# Wave propagation on asymptotically de Sitter and Anti de Sitter spaces

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There have been extensive studies of elliptic PDE on various complete non-compact manifolds; the goal is to extend results to hyperbolic PDE, such as the wave or Klein-Gordon equations.

Even for elliptic problems, there are often hyperbolic phenomena at infinity. A good example is scattering on asymptotically Euclidean spaces (or 'scattering metrics', to be precise), or indeed on $\mathbb{R}^n$ itself.

Thus, as shown by Melrose and refined by Melrose-Zworski and others, 'singularities' of solutions of $(\Delta - \lambda)u = 0$, here meaning microlocal lack of decay, propagate along bicharacteristics, and the scattering operator is a Fourier integral operator associated to the geodesic flow on the boundary at distance $\pi$.

One of the simplest elliptic settings is that of *conformally compact* manifolds. These are compact manifolds with boundary $\bar{X}$ equipped with a Riemannian metric $g$ on $X = \bar{X}^\circ$, such that

- $\hat{g} = x^2 g$ extends to be a $C^\infty$ positive-definite symmetric-cotensor up to $\partial X$ (so $\hat{g}$ is a Riemannian metric on $\bar{X}$), where $x$ is a defining function of $\partial X$, and such that
- $|dx|_{\hat{G}} = 1$; here $\hat{G}$ is the dual metric of $g$.

We can equivalently phrase this as follows: there is a product decomposition $[0,\epsilon)_x \times Y$ of a neighborhood $U$ of $Y = \partial X$, on which

$$g = \frac{dx^2 + h}{x^2}$$

with $h \in C^\infty(X; \mathrm{Sym}^2 T^*X)$, and $h|_Y$ is a section of $\mathrm{Sym}^2 T^*Y$.

The restriction on $|dx|_{\hat{G}}$ or on $h|_Y$ is a zeroth order product structure statement at $Y$, which is convenient, but not absolutely necessary for many considerations – see in particular the work of Borthwick.

A specific example is hyperbolic space $X$, given by the upper half ($z_{n+1} > 0$) of the two-sheeted hyperboloid

$$z_1^2 + \ldots + z_n^2 = z_{n+1}^2 - 1 \text{ in } \mathbb{R}^{n+1}$$

equipped with the negative of the pull-back of the Minkowski metric

$$dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2.$$

Since $z_{n+1} = \sqrt{z_1^2 + \ldots + z_n^2 + 1}$, $X$ is diffeomorphic to $\mathbb{R}^n$. To see its structure near infinity, we introduce polar coordinates $(R, \theta)$ in $(z_1, \ldots, z_n)$, hence $z_{n+1} = \sqrt{R^2 + 1}$, so the metric becomes

$$g = \frac{dR^2}{R^2 + 1} + R^2\, d\theta^2,$$

so with $x = R^{-1}$, we obtain

$$g = \frac{(1 + x^2)^{-1}\, dx^2 + d\theta^2}{x^2},$$

which is indeed conformally compact.

There are two immediate generalizations to Lorentzian metrics. Our convention is that the signature of Lorentzian metrics is $(1, n-1)$, so the maximal dimension of a subspace of $T_qX$ on which $g$ is positive definite is 1.

Recall that a Lorentzian metric on $X$ is a section $g$ of $C^\infty(X; \mathrm{Sym}^2 T^*X)$ such that for each $q \in X$, $g$ is a non-degenerate form on $T_qX$ (hence gives an isomorphism $T_qX \to T_q^*X$), and such that maximal dimension of a subspace of $T_qX$ on which $g$ is positive definite is 1.

Recall also that a Lorentzian metric $g$ still induces a non-degenerate density, $|dg|$, given by $\sqrt{|\det g_{ij}|}\,|dx_1 \ldots dx_n|$, hence an inner product on $C_c^\infty(X)$. Thus, there is an associated Laplace-Beltrami operator, called the d'Alembertian, given by $\Box_g = d^*d + dd^*$, or simply $d^*d$ on functions.

Assume that $\bar{X}$ is a manifold with boundary, $g$ a Lorentzian metric on $X = \bar{X}^\circ$, $x$ a defining function of $Y = \partial X$, and assume that $g$ is conformal to a Lorentzian metric on $\bar{X}$ in the sense that for some (hence any) boundary defining function $x$ of $Y$, $\hat{g} = x^2 g$ is (i.e. extends to) a Lorentzian metric on $\bar{X}$. There are two non-degenerate cases regarding the boundary behavior:

- $Y$ is space-like, i.e. $\langle dx, dx \rangle_{\hat{G}} > 0$,
- $Y$ is time-like, i.e. $\langle dx, dx \rangle_{\hat{G}} < 0$.

For convenience, we also adopt a zeroth order product structure assumption:

- $(\bar{X}, g)$ is asymptotically de Sitter if $Y$ is space-like and $\langle dx, dx \rangle_{\hat{G}} = 1$,
- $(\bar{X}, g)$ is asymptotically anti-de Sitter if $Y$ is time-like and $\langle dx, dx \rangle_{\hat{G}} = -1$.

A concrete example of the first class is de Sitter space, given by the hyperboloid

$$z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1 \text{ in } \mathbb{R}^{n+1}$$

equipped with the pull-back of the Minkowski metric

$$dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2.$$

Introduce polar coordinates $(R, \theta)$ in $(z_1, \ldots, z_n)$, noting $z_1^2 + \ldots + z_n^2 \geq 1$, write $\tau = z_{n+1}$, so the hyperboloid can be identified with $X = \mathbb{R}_\tau \times \mathbb{S}_\theta^{n-1}$ with the Lorentzian metric

$$g = \frac{d\tau^2}{\tau^2 + 1} - (\tau^2 + 1)\, d\theta^2.$$

For $\tau > 1$, let $x = \tau^{-1}$, so the metric becomes

$$\frac{(1+x^2)^{-1}\,dx^2 - (1+x^2)\,d\theta^2}{x^2}.$$

An analogous formula holds for $\tau < -1$, so if we compactify the real line as an interval $[0,1]_T$ (with $T = x$ for $x < \frac{1}{4}$, say), we obtain a compactification of de Sitter space on which the metric is conformal to a non-degenerate Lorentz metric.

Anti-de-Sitter space is defined analogously. Consider $\mathbb{R}^{n+1}$ with the pseudo-Riemannian metric of signature $(2, n-1)$ given by

$$-dz_1^2 - \ldots - dz_{n-1}^2 + dz_n^2 + dz_{n+1}^2,$$

with $(z_1, \ldots, z_{n+1})$ denoting coordinates on $\mathbb{R}^{n+1}$, and the hyperboloid

$$z_1^2 + \ldots + z_{n-1}^2 - z_n^2 - z_{n+1}^2 = -1$$

inside it. Anti-de-Sitter space is this hyperboloid $X$ equipped with the pull-back $g$ of the above pseudo-Riemannian metric.

Since $z_n^2 + z_{n+1}^2 \geq 1$ on the hyperboloid, so we can (diffeomorphically) introduce polar coordinates in these two variables, i.e. we let $(z_n, z_{n+1}) = R\theta$, $R \geq 1$, $\theta \in \mathbb{S}^1$. Then the hyperboloid is of the form

$$z_1^2 + \ldots + z_{n-1}^2 - R^2 = -1$$

inside $\mathbb{R}^{n-1} \times (0, \infty)_R \times \mathbb{S}^1_\theta$. As $dz_j$, $j = 1, \ldots, n-1$, $d\theta$ and $d(z_1^2 + \ldots + z_{n-1}^2 - R^2)$ are linearly independent at the hyperboloid,

$$z_1, \ldots, z_{n-1}, \theta$$

give local coordinates on it, and indeed these are global in the sense that the hyperboloid $X^\circ$ is identified with $\mathbb{R}^{n-1} \times \mathbb{S}^1$ via these.

The metric $g$ is indeed Lorentzian. Away from $\{0\} \times \mathbb{S}^1$, we obtain a convenient form of the metric by using polar coordinates $(r, \omega)$ in $\mathbb{R}^{n-1}$, so $R^2 = r^2 + 1$:

$$\begin{aligned} g &= -(dr)^2 - r^2\, d\omega^2 + (dR)^2 + R^2\, d\theta^2 \\ &= -(1+r^2)^{-1}\, dr^2 - r^2\, d\omega^2 + (1+r^2)\, d\theta^2, \end{aligned}$$

where $d\omega^2$ is the standard round metric; a similar description is easily obtained near $\{0\} \times \mathbb{S}^1$ by using the standard Euclidean variables.

We compactify the hyperboloid by compactifying $\mathbb{R}^{n-1}$ to a ball $\overline{\mathbb{B}^{n-1}}$ via inverse polar coordinates $(x,\omega)$, $x=r^{-1}$,

$$(z_1,\ldots,z_{n-1})=x^{-1}\omega,\ 0<x<\infty,\ \omega\in\mathbb{S}^{n-2}.$$

The interior of $\overline{\mathbb{B}^{n-1}}$ is identified with $\mathbb{R}^{n-1}$, and the boundary $\mathbb{S}^{n-2}$ of $\overline{\mathbb{B}^{n-1}}$ is added at $x=0$ to compactify $\mathbb{R}^{n-1}$. Let

$$\bar{X}=\overline{\mathbb{B}^{n-1}}\times\mathbb{S}^1$$

be this compactification of $X$; a collar neighborhood of $\partial X$ is identified with

$$[0,1)_x\times\mathbb{S}^{n-2}_\omega\times\mathbb{S}^1_\theta.$$

In this collar neighborhood the Lorentzian metric takes the form

$$g=\frac{1}{x^2}\Big(-(1+x^2)^{-1}\,dx^2-d\omega^2+(1+x^2)\,d\theta^2\Big),$$

which is of the desired form, and with respect to $\hat{g}$, the boundary, $\{x=0\}$, is indeed time-like. The induced metric on the boundary is $-d\omega^2+d\theta^2$.

For Lorentzian metrics $g$, the basic geometric concept is that of null-bicharacteristics in $T^*X$, whose projections to the base space are null-geodesics. (For elliptic equations, the bicharacteristics themselves have only relevance at a high energy limit.)

To define null-bicharacteristics, let $G$ be the dual metric, also considered as the metric function on $T^*X$, and let

$$\Sigma = \{\alpha \in T^*X : \ G(\alpha) = 0\}$$

be the characteristic set. Then null-bicharacteristics are integral curves of $\mathsf{H}_G$ in $\Sigma$, where $\mathsf{H}$ is the Hamilton vector field of $G$ (using that $T^*X$ is symplectic).

Since the map $G \mapsto \mathsf{H}_G$ is a derivation, $\mathsf{H}_{aG} = a\mathsf{H}_G$ at $\Sigma$, so null-bicharacteristics are merely reparameterized if $G$ is replaced by a conformal multiple (unlike non-null bicharacteristics!). In particular, in $X$, the null-bicharacteristics of $g$ and $\hat{g}$ are the same (up to reparameterization).

A basic difference between asymptotically De Sitter and anti de Sitter spaces is that in the former there are no null-bicharacteristics of $\hat{g}$ that are tangent to $Y$, while in the latter this is possible.

Even for $\hat{g}$, one has to be careful in defining generalized broken (null) bicharacteristics at $Y$ in general, which is an indication that AdS-like spaces are more complicated than dS-like ones.

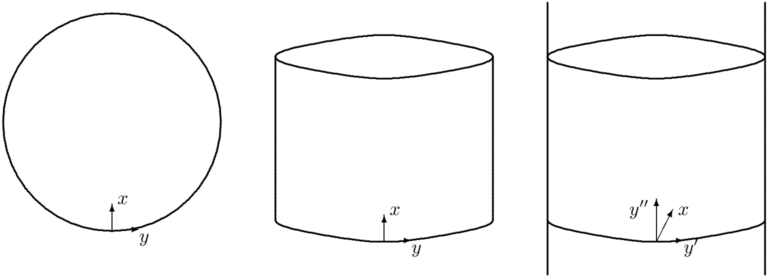
However, for AdS, $\hat{g}$ has the special feature that $Y$ is totally geodesic, unlike e.g. the case of $\mathbb{B}^{n-1} \times \mathbb{S}^1$ equipped with a product Lorentzian metric, with $\mathbb{B}^{n-1}$ carrying the standard Euclidean metric. This is a first order statement on $g$ at $Y$.

For hyperbolic PDE to have global solutions (rather than merely a local theory) one needs an extra assumption, such as global hyperbolicity. We take this to mean that there is a global 'time' function $t \in C^\infty(X)$, such that the surfaces $\mathcal{S}_{t_0} = \{t = t_0\}$ are spacelike, and every nullbicharacteristic crosses each $T^*_{\mathcal{S}_{t_0}} X$ in exactly one point. A spacelike hypersurface $\mathcal{S}$ such that each nullbicharacteristic intersects $T^*_{\mathcal{S}} X$ exactly once is a Cauchy hypersurface.

An example would be working on the universal cover of AdS, where $\mathbb{S}^1$ is replaced by $\mathbb{R}$.

We now move to the analysis of the Laplacian, resp. the d'Alembertian, on the three classes of spaces considered so far. It is helpful to think of these as more complicated versions of the conformally related problems, $\Delta_{\hat{g}}$ and $\Box_{\hat{g}}$, so we first consider the latter, for the Dirichlet boundary condition (DBC) when relevant for the sake of definiteness, global hyperbolicity for the hyperbolic equations, and without stating the function spaces.

- Riemannian: $(\Delta_{\hat{g}} - \lambda)u = f$ with DBC is well-posed for $\lambda < 0$.
- Lorentzian, $\partial X = Y_+ \cup Y_-$ is spacelike: $(\Box_{\hat{g}} - \lambda)u = 0$, $u$ and its normal derivative at $\mathcal{S}$ specified (IC), is well-posed. The solution is $C^\infty$ up to $Y_\pm$.
- Lorentzian, $\partial X$ is timelike: $(\Box_{\hat{g}} - \lambda)u = 0$, with DBC at $Y$, and $u$ and its normal derivative at $\mathcal{S}$ specified (IC), is well-posed.




On the left, a Riemannian example, $\overline{\mathbb{B}^2}$, in the middle, an example of spacelike boundary, $[0,1]_x \times \mathbb{S}^1_y$ with $x$ timelike, on the right, the case of timelike boundary, $\overline{\mathbb{B}^2_{x,y'}} \times \mathbb{R}_{y''}$, with $y''$ timelike.

We now go through the original problems. Let

$$s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}.$$

- Asymptotically hyperbolic, $\lambda \in \mathbb{C} \setminus [(n-1)^2/4, +\infty)$: There is a unique solution of $(\Delta_g - \lambda)u = f$, $f \in \dot{C}^\infty(\bar{X})$, such that $u = x^{s_+(\lambda)} v$, $v \in C^\infty(\bar{X})$. (Analogue of DBC; Mazzeo and Melrose.) (Indeed, $u = (\Delta_g - \lambda)^{-1} f$, and this can be extended to $\lambda \in [(n-1)^2/4, +\infty)$, and analytically continued further.)

- Asymptotically de Sitter, $\lambda \in \mathbb{C}$: There is a unique solution of $(\Box_g - \lambda)u = f$, $f \in \dot{C}^\infty(\bar{X})$, such that $u = x^{s_+(\lambda)}v_+ + x^{s_-(\lambda)}v_-$, $v_\pm \in C^\infty(\bar{X})$ and $v_\pm|_{Y_-}$ is specified, provided that $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$. (Analogue of IC: Vasy.)
- Asymptotically Anti de Sitter, $\lambda \in \mathbb{R} \setminus [(n-1)^2/4, +\infty)$: There is a unique solution of $(\Box_g - \lambda)u = f$ such that $u = x^{s_+(\lambda)}v$, $v \in C^\infty(\bar{X})$ and $u$ and its normal derivative at $\mathcal{S}$ are specified (in $\dot{C}^\infty(\mathcal{S})$). (Analogue of DBC: Vasy. Earlier work in special cases: Breitlohner-Freedman, Bachelot, Yagdjian-Galstian, Holzegel.)

Indeed, one has the following result for the homogeneous equation:

## Theorem

*Suppose that $X$ is asymptotically de Sitter, $\lambda \in \mathbb{C}$. The solution $u$ of the Cauchy problem $(\Box_g - \lambda)u = f$, $f \in \dot{C}^\infty(X)$, with $C^\infty$ initial data at $\mathcal{S}_{t_0}$, $0 < t_0 < 1$, has the form*

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-, \;\; v_\pm \in C^\infty(\bar{X}),$$

*if $s_+(\lambda) - s_-(\lambda) \notin \mathbb{N}$. If $s_+(\lambda) - s_-(\lambda)$ is an integer, the same conclusion holds if we replace $v_- \in C^\infty(\bar{X})$ by $v_- = C^\infty(\bar{X}) + x^{s_+(\lambda)-s_-(\lambda)} \log x \, C^\infty(\bar{X})$.*

*Conversely, given any $a_\pm \in C^\infty(Y_+)$ and $f \in \dot{C}^\infty(X)$, there exists a unique $u$ of the above form such that $(\Box_g - \lambda)u = f$ and $v_\pm|_{Y_+} = a_\pm$.*

Here the roles of $Y_+$ and $Y_-$ can be interchanged.

One of course wants to understand solutions of PDEs beyond their mere existence. We start with the conformal problem. First, we need a notion of singularity. This is Melrose's b-wave front set.

Recall that $\mathcal{V}_b(\bar{X})$ is the Lie algebra of vector fields tangent to $\partial X$, and $\mathrm{Diff}_b(\bar{X})$ is the corresponding algebra of differential operators.

The space of 'very nice' functions corresponding to $\mathcal{V}_b(\bar{X})$ and $\mathrm{Diff}_b(\bar{X})$, replacing $C^\infty(\bar{X})$, is the space of $L^2$ conormal functions to the boundary, i.e. functions $v \in L^2_{\mathrm{loc}}(\bar{X})$ such that $Qv \in L^2_{\mathrm{loc}}(\bar{X})$ for every $Q \in \mathrm{Diff}_b(\bar{X})$ (of any order).

One can also work relative to other spaces instead of $L^2$, such as $H^k(\bar{X})$. Thus, $H^k$ conormal functions to the boundary are functions $v \in H^k_{\mathrm{loc}}(\bar{X})$ such that $Qv \in L^2_{\mathrm{loc}}(\bar{X})$ for every $Q \in \mathrm{Diff}_b(\bar{X})$ (of any order).

We can microlocalize this using Melrose's $\Psi_b(\bar{X})$ (understood to be classical ps.d.o's). Recall that the principal symbol $\sigma_{b,s}(A)$ of $A \in \Psi_b^s(\bar{X})$ is then a homogenenous degree $s$ function on ${}^bT^*\bar{X} \setminus o$.

## Definition

*Suppose $u \in L^2_{\mathrm{loc}}(\bar{X})$. Then $q \in {}^bT^*\bar{X} \setminus o$ is* not *in $\mathrm{WF}_b(u)$ if there is an $A \in \Psi_b^0(\bar{X})$ such that $\sigma_{b,0}(A)(q)$ is invertible and $QAu \in L^2_{\mathrm{loc}}(\bar{X})$ for all $Q \in \mathrm{Diff}_b(\bar{X})$.*

Note that the definition of $\mathrm{WF}$ could be stated in a completely parallel manner: we would require (for $\bar{X}$ without boundary) $QAu \in L^2(\bar{X})$ for all $Q \in \mathrm{Diff}(\bar{X})$ – this is equivalent to $Au \in C^\infty(\bar{X})$ by the Sobolev embedding theorem.

In fact, technically it is useful to work with the space of functions conormal relative to $H^1_{\mathrm{loc}}(\bar{X})$, i.e. replace $L^2_{\mathrm{loc}}$ by $H^1_{\mathrm{loc}}$ above; for solutions of the wave equation, these two are equivalent.

For timelike boundaries we also need the notion of generalized broken bicharacteristics, due to Melrose and Sjöstrand. These encode the law of reflection at the boundary, as well as subtle phenomena when some light rays are tangent to the boundary.

Recall first that one has a natural map $\pi : T^*\bar{X} \to {}^bT^*\bar{X}$ given in local coordinates by

$$\pi(\xi\, dx + \eta\, dy) = (x\xi)\frac{dx}{x} + \eta\, dy,$$

i.e.

$$\pi(x, y, \xi, \eta) = (x, y, x\xi, \eta).$$

This is a diffeomorphism for $x \neq 0$, but is not invertible at $x = 0$. The characteristic set $\Sigma_{\hat{G}} \subset T^*\bar{X} \setminus o$ is the zero set of $\hat{G}$.

The compressed characteristic set, $\dot{\Sigma}$, is $\dot{\Sigma} = \pi(\Sigma_{\hat{G}}) \subset {}^bT^*\bar{X} \setminus o$. Let $\hat{\pi} = \pi|_{\Sigma}$.

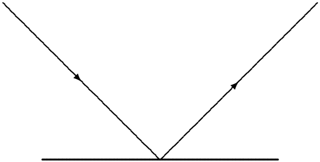
## Definition

*Generalized broken bicharacteristics, or GBB, are*

- *continuous maps* $\gamma : I \to \dot{\Sigma}$,
- *for* $f \in C^\infty({}^bT^*\bar{X})$ *real valued,*

$$\liminf_{s \to s_0} \frac{f \circ \gamma(s) - f \circ \gamma(s_0)}{s - s_0} \geq \inf\{H_p f(q) : \ q \in \hat{\pi}^{-1}(\gamma(s_0))\}.$$

In local coordinates, such $f$ are functions of $x, y, x\xi$ and $\eta$, but not of $\xi$, i.e. the normal component of momentum is allowed to change.



For the conformal problem we have:

- Riemannian: $(\Delta_{\hat{g}} - \lambda)u = f$ with DBC, then $\mathrm{WF}_b(u) \subset \mathrm{WF}_b(f)$ (microlocal elliptic regularity).
- Lorentzian, $\partial X = Y_+ \cup Y_-$ is spacelike: $(\Box_{\hat{g}} - \lambda)u = 0$, $\mathrm{WF}(u)$ is a union of maximally extended bicharacteristics (propagation of singularities, Duistermaat and Hörmander). Moreover, the (renormalized) map of Cauchy data at $Y_-$ to Cauchy data at $Y_+$ is an elliptic Fourier integral operator.
- Lorentzian, $\partial X$ is timelike, with DBC: $(\Box_{\hat{g}} - \lambda)u = 0$, with DBC at $Y$: $\mathrm{WF}_b(u)$ is a union of maximally extended GBB (propagation of singularities, Melrose, Sjöstrand, Taylor).

We now return to the original setting. Here it is useful to consider conormality relative to the 'energy space' $H_0^1(\bar{X})$ and its dual $H_0^{-1}(\bar{X})$. Recall that

- $\mathcal{V}_0(\bar{X})$ is the set of $C^\infty$ vector fields vanishing at $\partial X$,
- $\mathrm{Diff}_0(\bar{X})$ is the associated set of differential operators,
- $H_0^k(\bar{X})$ is the corresponding Sobolev space, i.e. for $k \geq 0$ integer, it is given by $u \in L_0^2(\bar{X})$ such that $Qu \in L_0^2(\bar{X})$ for $Q \in \mathrm{Diff}_0^k(\bar{X})$.
- $H_0^{-k}(\bar{X})$ is the dual of $H_0^k(\bar{X})$.

The general 0-Sobolev spaces are defined using the zero ps.d.o's of Mazzeo and Melrose.

**Definition**

*Suppose* $u \in H^k_{0,\mathrm{loc}}(\bar{X})$*. Then* $q \in {}^bT^*\bar{X} \setminus o$ *is* not *in* $\mathrm{WF}_b(u)$ *if there is an* $A \in \Psi^0_b(\bar{X})$ *such that* $\sigma_{b,0}(A)(q)$ *is invertible and* $QAu \in H^k_{0,\mathrm{loc}}(\bar{X})$ *for all* $Q \in \mathrm{Diff}_b(\bar{X})$*.*

More generally, one can ask whether one has $m$ b-derivatives in $H^k_{0,\mathrm{loc}}$; the resulting wave front set is denoted by $\mathrm{WF}_b^{k,m}$.

## Definition

*Suppose $\bar{X}$ is asymptotically de Sitter-like. The map* $S(\lambda) : C^\infty(Y_-)^2 \ni (v_+|_{Y_-}, v_-|_{Y_-}) \mapsto (v_+|_{Y_+}, v_-|_{Y_+}) \in C^\infty(Y_+)^2$ *is called the scattering operator.*

One can describe $S(\lambda)$ rather precisely.

## Theorem

*Suppose $\lambda \neq \left((n-1)^2 - m^2\right)/4$, $m \in \mathbb{N}$. Then the scattering operator is a Fourier integral operator associated to the null-geodesic flow on $\bar{X}$; which is elliptic after suitable renormalization. In particular, even for $a_\pm \in \mathcal{D}'(Y_-)$, $\mathrm{WF}(v_+|_{Y_+})$ and $\mathrm{WF}(v_-|_{Y_+})$ are subsets of the image of $\mathrm{WF}(a_+) \cup \mathrm{WF}(a_-)$ under the null-bicharacteristic flow of $\hat{G}$ in $\bar{X}$.*

More precisely, here we lift points $\alpha \in \mathrm{WF}(a_+) \cup \mathrm{WF}(a_-) \subset T^*Y_- \setminus o$ to the characteristic set in $T^*_{Y_-}\bar{X}$ (yielding two points!), flow them forward to $T^*_{Y_+}\bar{X}$, and pull back the result to $T^*Y_+ \setminus o$.

The condition $\lambda \neq \left((n-1)^2 - m^2\right)/4$ simply eliminates log terms that should show up in the construction in general.

In fact, one also understands the structure of the forward fundamental solution; this is due to the work of Dean Baskin: it is a paired Lagrangian distribution on a blow up of the zero-double-space $\bar{X}_0^2$, in which the boundary of the light cone (flow out of the 0-diagonal) is blown up.

The analogous result in the Anti de Sitter case is the following:

## Theorem

*Suppose that* $P = \Box_g + \lambda$, $\lambda \in (-\infty, (n-1)^2/4)$, *and* $m \in \mathbb{R}$ *or* $m = \infty$. *Suppose* $u \in H^1_{0,\mathrm{loc}}(X)$. *Then*

$$\mathrm{WF}_b^{1,m}(u) \setminus \dot{\Sigma} \subset \mathrm{WF}_b^{-1,m}(Pu).$$

*Moreover,*

$$(\mathrm{WF}_b^{1,m}(u) \cap \dot{\Sigma}) \setminus \mathrm{WF}_b^{-1,m+1}(Pu)$$

*is a union of maximally extended generalized broken bicharacteristics of the conformal metric* $\hat{g}$ *in*

$$\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,m+1}(Pu).$$

*In particular, if* $Pu = 0$ *then* $\mathrm{WF}_b^{1,\infty}(u) \subset \dot{\Sigma}$ *is a union of maximally extended generalized broken bicharacteristics of* $\hat{g}$.

We now return to de Sitter space, and the long time behavior of solutions of the wave equation there. Recall that these had the form (for $C^\infty$ Cauchy data)

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-, \ \ v_\pm \in C^\infty(\bar{X}),$$

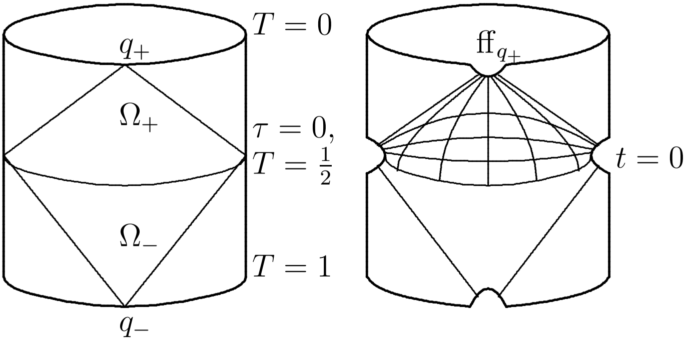
with possible logarithmic modifications for the more decaying term, where

$$s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}.$$

In particular, for $\lambda = 0$, $s_-(\lambda) = 0$, so the result is

$$u = v_+ + x^{n-1}(\log x) v_-, \ \ v_\pm \in C^\infty(\bar{X}).$$

Rather than looking at the asymptotics on the whole spacetime, $\bar{X}$, it is sometimes useful to study it only on a domain in $\bar{X}$, whose boundaries are characteristic for the PDE (lightlike).




On the left, the compactification of de Sitter space with the backward light cone from $q_+$ and forward light cone from $q_-$ are shown. $\Omega_+$, resp. $\Omega_-$, denotes the intersection of these light cones with $T > 0$, resp. $T < 0$. On the right, the blow up of de Sitter space $\bar{M}'$, together with the spatial and temporal coordinate lines of the static model in $\Omega_+$. The interior of the light cone inside the front face $\mathrm{ff}_{q_+}$ can be identified with the spatial part of the static model of de Sitter space.

The *static model* of de Sitter space arises by singling out a point on $\mathbb{S}_\theta^{n-1}$, e.g. $q_0 = (1, 0, \ldots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$:

- it is the intersection of the backward lightcone from $q_0$ considered as a point $q_+$ at $T = 0$, and the forward light cone from $q_0$ considered as a point $q_-$ at $T = 1$.
- These intersect the equator $T = 1/2$ (here $\tau = 0$) in the same set, and altogether form a 'diamond'.
- Explicitly this region is given by $z_2^2 + \ldots + z_n^2 \leq 1$ inside the hyperboloid.
- Blow up the corner where the light cones intersect $\tau = 0$, as well as $q_+$ and $q_-$; call the resulting space $\bar{M}'$.

$\bar{M}$ can also be obtained as follows.

- Consider $[0,1]_T \times \overline{\mathbb{B}^3}$, with $T = e^{-2t}$ for $t > 4$, say.
- In polar coordinates $(r, \omega)$ on $\overline{\mathbb{B}^3}$, consider the Lorentz metric

$$(1 - r^2)\, dt^2 - (1 - r^2)^{-1} dr^2 - r^2\, d\omega^2.$$

- Blow up the corners to obtain $\bar{M}$.
- It is straightforward to see that $\bar{M}$ and $\bar{M}'$ are (almost) diffeomorphic and isometric.
- 'Almost' refers to this approach gives that the defining function of $\mathrm{ff}_{q_+}$ in $\bar{M}$ is $x^2$ – this corresponds to an evenness statement for the Lorentz metric in the sense of Guillarmou.

While one *can* analyze the solutions of the wave equations on de Sitter space at points inside the 'diamond' by considering the diamond only (in view of the finite propagation speed for the wave equation), the resulting picture does include rather artificial limitations.

For instance, the local static asymptotics, corresponding to the tip of the diamond at $Y_+$, describes only a small part of the asymptotics of solutions of the Cauchy problem on de Sitter space.

Note though that restricted to the diamond the asymptotics is particularly simple: as there is only one point on $Y_+$ in the diamond, one sees that for the wave equation ($\lambda = 0$) the solution decays to a constant, and the decay rate is 'exponential' (i.e. modulo $O(x)$).

Black holes, especially de Sitter-Schwarzschild space, are very closely connected to de Sitter space: the event horizon for a black hole plays the same role as for the backward light cone in de Sitter space.

In these settings, energy decay was proved by Dafermos and Rodnianski (polynomial decay) and Melrose-Sá Barreto-V. (exponential decay).