Wave propagation and high energy resolvent estimates for De Sitter-Schwarzschild space

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Goal: description of the asymptotics of solutions of the wave equation on de Sitter-Schwarzschild space, with exponential decay rate.

Features:

- Requires high energy estimates on the analytic continuation of the resolvent of an operator asymptotic to the hyperbolic Laplacian acting on *slightly* weighted spaces,
- *and* a good understanding of the geometry.

The latter ensures, for instance, that one needs to apply the resolvent to functions in these slightly weighted spaces (which is not a priori the case).

A simpler case, that already has almost all of the geometric features is de Sitter space, given by the hyperboloid

$$z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1 \text{ in } \mathbb{R}^{n+1}$$

equipped with the pull-back of the Minkowski metric $dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2$. Introduce polar coordinates (R, θ) in $(z_1, \ldots, z_n)$, write $\tau = z_{n+1}$, so the hyperboloid can be identified with $\mathbb{R}_\tau \times \mathbb{S}_\theta^{n-1}$ with the Lorentzian metric

$$\frac{d\tau^2}{\tau^2 + 1} - (\tau^2 + 1)\, d\theta^2.$$

For $\tau > 1$, let $x = \tau^{-1}$, so the metric becomes

$$\frac{(1 + x^2)^{-1}\, dx^2 - (1 + x^2)\, d\theta^2}{x^2}.$$

An analogous formula holds for $\tau < -1$, so if we compactify the real line as an interval $[0, 1]_T$ (with $T = x$ for $x < \frac{1}{4}$, say), we obtain a compactification of de Sitter space on which the metric is conformal to a non-degenerate Lorentz metric.

A natural generalization is *asymptotically de Sitter-like spaces*, $\widehat{\mathrm{M}}$, which are diffeomorphic to $[0,1]_T \times Y$ for a compact manifold without boundary Y with a Lorentz metric g on the interior conformal to a Lorentz metric $\bar{g}$ smooth up to the boundary, $|dx|_{\bar{G},\ x=0} = 1$ ($x = T$, $1 - T$), one has the following result:

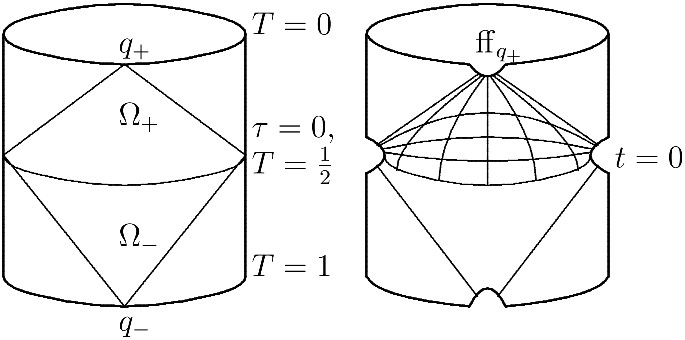
Theorem

Let $s_\pm(\lambda) = \frac{n-1}{2} \pm (\frac{(n-1)^2}{4} - \lambda)^{1/2}$. The solution u of the Cauchy problem for $\Box_g - \lambda$ with C^∞ initial data at $T = 1/2$ has the form

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-,\ \ v_\pm \in C^\infty(\widehat{\mathrm{M}}),$$

if $s_+(\lambda) - s_-(\lambda) \notin \mathbb{N}$. If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in C^\infty(\widehat{\mathrm{M}})$ by $v_- = C^\infty(\widehat{\mathrm{M}}) + x^{s_+(\lambda) - s_-(\lambda)} \log x\, C^\infty(\widehat{\mathrm{M}})$.

In fact, one also knows that the scattering matrix is an FIO, and one understands the strucrture of the forward fundamental solution (the latter is the work of Dean Baskin).



On the left, the compactification of de Sitter space with the backward light cone from q_+ and forward light cone from q_- are shown. Ω_+, resp. Ω_-, denotes the intersection of these light cones with $T > 0$, resp. $T < 0$. On the right, the blow up of de Sitter space $\bar{M}'$, together with the spatial and temporal coordinate lines of the static model in Ω_+. The interior of the light cone inside the front face ff_{q_+} can be identified with the spatial part of the static model of de Sitter space.

The *static model* of de Sitter space arises by singling out a point on $\mathbb{S}^{n-1}_\theta$, e.g. $q_0 = (1, 0, \ldots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$:

- it is the intersection of the backward lightcone from q_0 considered as a point q_+ at $T = 0$, and the forward light cone from q_0 considered as a point q_- at $T = 1$.
- These intersect the equator $T = 1/2$ (here $\tau = 0$) in the same set, and altogether form a 'diamond'.
- Explicitly this region is given by $z_2^2 + \ldots + z_n^2 \leq 1$ inside the hyperboloid.
- Blow up the corner where the light cones intersect $\tau = 0$, as well as q_+ and q_-; call the resulting space $\bar{M}'$.

$\bar{M}$ can also be obtained as follows.

- Consider $[0,1]_T \times \overline{\mathbb{B}^3}$, with $T = e^{-2t}$ for $t > 4$, say.
- In polar coordinates (r, ω) on $\overline{\mathbb{B}^3}$, consider the Lorentz metric

$$(1 - r^2)\, dt^2 - (1 - r^2)^{-1} dr^2 - r^2\, d\omega^2.$$

- Blow up the corners to obtain $\bar{M}$.
- It is straightforward to see that $\bar{M}$ and $\bar{M}'$ are (almost) diffeomorphic and isometric.
- 'Almost' refers to this approach gives that the defining function of ff_{q_+} in $\bar{M}$ is x^2 – this corresponds to an evenness statement for the Lorentz metric in the sense of Guillarmou.

While one *can* analyze the solutions of the wave equations on de Sitter space at points inside the ‘diamond’ by considering the diamond only (in view of the finite propagation speed for the wave equation), the resulting picture does include rather artificial limitations.

For instance, the local static asymptotics, corresponding to the tip of the diamond at Y_+, describes only a small part of the asymptotics of solutions of the Cauchy problem on de Sitter space.

We now turn to de Sitter-Schwarzschild space.
Static model: let

$$M = \mathbb{R}_t \times X,\ X = (r_{\mathrm{bh}}, r_{\mathrm{dS}})_r \times \mathbb{S}^2_\omega$$

with the Lorentzian metric

$$g = \alpha^2\, dt^2 - \alpha^{-2}\, dr^2 - r^2\, d\omega^2,$$

where

- $$\alpha = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{1/2},$$

  with Λ and m suitable positive constants, $0 < 9m^2\Lambda < 1$,
- $r_{\mathrm{bh}}, r_{\mathrm{dS}}$ the two positive roots of α,
- $d\omega^2$ the standard metric on $\mathbb{S}^2$.
- the static model of De Sitter space is given by $m = 0$, $\Lambda = 3$.

We also consider the compactification of X:

$$\bar{X} = [r_{\mathrm{bh}}, r_{\mathrm{dS}}]_r \times \mathbb{S}^2_\omega.$$

- $\mu = \alpha^2$ is a C^∞ defining function for the two boundary hypersurfaces of $\bar{X}$,
- can also consider α as a boundary defining function of $\bar{X}$, amounting to a change in the C^∞ structure of $\bar{X}$. Denote the new manifold by $\bar{X}_{\frac{1}{2}}$.

The d'Alembertian with respect to this metric is

$$\Box = \alpha^{-2}(D_t^2 - \alpha^2 r^{-2} D_r(r^2\alpha^2 D_r) - \alpha^2 r^{-2}\Delta_\omega),$$

where Δ_ω is the Laplacian on $\mathbb{S}^2$.

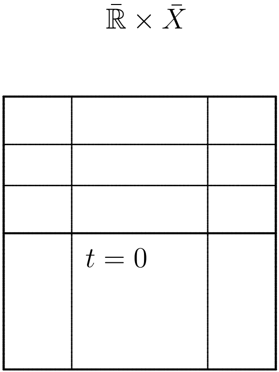
For the wave equation, we also compactify time to an interval $I = [0,1]_T$:

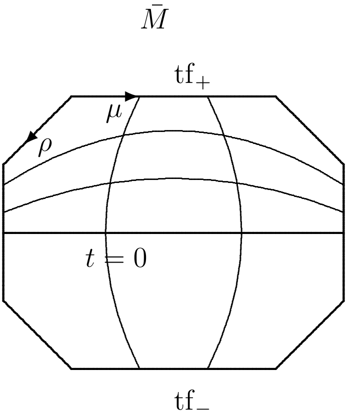
- let
$$T_+ = T_{\lambda,+} = e^{-2\lambda t} \text{ in } t > C,$$
with λ to be decided,
- let
$$T = T_+ \text{ in } t > C,$$
- similarly, $T_- = e^{2\lambda t}$, let $T = 1 - T_-$ in $t < -C$.

It turns out that

$$\bar{\mathbb{R}}_t \times \bar{X} = [0,1]_T \times [r_{\mathrm{bh}}, r_{\mathrm{dS}}]_r \times \mathbb{S}^2_\omega.$$

is not the best space to consider the asymptotics.





The space-time product compactification of de Sitter-Schwarzschild space and its blow up $\bar{M}$ are shown, with the time and space coordinate lines indicated by thin lines. These are no longer valid coordinates on $\bar{M}$. Valid coordinates near the top left corner are ρ and μ.

Blow up the corners

$$\{0\} \times \{r_{\mathrm{bh}}\} \times \mathbb{S}^2,\ \{0\} \times \{r_{\mathrm{dS}}\} \times \mathbb{S}^2$$

(and analogously at $T = 1$, i.e. $T_- = 0$), denoting this space by $\bar{M}$.

Thus, a neighborhood $U = U_{\lambda,+}$ of the 'future temporal face' tf_+, given by $T = 0$, is diffeomorphic to

$$[0, \epsilon)_\rho \times [r_{\mathrm{bh}}, r_{\mathrm{dS}}] \times \mathbb{S}^2_\omega,\ \ \rho = T_{\lambda,+}/\mu.$$

In the interior of tf_+, this is in turn diffeomorphic to an open subset of

$$[0, \epsilon)_T \times (r_{\mathrm{bh}}, r_{\mathrm{dS}}) \times \mathbb{S}^2_\omega.$$

If we replace μ by α as the defining function, we obtain $\bar{M}_{1/2}$. Note that tf_+ is naturally diffeomorphic to $\bar{X}$ in $\bar{M}$, and to $\bar{X}_{1/2}$ in $\bar{M}_{1/2}$.

Let

$$\mathcal{A}^m_{\mathrm{tf}_+}(\bar{M})$$

denote the space of functions v which are C^∞ on $\bar{M}$ away from tf_+, and are conormal at tf_+, including smoothness up to the boundary of tf_+. That is, for any k and smooth vector fields $V_1, \dots, V_k$ on $\bar{M}$ which are tangent to tf_+,

$$V_1 \dots V_k v \in \rho^m L^2_{\mathrm{b},\mathrm{tf}_+}(\bar{M}),$$

where $L^2_{\mathrm{b},\mathrm{tf}_+}(\bar{M})$ is the L^2-space with respect to ρ^{-1} times smooth non-degenerate densities on $\bar{M}$.

Thus,

- $\rho^m C^\infty(\bar{M}) \subset \mathcal{A}^{m-\epsilon}_{\mathrm{tf}_+}(\bar{M})$ for all $\epsilon > 0$,
- $\mathcal{A}^m_{\mathrm{tf}_+}(\bar{M}) \subset \rho^m L^\infty(\bar{M})$.

The first (and technically main) result on wave propagation is the following.

Theorem

Suppose $\Box u = 0$, u is C^∞ in $\bar{M} \cap \{\rho \in (0,1)\}$. Then there exists a constant c and $\epsilon > 0$ such that $u - c$ is in $\mathcal{A}^{\epsilon}_{\mathrm{tf}_+}(\bar{M}) = \rho^\epsilon \mathcal{A}^0_{\mathrm{tf}_+}(\bar{M})$ near tf_+.

Thus,

- there is an asymptotic limit of u, uniformly on $\bar{X}$,
- the convergence is exponentially fast in terms of 'time',
- the constant corresponds to the zero resonance of the 'spatial Laplacian', described below.

We would like to phrase the assumption on u in terms of initial conditions. To do so, we note that one can blow down the spatial faces, i.e. there is a manifold $\overline{\mathrm{M}}$ and a C^∞ map β, $\beta : \bar{M} \to \overline{\mathrm{M}}$ such that

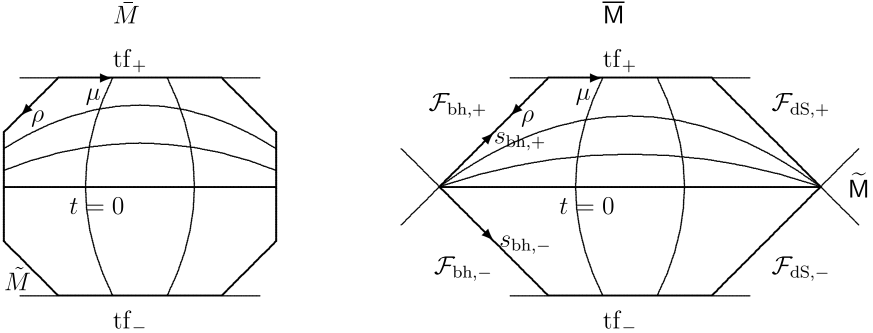
- β is a diffeomorphism away from spatial infinity,
- the metric g lifts to a C^∞ Lorentz b-metric on $\overline{\mathrm{M}}$, b at the temporal face, smooth at the other faces, with respect to which the non-temporal faces are characteristic.

One valid coordinate system in a neighborhood of the black hole end of spatial infinity, disjoint from temporal infinity, is given by:

$$s_{\mathrm{bh},+} = \alpha/T^{1/2}_{\lambda_{\mathrm{bh}},+} = \rho^{-1/2}_{\mathrm{bh},+},$$
$$s_{\mathrm{bh},-} = \alpha/T^{1/2}_{\lambda_{\mathrm{bh}},-} = \alpha T^{1/2}_{\lambda_{\mathrm{bh}},+} = \mu\rho^{1/2}_{\mathrm{bh},+},\ \omega,$$

where as usual ω denotes coordinates on $\mathbb{S}^2$.

- $$\mathcal{F}_{\mathrm{bh},+} = \{s_{\mathrm{bh},-} = 0\}$$
  is the characteristic surface given by $\mu = 0$ in $T > 0$ (i.e. the front face of the blow up of the corner),
- $$\mathcal{F}_{\mathrm{bh},-} = \{s_{\mathrm{bh},+} = 0\}$$
  is its negative time analogue,
- the change of coordinates $(\rho_{\mathrm{bh},+}, \mu) \mapsto (s_{\mathrm{bh},+}, s_{\mathrm{bh},-})$ is a diffeomorphism from $(0,\infty) \times (0,\delta)$ onto its image, i.e. these coordinates are indeed compatible.



On the left, the $\bar{M}$ is shown, while on the right its blow-down $\overline{\mathsf{M}}$. The time and space coordinate lines corresponding to the $\mathbb{R} \times X$ decomposition are indicated by thin lines in the interior. Certain boundary hypersurfaces of $\bar{M}$ are continued by thin lines to show that the Lorentz metric extends smoothly along these (but not across tf_+ and tf_-!). The extended spaces are denoted by $\tilde{M}$ and $\widetilde{\mathsf{M}}$.

The main theorem then is:

Theorem

Suppose u solves $\Box u = 0$ with C^∞ Cauchy data on a space-like Cauchy surface Σ in $\widetilde{\mathrm{M}} \cap \{t \geq 0\}$, say $\Sigma = \{t = 0\}$ (i.e. $s_{\mathrm{bh},+} = s_{\mathrm{bh},-}$).

Then there exists a constant c and $\epsilon > 0$ such that $u - c$ is in $\mathcal{A}^{\epsilon}_{\mathrm{tf}_+}(\overline{\mathrm{M}}) = \rho^{\epsilon}\mathcal{A}^{0}_{\mathrm{tf}_+}(\overline{\mathrm{M}})$ near the future temporal face, tf_+.

Dafermos and Rodnianski proved a version of this theorem, with a logarithmic decay rate, i.e. $u - c \in (\log \rho)^{-N}\mathcal{A}^{0}_{\mathrm{tf}_+}(\overline{\mathrm{M}})$ for every N, instead of ρ^{ϵ}, by rather different methods.

In terms of our methods, the logarithmic convergence follows by obtaining polynomial bounds on the resolvent of the 'spatial Laplacian' and its derivatives *at the real axis*, rather than in a strip for the analytic continuation.

Writing $T = T_+ = T_{\lambda,+}$ in $t > C$, the dual metric has the form

$$G = 4\alpha^{-2}\lambda^2 T^2\partial_T^2 - \alpha^2\partial_r^2 - r^{-2}\partial_\omega^2$$

in the original product compactification, with

$$\partial_r = \frac{d\mu}{dr}\partial_\mu = 2\beta\partial_\mu.$$

The change of variables from r to μ is smooth and non-degenerate, i.e. $2\beta = d\mu/dr \neq 0$ for μ close to 0, i.e. r close to r_{bh} or r_{dS}.

Upon the blow-up, in $\bar{M}$, near tf_+, in the coordinates (ρ, μ, ω), we have

$$G = 4\mu^{-1}\lambda^2\rho^2\partial_\rho^2 - 4\mu\beta^2(\partial_\mu - \mu^{-1}\rho\partial_\rho)^2 - r^{-2}\partial_\omega^2.$$

Thus,

$$G = \mu^{-1}\left(4\lambda^2\rho^2\partial_\rho^2 - 4\beta^2(\mu\partial_\mu - \rho\partial_\rho)^2\right) - r^{-2}\partial_\omega^2.$$

If we let $\lambda = \beta(r_{\rm bh}) > 0$ or $\lambda = -\beta(r_{\rm dS}) > 0$ then the $\rho^2\partial_\rho^2$ terms cancel, so *locally near* $r_{\rm bh}$

$$G = 4\gamma\rho^2\partial_\rho^2 + 8\beta^2\rho\partial_\rho\partial_\mu - 4\beta^2\mu\partial_\mu^2 - r^{-2}\partial_\omega^2,\ \gamma = \mu^{-1}(\beta(r_{\rm bh})^2 - \beta^2),$$

γ C^∞, with a similar expansion at $r_{\rm dS}$.

We remark that in fact the choice of λ determines the compactification $\bar{M}$, i.e. it is only at this point that the compactification has been specified!

The metric is a C^∞ Lorentzian b-metric on

$$[0,\epsilon)_\rho \times (r_{\mathrm{bh}}-\epsilon, r_{\mathrm{bh}}+\epsilon)_r \times \mathbb{S}^2_\omega,$$

i.e. is non-degenerate as a quadratic form on the b-cotangent bundle, in particular it is C^∞ across $\mu = 0$.
Write $\mathcal{F}$ for the set given by $\mu = 0$, i.e. the boundary hypersurface of $\bar{M}$ that is no longer a boundary hypersurface of $\tilde{M}$.

For this metric $\mathcal{F}$ is characteristic, and one has the standard propagation of singularities in $\rho > 0$. In particular, for C^∞ initial data specified at a Cauchy surface such as $\rho = \rho_0$ constant, the solution is smooth in $\rho > 0$ up to $\mu = 0$.

To see this:

- write covectors as

$$\xi \frac{d\rho}{\rho} + \zeta\, d\mu + \sum \eta_j\, d\omega_j,$$

- i.e. $(\rho, \mu, \omega, \xi, \zeta, \eta)$ are coordinates on ${}^bT^*\tilde{M}$,
- the dual metric (which is the principal symbol of $\Box$), considered as a function on ${}^bT^*\tilde{M}$, is

$$G = 4\gamma\xi^2 + 8\beta^2\xi\zeta - 4\beta^2\mu\zeta^2 - r^{-2}|\eta|^2,$$

- the Hamilton vector field of G is

$$\begin{aligned} H_G =& 8(\gamma\xi + \beta^2\zeta)\rho\partial_\rho - 8\beta^2(\mu\zeta - \xi)\partial_\mu \\ & - (4\frac{\partial\gamma}{\partial\mu}\xi^2 + 8\beta\frac{\partial\beta}{\partial\mu}(2\xi\zeta - \mu\zeta^2) \\ & \quad - 4\beta^2\zeta^2 - \frac{\partial r^{-2}}{\partial\mu}|\eta|^2)\partial_\zeta - r^{-2}H_{(\omega,\eta)}, \end{aligned}$$

with $H_{(\omega,\eta)}$ denoting the Hamilton vector field of the standard metric on the sphere.

The conormal bundle of $\mu = 0$ is $\mu = 0$, $\xi = 0$, $\eta = 0$, so the vector field is

$$8\beta^2\zeta\rho\partial_\rho + 4\beta^2\zeta^2\partial_\zeta$$

which is indeed tangent to the conormal bundle, and is non-radial off the zero section (where $\zeta \neq 0$) as long as $\rho \neq 0$.

At $\partial\mathcal{F}$, i.e. at $\rho = 0$, however we have radial points over the conormal bundle of $\mathcal{F}$. Rather than dealing with them directly, we reduce the problem to the study of the resolvent of the spatial Laplacian.

The study of this operator has a relatively long history, with poles of the analytic continuation of the resolvent studied by Sá Barreto and Zworski, and the cutoff resolvent studied by Bony and Häfner.

We now study the asymptotics at tf_+. Set $T = e^{-t}$. Then

$$\begin{aligned}\Box &= \alpha^{-2}\left((TD_T)^2 - \alpha^2 r^{-2} D_r \alpha^2 r^2 D_r - \alpha^2 r^{-2}\Delta_\omega\right)\\ &= \alpha^{-2}((TD_T)^2 - \Delta_X),\end{aligned}$$

where we introduced the spatial 'Laplacian'

$$\begin{aligned}\Delta_X &= \alpha^2 r^{-2} D_r \alpha^2 r^2 D_r + \alpha^2 r^{-2}\Delta_\omega\\ &= \beta r^{-2}\alpha D_\alpha \beta r^2 \alpha D_\alpha + \alpha^2 r^{-2}\Delta_\omega \text{ near } \alpha = 0.\end{aligned}$$

We also recall that not T, but rather $T^{\lambda_{\mathrm{bh}}}$ and $T^{\lambda_{\mathrm{dS}}}$ were used above to construct the compactification.

Thus,

$$\Delta_X \in \mathrm{Diff}_0^2(\bar{X}_{1/2}),$$

where, as usual, $\mathrm{Diff}_0(N)$ on a manifold with boundary N is the algebra of differential operators generated by vector fields *vanishing* at the boundary over $C^\infty(N)$ (i.e. $x\partial_x$, $x\partial_{y_j}$ in local coordinates in which x is a boundary defining function).

Although this is not the Laplacian of a Riemannian metric on X, it is not far from it: it is $d^* d$ with respect to

- the inner product on one-forms given by the fiber inner product with respect to the 'spatial part' $H = \alpha^2 \partial_r^2 + r^{-2} \partial_\omega^2$ of G
- and density on X given by $dh = \alpha^{-2} r^2 \, dr \, d\omega$.

Thus, Δ_X is self-adjoint on

$$L^2(X, |dh|), \; |dh| = \alpha^{-2} r^2 \, |dr| \, |d\omega| = \alpha^{-1} |\beta|^{-1} r^2 \, |d\alpha| \, |d\omega|,$$

and we will use the techniques of Mazzeo and Melrose to study its resolvent.

Sá Barreto and Zworski showed that its resolvent

$$R(\sigma) = (\Delta_X - \sigma^2)^{-1},\ \operatorname{Im}\sigma < 0,$$

admits an analytic continuation from the 'physical half plane', $\operatorname{Im}\sigma < 0$, with only one pole in a half plane $\operatorname{Im}\sigma < \epsilon$, ϵ sufficiently small, which is the pole 0.

Bony and Häfner obtained polynomial bounds on the cutoff resolvent, $\chi R(\sigma)\chi$, $\chi \in C_c^\infty(X)$, as $|\sigma| \to \infty$ in the strip $|\operatorname{Im}\sigma| < \epsilon$.

This implies that the *local energy* (essentially the behavior of the solution in a compact subset of the interior of tf_+) for $C_c^\infty(X)$ initial data decays to the energy corresponding to the 0-resonance.

Our extension of their result is both to allow more general initial data (not compactly supported ones) and to study the asymptotics uniformly up to $\partial\mathrm{tf}_+$.

It is also useful to introduce the operator

$$L = \alpha\Delta_X\alpha^{-1} \in \mathrm{Diff}_0^2(\bar{X}_{1/2}),$$

which is self-adjoint on

$$L^2(X, \alpha^{-2}\,|dh|) = \alpha L^2(X, dh),\ \alpha^{-2}|dh| = \alpha^{-3}|\beta|^{-1}r^2\,|d\alpha|\,|d\omega|.$$

Thus, this space is $L_0^2(X)$ as a Banach space, up to equivalence of norms.

The normal operators $N_{0,\mathrm{bh}}(L)$, $N_{0,\mathrm{dS}}(L)$ of L in $\mathrm{Diff}_0^2(\bar{X}_{1/2})$ at $r = r_{\mathrm{bh}}$, resp. $r = r_{\mathrm{dS}}$, are

$$N_{0,\mathrm{bh}}(L) = \lambda_{\mathrm{bh}}^2 N_{0,\mathrm{bh}}(\Delta_{\mathbb{H}^3}),\ N_{0,\mathrm{dS}}(L) = \lambda_{\mathrm{dS}}^2 N_{0,\mathrm{dS}}(\Delta_{\mathbb{H}^3}),$$

where $\Delta_{\mathbb{H}^3}$ is the hyperbolic Laplacian, explaining the usefulness of this conjugation.

Let

$$\tilde\alpha \in C^\infty(X),\ \tilde\alpha > 0,$$
$$\tilde\alpha = \alpha^{1/\lambda_{\rm bh}} \text{ for } r \text{ near} r_{\rm bh},$$
$$\tilde\alpha = \alpha^{1/\lambda_{\rm dS}} \text{ for } r \text{ near} r_{\rm dS}.$$

From the work of Mazzeo-Melrose (with improvements by Guillarmou) the resolvent

$$\mathcal{R}(\sigma) = (L-\sigma^2)^{-1},\ \text{on } L^2_0(\bar X_{1/2}),\ \operatorname{Im}\sigma < 0,$$

continues meromorphically to a strip $|\operatorname{Im}\sigma| < \epsilon$ as an operator between weighted L^2-spaces (as well as other spaces):

$$\mathcal{R}(\sigma) : \tilde\alpha^{\delta} L^2_0(\bar X_{1/2}) \to \tilde\alpha^{-\delta} L^2_0(\bar X_{1/2}),\ \delta > \epsilon;$$

we keep denoting the analytic continuation by $\mathcal{R}(\sigma)$.

Thus,

$$R(\sigma) = (\Delta_X - \sigma^2)^{-1} = \alpha^{-1}\mathcal{R}(\sigma)\alpha \text{ on } L^2(X, |dh|),\ \operatorname{Im}\sigma < 0,$$

continues meromorphically to a strip $|\operatorname{Im}\sigma| < \epsilon$ (again, we keep denoting the analytic continuation by $R(\sigma)$):

$$R(\sigma) : \tilde{\alpha}^{\delta} L^2(X, |dh|) \to \tilde{\alpha}^{-\delta} L^2(X, |dh|),\ \delta > \epsilon.$$

The main result we need about this resolvent is the following bounds on its behavior as $|\operatorname{Re}\sigma| \to \infty$ in the strip (with the analogous bounds outside the strip in the lower half plane being much easier):

Theorem

There exists $\epsilon > 0$ with the following properties.
The only pole of the analytic continuation of the resolvent $R(\sigma)$ in $\operatorname{Im}\sigma < \epsilon$ is $\sigma = 0$, which is a simple pole, with residue given by a constant γ.
Moreover, for each k and $\delta > \epsilon$ there exist $m > 0$ and $C > 0$ such that

$$\|\tilde{\alpha}^{-i\sigma} R(\sigma)\|_{\mathcal{L}(\alpha^{-1}\tilde{\alpha}^{\delta} H_0^m(\bar{X}_{1/2}), C^k(\bar{X}))} \leq C|\sigma|^C,$$

for $|\sigma| > 1$, $\operatorname{Im}\sigma < \epsilon$.

This theorem in fact follows from just the corresponding weighted L^2-estimates and the fact that in the Mazzeo-Melrose parametrix construction in the non-semiclassical (i.e. non-high-energy) setting the pseudo-differential seminorms blow up like σ^ℓ, ℓ dependent on the seminorm.

Theorem

$$(\exists\epsilon > 0)(\forall\delta > \epsilon)(\exists C > 0)$$
$$|\sigma| > 1,\ \operatorname{Im}\sigma < \epsilon \Rightarrow \|\tilde{\alpha}^\delta R(\sigma)\tilde{\alpha}^\delta\|_{\mathcal{L}(L^2(X_{1/2},|dh|))} \leq C|\sigma|^C.$$

The proof has two ingredients:

- a semiclassical parametrix construction for the analytic continuation of the resolvent near infinity, i.e. $\partial\bar{X}_{1/2}$
- propagation estimates in compact sets, which are conveniently available from the work of Bony and Häfner in terms of estimates on the cutoff resolvent.

If one wanted weighted L^2 estimates *only at the real axis*, one could do it quite a bit more easily:

- take the estimates for the cutoff resolvent, due to Bony and Häfner, or analogous semiclassical propagation estimates at the trapped set,
- paste this with well-known high energy resolvent estimates localized near infinity, studied particularly by Cardoso and Vodev, but with origins in the work of Burq,
- e.g. using the method of Bruneau and Petkov.

This gives the Dafermos-Rodnianski result without having to construct a high energy parametrix for the analytic continuation of the resolvent.

We now sketch the high energy parametrix construction for the analytic continuation of the resolvent.

Here $X = \mathbb{B}^n$ the closed ball, with an operator L which

- has principal symbol given by a 0-metric,
- is asymptotic to $\Delta_{\mathbb{H}^n}$.

An example is obtained by taking a neighborhood of either end of $\bar{X}_{1/2}$ with the operator $\lambda^{-2}\alpha\Delta_X\alpha^{-1}$, and transferring it to a ball, via a smooth cutoff.

Let $X_0^2 = [X \times X; \partial\mathrm{diag}]$ be the zero double space. First, we construct the distance function on $X^\circ \times X^\circ$, uniformly to infinity, using this compactification. We write

$$\rho_L, \ \rho_R, \ \rho_{\mathrm{ff}}$$

for the defining functions of the lifts of $\partial X \times X$, $X \times \partial X$, $\partial\mathrm{diag}$. Thus, $\pi_L^* x$ is a smooth non-vanishing multiple of $\rho_L \rho_{\mathrm{ff}}$, if x is a defining function for ∂X.

Local coordinates on ${}^0T^*X$, the dual bundle of ${}^0TX$, induced by local coordinates $(x, y_1, \ldots, y_{n-1})$ on X arise by writing one-forms as

$$\lambda \frac{dx}{x} + \sum_j \mu_j \frac{dy_j}{x},$$

as opposed to the induced local coordinates on T^*X:

$$\xi\, dx + \sum_j \eta_j\, dy_j.$$

Thus, over X° we have the canonical identification of T^*X° with ${}^0T^*X^\circ$

$$T^*X^\circ \ni (x, y, \xi, \eta) \mapsto (x, y, x\xi, x\eta) \in {}^0T^*X,$$

which is C^∞ as a map $T^*X \to {}^0T^*X$, but is *not* a diffeomorphism at ∂X.

If g is a 0-metric, i.e. has the form $x^{-2}\bar{g}$, $\bar{g}$ is C^∞ Riemannian, $|dx|_{\bar{G}} = 1$ (so $\bar{g} = dx^2 + h$, where $h|_{x=0}$ a metric on ∂X), Δ_g its Laplacian, then the principal symbol of Δ_g is the dual metric

$$p \in C^\infty({}^0T^*X), \text{ and } p|_{x=0} = \lambda^2 + H(y,\mu),$$

H the dual metric of $h|_{x=0}$. The lift of the Hamilton vector field of p from T^*X° is

$$\mathsf{H}_p = x\frac{\partial p}{\partial \lambda}\partial_x + x\frac{\partial p}{\partial \mu}\cdot\partial_y - \left(\mu\cdot\frac{\partial p}{\partial \mu} + x\frac{\partial p}{\partial x}\right)\partial_\lambda - \left(-\frac{\partial p}{\partial \lambda}\mu + x\frac{\partial p}{\partial y}\right)\cdot\partial_\mu,$$

which equals

$$\frac{\partial p}{\partial \lambda}(x\partial_x) - \mu\cdot\frac{\partial p}{\partial \mu}\partial_\lambda + \frac{\partial p}{\partial \lambda}\mu\cdot\partial_\mu$$

modulo vector fields tangent to ${}^0T^*_{\partial X}X$. In particular, for our p, this is equal to

$$2\lambda(x\partial_x + \mu\partial_\mu) - 2H\partial_\lambda,$$

which at $p = 1$ vanishes exactly at the *radial set* $x = 0$, $\mu = 0$, which is a source/sink.

- If we blow up the radial set (inside $p = 1$), H_p lifts to a vector field of the form $\rho_{\mathrm{ff}} V$, V transversal to the front face.
- Thus, the renormalized flow of H_p, i.e. the flow of $\rho_{\mathrm{ff}}^{-1} H_p = V$, reaches the front face in finite time.

To construct the distance function, we lift the argument to X_0^2, or more precisely to the pull-back of ${}^0T^*X \otimes {}^0T^*X$ to X_0^2.

- The 0-Hamilton vector field of the metric lifted from the left factor, H_L, has radial points at the left face, and similarly with the right Hamilton vector field, H_R.
- Upon blowing up the set of radial points, and factoring out the defining function of the front face, the rescaled Hamilton vector fields are transversal to the front face.
- *If* the metrics are close, the flowout of $N^*\mathrm{diag}$ under the joint flow of H_L and H_R is close to the flowout of $N^*\mathrm{diag}$ for the hyperbolic metric.
- Thus, the flowout is the graph of the differential of the distance function.

One concludes that, as on hyperbolic space,

$$d(z,z') = -\log(\rho_L \rho_R) + F, \ \ F \in C^\infty(X_0^2 \setminus \mathrm{diag}_0).$$

The required semiclassical parametrix, *if the metric is close to the hyperbolic metric*, can now be constructed on

$$[X_0^2 \times [0,1)_h; \mathrm{diag}_0 \times \{0\}], \ \ h = |\operatorname{Re}\sigma|^{-1},$$

by considering the ansatz (for $n=3$)

$$G(\sigma,z,z') = \frac{e^{-i\sigma d(z,z')}}{4\pi \sinh d(z,z')} U(\sigma,z,z').$$

motivated by the hyperbolic resolvent, which has the same form, with d replaced by $d_{\mathbb{H}^3}$, and $U = U_{\mathbb{H}^3} \equiv 1$.

We want U to be smooth on

$$[X_0^2 \times [0,1)_h \times (-\epsilon,\epsilon)_{\mathrm{Im}\,\sigma}; \mathrm{diag}_0 \times \{0\} \times (-\epsilon,\epsilon)_{\mathrm{Im}\,\sigma}].$$

Thus, $\mathrm{Im}\,\sigma$ is regarded as a bounded parameter; the semiclassical rescaling is in $\mathrm{Re}\,\sigma$.

We also want $U \equiv 1$ on the semiclassical front face, and on the 0-front face, where the 'error' $(L-\sigma^2)\mathcal{R}(\sigma) - \mathrm{Id}$ already vanishes to leading order, with G the Schwartz kernel of $\mathcal{R}(\sigma)$.

Notice that (for $\mathrm{Re}\,\sigma > 1$)

$$L - \sigma^2 = h^{-2}(h^2 L - (1 + ih\,\mathrm{Im}\,\sigma)^2),$$

so $\mathrm{Im}\,\sigma$ contributes a subprincipal term only in the semiclassical sense. However, at infinity, it has a rather important impact.

We only need analytic continuation to a small strip: there are resonances farther away anyway. In the non-high energy framework, this would merely necessitate arranging

- the correct principal symbol and
- the correct normal operator,

which our choice of U would imply (i.e. there is no need for iteration).

In the semiclassical setting:

- One can solve away the error at the semiclassical front face as usual.
- The transport equations can be solved to give U at $h = 0$.
- The ansatz, with $U = 1$ at the zero front face, assures that there is no error there to leading order.
- At the left face, the error term as a right parametrix is better than a priori expected as usual.

Returning to de Sitter-Schwarzschild space, one can combine

- the resolvents on the de Sitter and black hole ends, for which we had obtained a parametrix, and
- high energy cutoff resolvent estimates (or instead semiclassical propagation estimates at the trapped set), as obtained by Bony and Häfner,
- using for instance the method of Bruneau and Petkov,

to prove polynomial bounds on the resolvent acting on weighted spaces.

Suppose u solves the wave equation, $\Box u = 0$, and u is smooth on $\bar{M}$ in $\rho > 0$. Energy estimates show that u is necessarily tempered, e.g. in the sense that $u \in \rho^{-s} L^2(\bar{M})$ for some $s > 0$, but in fact this can be shown more directly, as done below.

Let $\phi \in C^\infty(\bar{M})$ be a cutoff function, ϕ supported near tf_+, identically 1 in a neighborhood of tf_+: we can (and do) take $\phi = \phi_0(\rho)$ with $\rho \in C_c^\infty([0,1))$, identically 1 near 0.

If u solves $\Box u = 0$ and u is smooth in μ (i.e. across the side faces), then ϕu is smooth in μ, and

$$\Box(\phi u) = [\Box, \phi] u = f$$

where f is also smooth in μ, and vanishes in a neighborhood of the temporal face in view of $[\Box, \phi] \in \mathrm{Diff}_b^1(\tilde{M})$, supported away from tf_+. Moreover, $v = \phi u$ is the *unique* solution of $\Box v = f$ in $\bar{M}^\circ$ with $v = 0$ for ρ sufficiently large.

Consider the Mellin transform in $T = e^{-t}$, for functions supported in a neighborhood of tf_+, namely in $U = \{\rho < 1\}$:

- U is equipped with a fibration $U \to \bar{X}$, extending the fibration $U^\circ \to X$ in the interior,
- there is a natural density $|dt| = \frac{|dT|}{T}$ on the fibers,
- which in coordinates $(\tilde{\rho}, \alpha, \omega) = (T^{\lambda_{\mathrm{bh}}}/\alpha, \alpha, \omega)$ valid near the boundary of tf_+ at $r = r_{\mathrm{bh}}$ takes the form $\frac{|d\tilde{\rho}|}{\lambda_{\mathrm{bh}}\tilde{\rho}}$; $\tilde{\rho} = \rho^{1/2}$.

The Mellin transform with respect to this fibration and density is the map $v \mapsto \hat{v}$ from functions supported in U (i.e. near tf_+) to functions on $\Omega \times \bar{X}$, $\Omega \subset \mathbb{C}$,

$$\hat{v}(\sigma, z) = \int T^{i\sigma} v(T, z) \frac{|dT|}{T}.$$

If v is polynomially bounded in T, supported in $T \geq 0$, with values in a function space $\mathcal{H}$ in z, this transform gives an analytic function in a lower half plane (depending on the order of growth of v) with values in $\mathcal{H}$.

Rewrite the integral near $\partial\mathrm{tf}_+$, $\tilde{\rho} = T^{\lambda_{\mathrm{bh}}}/\alpha$:

$$\hat{v}(\sigma,\alpha,\omega) = \alpha^{i\sigma/\lambda_{\mathrm{bh}}}\lambda_{\mathrm{bh}}^{-1}\int \tilde{\rho}^{i\sigma/\lambda_{\mathrm{bh}}} v(\tilde{\rho},\alpha,\omega)\,\frac{|d\tilde{\rho}|}{\tilde{\rho}},$$

and the integral is merely the Mellin transform of v with respect to $\tilde{\rho}$ evaluated at $\sigma/\lambda_{\mathrm{bh}}$.

Thus, if v is smooth on $\tilde{M}^\circ$, supported in $\{0 < \tilde{\rho} < 1\}$ then $\hat{v}$ is in fact analytic in $\mathbb{C}$ with values in functions of the form $\tilde{\alpha}^{i\sigma}C^\infty$, with C^∞ seminorms all bounded by $C_k\langle\sigma\rangle^{-k}$, k arbitrary.

If v is supported in U with polynomial bounds at tf_+, then $\hat{v}$ is analytic in a lower half plane.

If ϕu is polynomially bounded in T, $\Box(\phi u) = f$ becomes

$$(\sigma^2 - \Delta_X)\,\widehat{\phi u} = \alpha^2 \hat{f}.$$

If ϕu is polynomially growing in T, then both $\hat{f}$ and $\widehat{\phi u}$ are analytic in $\operatorname{Im}\sigma < -C$, and as f is compactly supported in $\tilde{\rho}$, $\hat{f}$ is in fact analytic in all of $\mathbb{C}$, with values in functions of the form $\tilde{\alpha}^{i\sigma} C^\infty$, with C^∞ seminorms all bounded by $C_k \langle\sigma\rangle^{-k}$, k arbitrary. Thus,

$$\widehat{\phi u} = R(\sigma)(\alpha^2 \hat{f}),\ \operatorname{Im}\sigma < -C,$$

and we recover ϕu by taking the inverse Mellin transform.

Now we drop the a priori polynomial bound assumption on u, and let $f = \Box(\phi u)$, as above. Then

- $\hat{f}$ is analytic in all of $\mathbb{C}$, with values in functions of the form $\tilde{\alpha}^{i\sigma} C^\infty$, with C^∞ seminorms all bounded by $C_k \langle \sigma \rangle^{-k}$, k arbitrary,
- observe that the inclusion

$$\alpha^{1+s} L^\infty(X) \hookrightarrow L^2_0(\bar{X}_{1/2})$$

  is continuous for every $s > 0$,
- use this to conclude

$$\|\alpha^2 \hat{f}\|_{\alpha^{-1}\tilde{\alpha}^\delta H^m_0(\bar{X}_{1/2})} \leq C_k \langle \sigma \rangle^{-k}$$

  for all k in $\operatorname{Im} \sigma < \epsilon < \delta$ (with new constants), $0 < \epsilon < \delta$ sufficiently small.

- Our resolvent estimates show that, for $\epsilon > 0$ sufficiently small and for all N and k,

$$\|\tilde{\alpha}^{-i\sigma} R(\sigma)(\alpha^2 \hat{f})\|_{C^N(\bar{X})} \leq C_k |\sigma|^{-k}, \operatorname{Im}\sigma < \epsilon.$$

- The inverse Mellin transform of $w = R(\sigma)(\alpha^2 \hat{f})$ is

$$\check{w}(T, z) = (2\pi)^{-1} \int T^{-i\sigma} w(\sigma, z)\, d\sigma.$$

- Thus,

$$\check{w}(\tilde{\rho}, \alpha, \omega) = (2\pi)^{-1} \int \tilde{\rho}^{-i\sigma/\lambda_{\mathrm{bh}}} \alpha^{-i\sigma/\lambda_{\mathrm{bh}}} w(\sigma, \alpha, \omega)\, d\sigma.$$

- In view of the analyticity of $\check{w}$ in the lower half plane with the stated estimates, $w = 0$ for $T < 0$.
- Since the unique solution of $\Box v = f$, v supported in $T \geq 0$, is ϕu, we deduce that $\check{w} = \phi u$.

Shifting the contour for the inverse Mellin transform for w to $\operatorname{Im}\sigma = \epsilon$ gives a residue term at 0, and shows that for $\epsilon' < \epsilon$,

$$\rho^{-\epsilon'}(\phi u - c) \in L^2([0,\delta)_\rho; C^\infty(\bar{X})),$$

where c arises from the residue at 0, hence is a constant.

The D_t derivatives also satisfy similar estimates, i.e. the same estimates hold for the conormal derivatives with respect to ρ. We thus deduce the leading part of the asymptotics of u at tf_+.

Theorem

Suppose $\Box u = 0$, u is C^∞ in $\bar{M}$ in $\rho > 0$. Then there exists a constant c and $\epsilon > 0$ such that $u - c$ is in $\mathcal{A}^{\epsilon}_{\mathrm{tf}_+}(\bar{M}) = \rho^\epsilon \mathcal{A}^0_{\mathrm{tf}_+}(\bar{M})$ near tf_+.

The main result, which we now recall, immediately follows:

Theorem

If u solves $\Box u = 0$ with C^∞ Cauchy data on a space-like Cauchy surface Σ in $\widetilde{\mathrm{M}} \cap \{t \geq 0\}$, say $\Sigma = \{t = 0\}$ (i.e. $s_{\mathrm{bh},+} = s_{\mathrm{bh},-}$), then there exists a constant c and $\epsilon > 0$ such that $u - c$ is in $\mathcal{A}^{\epsilon}_{\mathrm{tf}_+}(\overline{\mathrm{M}}) = \rho^{\epsilon}\mathcal{A}^{0}_{\mathrm{tf}_+}(\overline{\mathrm{M}})$ near the future temporal face, tf_+.