V$ on differential forms, which has scalar principal symbol given by that of V. The key ingredient to the meromorphic continuation of the dynamical zeta function, which can be expressed as a (regularized) trace, is the analysis of $(P-\lambda)^{-1}$ on appropriate function spaces. But with $E_s^*(x)$ and $E_u^*(x)$ the dual bundles, they are sources/sinks for the Hamilton flow (which is just the flow of V lifted to $T^*X \setminus o$ from the homogeneous perspective), and the microlocal analysis we discussed yields the desired analytic Fredholm statement for the family $\lambda \mapsto P-\lambda$. A wave front set analysis then allows Dyatlov and Zworski to complete the proof of Theorem 1.3.

4.2. Asymptotically hyperbolic and de Sitter spaces. As a more involved application, the results discussed so far by themselves suffice to show the meromorphic extension of the resolvent of an asymptotically hyperbolic Laplacian together with high energy estimates using $\Psi_\hbar(M)$. We start by recalling the definition of manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of an n-dimensional compact manifold with boundary X_0 such that near the boundary Y, with a product decomposition $[0,\epsilon)_x \times Y$ of a neighborhood U of Y and a boundary defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2}$$

where h is a family of metrics on $Y = \partial X_0$ depending on x in an even manner, i.e. all odd derivatives of h with respect to x vanish at Y. (There is a much more natural way to phrase the evenness condition due to Guillarmou [29].) Then the dual metric is

$$G_0 = x^2(\partial_x^2 + H),$$

with H the dual metric family of h (depending on x as a parameter), and so

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-1+x^2\gamma)(xD_x) + x^2\Delta_h,$$

with γ even, and Δ_h the x-dependent family of Laplacians of h on Y. We then consider the spectral family

$$\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2$$

of the Laplacian. In addition to working with finite σ, or σ in a compact set, we also want to consider $\sigma \to \infty$, mostly in strips, with $|\operatorname{Im}\sigma|$ bounded. In that case we should consider σ as a 'large parameter' in the sense of [61]. Such a setting can be converted into a semiclassical one by writing $\sigma = h^{-1}z$, where $h \sim |\sigma|^{-1}$ (one can even take $h = |\sigma|^{-1}$, but this is often not convenient); then the spectral family becomes $h^{-2}(h^2\Delta_{g_0} - h^2\frac{(n-1)^2}{4} - z^2)$.

We show now that if we change the smooth structure on X_0 by declaring that only even functions of x are smooth, i.e. introducing $\mu = x^2$ as the boundary defining function, then after a suitable conjugation and division by a vanishing factor the resulting operator smoothly and non-degenerately continues across the boundary, i.e. continues to $X_{-\delta_0} = (-\delta_0, 0)_\mu \times Y \sqcup X_{0,\mathrm{even}}$, where $X_{0,\mathrm{even}}$ is the manifold X_0 with the new smooth structure. At the level of the principal symbol, i.e. the dual metric, the conjugation is irrelevant, so we can easily see what happens: changing to coordinates (μ, y), $\mu = x^2$, as $x\partial_x = 2\mu\partial_\mu$,

$$G_0 = 4\mu^2\partial_\mu^2 + \mu H = \mu(4\mu\partial_\mu^2 + H),$$

so after dividing by μ, we obtain $\mu^{-1}G_0 = 4\mu\partial_\mu^2 + H$. This is a quadratic form that is positive definite for $\mu > 0$, is Lorentzian for $\mu < 0$, and has a transition at $\mu = 0$ that as we shall see involves radial points. In fact, a similar argument would show that in $\mu < 0$, this dual metric is obtained by similar manipulations performed on the negative of a signature $(1, n-1)$ even *asymptotically de Sitter metric*, i.e. one of the form $\tilde{x}^{-2}(d\tilde{x}^2 - h)$, with $\tilde{x}$ the boundary defining function, and h positive definite at $\tilde{x} = 0$. Then $\mu = -\tilde{x}^2$ gives this form of the metric. Notice that $-\tilde{x}^2$ and x^2 are formally the 'same', i.e. $\tilde{x}$ is formally like $\imath x$, which means that this extension across the boundary is a mathematically precise general realization of a 'Wick rotation'. Correspondingly, in addition to providing a new method of analysis for asymptotically hyperbolic spaces, extension across the boundary also provides a new approach to asymptotically de Sitter analysis, providing an alternative to [81, 6].

To see that the full spectral family of the Laplacian is well behaved, first, changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n - 1 + \mu\gamma)(\mu D_\mu) + \mu\Delta_h.$$

Now we conjugate by $\mu^{-\imath\sigma/2+(n+1)/4}$, and multiply by $\mu^{-1/2}$ from both the left and right

$$\begin{aligned}\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}&(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}\\ &= 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4).\end{aligned}$$

This operator is in $\mathrm{Diff}^2(X_{0,\mathrm{even}})$, and now it continues smoothly across the boundary, by extending h and γ in an arbitrary smooth manner; it is now of the form

$$P_\sigma = 4(1 + a_1)\mu D_\mu^2 - 4(1 + a_2)\sigma D_\mu - a_3\sigma^2 + \Delta_h - 4\imath D_\mu + b_1\mu D_\mu + b_2\sigma + c_1$$

with a_j smooth, real, vanishing at $\mu = 0$, b_j and c_1 smooth. This form suffices for analyzing the problem for σ in a compact set, or indeed for σ going to infinity in a strip near the reals. (In [74] a further modification is made to obtain semiclassical ellipticity when σ leaves this strip in an appropriate manner.)

Writing covectors as $\xi\,d\mu + \eta\,dy$, the principal symbol of $P_\sigma \in \mathrm{Diff}^2(X_{-\delta_0})$, including in the high energy sense ($\sigma \to \infty$), is

$$p_{\mathrm{full}} = 4(1+a_1)\mu\xi^2 - 4(1+a_2)\sigma\xi - a_3\sigma^2 + |\eta|^2_{\mu,y},$$

and is real for σ real. Correspondingly, the standard principal symbol is

$$p = \sigma_2(P_\sigma) = 4(1+a_1)\mu\xi^2 + |\eta|^2_{\mu,y},$$

which is real, independent of σ, and elliptic for $\mu > 0$.

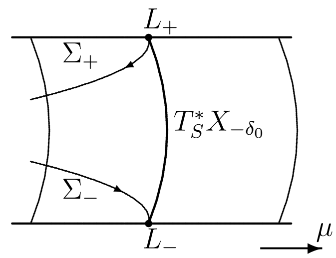

Figure 1. The cotangent bundle of $X_{-\delta_0}$ near $S = \{\mu = 0\}$ in a fiber-radially compactified view. The boundary of the fiber compactification is the cosphere bundle $S^*X_{-\delta_0}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \le 0$, only meeting $S^*_S X_{-\delta_0}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^* X_{-\delta_0}$, for $z = h^{-1}\sigma > 0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu > 0$. This is reversed for $z < 0$.

Let

$$N^*S \setminus o = \Lambda_+ \cup \Lambda_-, \qquad \Lambda_\pm = N^*S \cap \{\pm\xi > 0\}, \qquad S = \{\mu = 0\};$$

thus $S \subset X_{-\delta_0}$ can be identified with $Y = \partial X_0 (= \partial X_{0,\mathrm{even}})$. Note that $p = 0$ at $\Lambda_\pm$ and H_p is radial there since

$$N^*S = \{(\mu, y, \xi, \eta) :\ \mu = 0,\ \eta = 0\},\ \text{so}\ H_p|_{N^*S} = -4\xi^2\partial_\xi.$$

This corresponds to $dp = 4\xi^2\,d\mu$ at N^*S, so the characteristic set $\Sigma = \{p = 0\}$ is smooth at N^*S.

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Then L_- is a sink and L_+ is a source in the sense that all bicharacteristics nearby converge to $L_\pm$ as the parameter goes to $\mp\infty$. Further, one computes that $\tilde\beta|_{L_\pm} = \mathrm{Im}\,\sigma$. In the other direction, all bicharacteristics reach $\mu = -\epsilon_0$, $\epsilon_0 > 0$ small, so adding complex absorption there assures that we have a Fredholm problem if we make the choice of *propagating all estimates away from L_+ and L_- for P_σ, and towards L_+ and L_- in P^*_σ*. To be precise, we take two copies of $X_{-\delta_0}$, smoothly glued at $\mu = -\epsilon_0$, where complex absorption is introduced, to obtain a compact manifold without boundary X; alternatively, one can work with a single copy, and replace the complex absorption by a boundary working with spaces of extendible distributions for P_σ, and supported distributions for P_σ^*, see [41, Section 2]. This requires that the order of the Sobolev space (for P_σ) be sufficiently high, namely the more negative $\operatorname{Im}\sigma$ becomes, the more positive the Sobolev order must be. Indeed, if $\tilde f \in \mathcal{C}^\infty(X)$, then $(P_\sigma - \imath Q_\sigma)^{-1}\tilde f \in \mathcal{C}^\infty(X)$ as well (away from poles of this operator). If the geodesic flow is non-trapping then in fact we have semiclassical propagation/radial point estimates, which in turn imply the non-trapping statement of Theorem 1.1.

While this explains why $(P_\sigma - \imath Q_\sigma)^{-1}$ is a well-behaved operator, it may not be obvious how this helps with understanding the resolvent of the Laplacian, $R(\sigma)$. However, this is not hard to see. To make the extension from $X_{0,\mathrm{even}}$ to X more systematic, let $E_s : H^s(X_{0,\mathrm{even}}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(X_{0,\mathrm{even}})$ the restriction map. Then in $\operatorname{Im}\sigma > 0$, when σ is not a pole of either $R(\sigma)$ or $(P_\sigma - \imath Q_\sigma)^{-1}$, we have for $f \in \dot{\mathcal{C}}^\infty(X_0)$,

$$R(\sigma)f = x^{(n+1)/2-i\sigma}x^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}x^{-(n+1)/2+i\sigma}x^{-1}f, \tag{4.1}$$

since a simple computation shows that the right hand side is an element of $L^2(X_0, dg)$ (indeed, it is of the form $x^{(n-1)/2-\imath\sigma}\mathcal{C}^\infty(X_{0,\mathrm{even}})$, since after the application of $(P_\sigma - \imath Q_\sigma)^{-1}$ in the formula, the result is in $\mathcal{C}^\infty(X)$) with $\Delta_{g_0} - (n-1)^2/4 - \sigma^2$ applied to it yielding f, so by the self-adjointness of Δ_{g_0}, it is indeed $R(\sigma)f$. Notice that this uses very strongly that Q_σ has Schwartz kernel supported away from $X_{0,\mathrm{even}} \times X_{0,\mathrm{even}}$ (i.e. more than just $\mathrm{WF}'(Q) \cap S^*X_{0,\mathrm{even}} = \emptyset$).

In fact, in this unified treatment of asymptotically hyperbolic and de Sitter spaces one can even arrange a set up which does not need complex absorption at all, and does not need an artificially added boundary. To do so, given an asymptotically hyperbolic space (X_0, g_0), one can construct a compact manifold X without boundary containing two (disjoint) copies of X_0, connected by an asymptotically de Sitter space; one may call the two copies the 'future' and 'past' copies. (Vice versa, given an asymptotically de Sitter space, one can cap it off by two asymptotically hyperbolic spaces; one may need to take two copies of the de Sitter space, however, for topological reasons.) This is motivated by the structure of the boundary of radially compactified Minkowski space, which has two copies of hyperbolic space in the interior of the future and past light cones at infinity, and a copy of de Sitter space outside these light cones. However, the construction can be made in full generality, see [76, Section 3]. In this case one propagates estimates from the conormal bundle of the boundary of one of the copies of $X_{0,\mathrm{even}}$ (say, the past one) to the other one; for the adjoint the estimates propagate in the opposite direction. Since the threshold regularity for the radial points is the same for both the future and the past copies, this requires variable order Sobolev space; in this case one can actually arrange that the order varies only in the interior of the asymptotically de Sitter space, and depends only on the base, X (not on the location within the fiber of S^*X).

4.3. Kerr-de Sitter spaces. Next we turn to Kerr-de Sitter spaces, which are 4-dimensional Lorentzian space-times. Here an appropriate bordification of the space-time is

$$M_\delta = X_\delta \times [0, \infty)_\tau, \qquad X_\delta = (r_- - \delta, r_+ + \delta)_r \times \mathbb{S}^2,$$

where $r_\pm$ are specified later, and where $\tau = e^{-t}$ in $\tau > 0$ for a more conventional 'time' variable t (that is essentially equivalent to the usual time far from $r_\pm$ in (r_-, r_+)). On this the signature $(1, 3)$ dual metric G has the form

$$\begin{aligned} G = -\rho^{-2}\Big(\mu\big(\partial_r \pm c\tau\partial_\tau\big)^2 &\pm 2(1+\gamma)(r^2+a^2)\big(\partial_r \pm c\tau\partial_\tau\big)\tau\partial_\tau \\ &\mp 2(1+\gamma)a\big(\partial_r \pm c\tau\partial_\tau\big)\partial_\phi + \kappa\partial_\theta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta\tau\partial_\tau + \partial_\phi)^2\Big), \end{aligned}$$

where r_s, Λ, a constants, $r_s, \Lambda \geq 0$, $\kappa = 1 + \gamma\cos^2\theta$, $\gamma = \frac{\Lambda a^2}{3}$,

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \mu = (r^2+a^2)(1 - \frac{\Lambda r^2}{3}) - r_s r,$$

and $\mu(r) = 0$ has two positive roots $r = r_\pm$, $r_+ > r_-$, with $F_\pm = \mp\frac{\partial\mu}{\partial r}|_{r=r_\pm} > 0$; r_+ is the de Sitter end (cosmological horizon), r_- is the Kerr end (event horizon). Physically, r_s is twice the black hole mass, Λ is the cosmological constant, a is the angular momentum of the black hole. Thus, de Sitter space is the case $a = 0$, $r_s = 0$, and Λ can be normalized to be 3; in this case $r_- = 0$ can be removed, and the space-time becomes $M_\delta = X_\delta \times [0, \infty)_\tau$, $X_\delta = \mathbb{B}_{r_+ + \delta}$, with $\mathbb{B}_{r_+ + \delta}$ the ball of radius $r_+ + \delta$. If $a = 0$ still, but $r_s > 0$, then one obtains non-rotating de Sitter-Schwarzschild black holes.

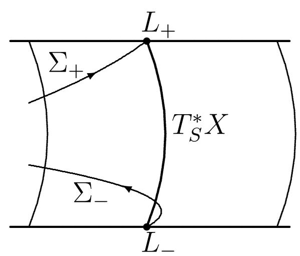

Figure 2. The fiber-radially compactified cotangent bundle near the event horizon $S = \{\mu = 0\}$. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. The characteristic set crosses the event horizon on both components; here the part near L_+ is hidden from view. The projection of this region to the base space is the ergoregion. Semiclassically, i.e. the interior of $\overline{T}^* X_\delta$, for $z = h^{-1}\sigma > 0$, only $\Sigma_{\hbar,+}$ can enter $\mu > a^2$.

Mellin transforming $\Box_g$ in τ (i.e. Fourier transforming in e^{-t}), with dual parameter σ, one obtains a family of operators P_σ, whose principal symbol in the large parameter sense is given by this dual metric function. One can now check that P_σ *almost* has the same structure as the conjugated extended asymptotically hyperbolic case. Most importantly, if $a = 0$, $N^*\{r = r_\pm\}$ in X_δ consists of radial points which are sources or sinks; if $a \neq 0$, then instead $N^*\{r = r_\pm\}$ are still 'normally source/sink bundles', as required for our generalized radial points results, but there is non-trivial dynamics within $N^*\{r = r_\pm\}$ corresponding to the black hole rotation (closed orbits, along which ϕ varies and θ is fixed). Further, if one adds complex absorption where μ is small and negative, i.e. just outside $[r_-, r_+] \times \mathbb{S}^2$, then all classical bicharacteristics reach $N^*\{r = r_\pm\}$ or the elliptic set of the complex absorption in both the forward and backward direction, i.e. the problem is *classically non-trapping*, and thus one can make it Fredholm by capping off the space X_δ (e.g. by working with two copies of the space, glued via complex absorption, so the problem is elliptic). Again a useful, but less microlocal, alternative is to add a boundary at $\mu = -\delta_0$, and set up appropriate Fredholm Cauchy-type problems. Although it does not play a role in our analysis, one interesting feature of Kerr-de Sitter wave operators with $a \neq 0$ is that the projection of the characteristic set of P_σ to the base space enters (r_-, r_+); this is called the *ergoregion* – the operator is thus *not* elliptic everywhere between the event horizons. This was considered a major difficulty for the analysis, and was first overcome by Dyatlov [22] by a separation of variables argument; the microlocal analysis described here achieves a similar result in a systematic manner.

However, the operator is semiclassically trapping due to the *photon sphere* in the de Sitter-Schwarzschild case, and its no longer spherically symmetric replacement in general. This trapped set is, however, *normally hyperbolic*. The works of Wunsch and Zworski [83], Hintz and Vasy [39] and Dyatlov [23, 21] give microlocal control at this trapped set, which, combined with gluing constructions of Datchev and Vasy [16], suffices to prove Theorem 1.2.

While Kerr-de Sitter space had not been intensively studied, though there have been works on de Sitter-Schwarzschild space ($a = 0$) [4, 59, 9] and further references in [75], we mention that Kerr space-time has been the subject of intensive research. For instance, polynomial decay on Kerr space was shown recently by Tataru and Tohaneanu [69, 68] and Dafermos, Rodnianski and Shlapentokh [13, 12, 14], while electromagnetic waves were studied by Andersson and Blue [3], after pioneering work of Kay and Wald in [43] and [82] in the Schwarzschild setting. While some of these papers employ microlocal methods at the trapped set, they are mostly based on physical space where the phenomena are less clear than in phase space (unstable tools, such as separation of variables, are often used in phase space though). Kerr space is less amenable to immediate microlocal analysis to attack the decay of solutions of the wave equation due to the singular/degenerate behavior at zero frequency; in some sense it combines the scattering and b-analysis.

4.4. Melrose's b-analysis. While here we used the dilation invariance to reduce to a problem on X_δ, this is easily eliminated. The framework then is Melrose's b-pseudodifferential operator algebra $\Psi_{\mathrm{b}}(M)$, introduced in [53] to study hyperbolic boundary value problems; see [54] for a general setup. On a general manifold M, this microlocalizes the earlier mentioned Lie algebra $\mathcal{V}_{\mathrm{b}}(M)$ of vector fields tangent to the boundary, which are locally of the form $a(x\partial_x) + \sum b_j \partial_{y_j}$. These are the smooth sections of a vector bundle; the dual bundle ${}^{\mathrm{b}}T^*M$ has a local basis $\frac{dx}{x}, dy_j$ over $\mathcal{C}^\infty(M)$. Now from the homogeneous perspective the (standard) principal symbol is a homogeneous function on ${}^{\mathrm{b}}T^*M \setminus o$; for 0th order operators, it can be considered as a function on ${}^{\mathrm{b}}S^*M = ({}^{\mathrm{b}}T^*M \setminus o)/\mathbb{R}^+$.

Much like for $\Psi_{\mathrm{sc}}(M)$, the standard principal symbol does not capture operators modulo (relatively) compact ones. However, unlike the scattering case, there is no other function making up for this deficit (in the case of the scattering algebra, the symbol on ${}^{\mathrm{sc}}T^*_{\partial M}M$), rather it is an operator, called the *normal operator*. This is obtained by 'freezing coefficients' at ∂M to obtain a dilation invariant operator. Together the principal symbol and the Mellin transformed (as it is dilation invariant!) normal operator $\hat{L}(\sigma)$ do allow for a development of Fredholm theory. However, this is a bit more intricate: one has to work with b-Sobolev spaces $H_{\mathrm{b}}^{s,r}(M)$ which have constant weights r, which on the Mellin transform side corresponds to working on the line $\operatorname{Im}\sigma = -r$, but s variable for many non-elliptic problems of interest (though Kerr-de Sitter allows for constant s). Now, at the principal symbol level there are analogues of all of the microlocal ingredients described above; indeed, one also has to allow $L_\pm$ to have a 'normally saddle' structure for Kerr-de Sitter type settings, see [41]. This allows one to conclude that $\hat{L}(\sigma)$ is a Fredholm family on induced spaces. However, in order to have a Fredholm problem on M, one needs that $\hat{L}(\sigma)$ is *invertible* for $\operatorname{Im}\sigma = -r$, so non-symbolic, or 'quantum' objects determine Fredholm properties of L. On the other hand, under this assumption, one indeed has a Fredholm problem, which is perturbation stable in the appropriate sense. This gives the stability of the Kerr-de Sitter problem. In fact, the earlier mentioned Lorentzian scattering metrics, studied by Baskin, Vasy and Wunsch [5], fit into the same general framework.

4.5. Non-linear equations. The final topic we discuss is non-linear PDE. Small data semilinear problems in either non-trapping or, with lower order semilinear terms, normally hyperbolic Kerr-de Sitter type settings can be easily solved by the contraction mapping principle as long as one can work with Sobolev spaces with non-growing weights (i.e. one can choose such a weight r with no resonances σ with $\operatorname{Im}\sigma = -r$), or one has special properties of the resonances for Sobolev spaces $H_{\mathrm{b}}^{s,0}(M)$. In this case, for instance polynomial semilinear terms (for second order equations, to be definite) map $H_{\mathrm{b}}^{s,r}$ to $H_{\mathrm{b}}^{s-1,r}$ for $s > n/2+1$, and thus the Fredholm structure we discussed provides for a Picard iteration for small data; see [41]. The same setting for Lorentzian scattering metrics, generalizing results of Klainerman [44, 45] and Christodoulou [11] is more delicate, both because unlike the saddle points of Kerr-de Sitter space, the radial source/sinks in Minkowski space limit regularity when one is propagating estimates towards them, and also because the reduction to a b-problem involves weights, so there is a more complicated numerology, and rely on additional microlocal regularity relative to a pseudodifferential module (which generalizes Klainerman's vector fields), see [41].

While the linear setting of asymptotically Minkowski spaces had well-behaved global dynamics and thus no artificial tools such as complex absorption was needed (we only needed variable order Sobolev spaces), in the Kerr-de Sitter, and indeed localized de Sitter, type settings one needs to 'cut off' the problem. Such a cut off is possible due to the hyperbolic nature of the equations. From the perspective of microlocal analysis it is most conveniently done via complex absorption as discussed above, but this may not provide complete control: one only gets the *exact* solution operator one wants (with supports) if each bicharacteristic is controlled at least at one end by other means, such as radial points or a boundary. Thus, in the non-dilation invariant de Sitter and Kerr-de Sitter type setting it is convenient to consider domains Ω in the manifold M whose (artificial) boundary hypersurfaces (other than those of M, that is) are space-like. (Note that such a Cauchy hypersurface is just as artificial as complex absorption, is less well-behaved microlocally, but has the advantage of giving the supports one wants for time-oriented problems by standard local energy estimates!) As all the complicated phenomena, such as radial points or trapping, or indeed even variable orders of Sobolev spaces, are located away from these artificial boundaries, including these artificial boundaries (as done in [41]) in the framework does not pose significant complications. One obtains, for instance, the small data (here f) well-posedness, with vanishing Cauchy data at the appropriate boundary hypersurface, of Klein-Gordon equations

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du),$$

where q is a polynomial with second order vanishing at $(0, 0)$ (so quadratic terms are allowed) if $m > 0$ and the metric g is non-trapping, such as perturbations of asymptotically de Sitter type spaces. Here ${}^{\mathrm{b}}du$ denotes derivatives relative to the b-structure, i.e. the derivatives are given by b-vector fields. If the metric has normally hyperbolic trapping such as Kerr-de Sitter metrics, the losses in derivatives provided by the normally hyperbolic estimates only allow for general non-linearities independent of ${}^{\mathrm{b}}du$ for the contraction mapping argument to go through, though non-linearities depending on derivatives with a particular structure are allowed as well since the loss of derivatives is only microlocally at the trapped set. If $m = 0$ the issue is the 0-resonance, which has resonant state 1, and thus non-linearities q which *only contain derivatives*, and thus annihilate the resonant state, are allowed in the non-trapping asymptotically de Sitter type settings. In either case, one also obtains an expansion at infinity which is generated by the resonances of the Mellin transformed normal operator of the linear problem.

Quasilinear problems require more work. Hintz [38] has developed a framework for b-pseudodifferential operators with Sobolev coefficients, modeled on the Sobolev pseudodifferential operators of Beals and Reed [8]. This framework is sufficient in non-trapping settings, such as perturbations of de Sitter space, to achieve this. More recently, Hintz and Vasy [40] extended this analysis even to normally hyperbolic problems. In this case the contraction mapping is replaced by a use of the Nash-Moser iteration due to the losses in derivatives; the conclusion is a small data global well-posedness and decay result for quasilinear wave equations on Kerr-de Sitter space: for the small mass Klein-Gordon equation without further restrictions (since there is no 0-resonance), while for the actual wave equation for non-linearities containing derivatives (due to the 0-resonance).

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