MATH 174A: LECTURE NOTES ON THE INVERSE FUNCTION THEOREM

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^n$ is open, $F : \Omega \to \mathbb{R}^n$ is C^k, $k \geq 1$, $p_0 \in \Omega$, $q_0 = F(p_0)$. Suppose that $DF(p_0)$ is invertible. Then there is a neighborhood U of p_0 and a neighborhood V of q_0 such that $F : U \to V$ is a bijection and $F^{-1} : V \to U$ is C^k. (One says that F is a C^k diffeomorphism.) Moreover, $D(F^{-1})(q) = ((DF)(F^{-1}(q)))^{-1}$ for $q \in V$.*

Proof. First, it is easy to reduce the theorem to the special case where $q_0 = p_0 = 0$ and $DF(p_0) = \mathrm{Id}$; one simply needs an affine diffeomorphism to accomplish this. So with F in the theorem, let

$$G(x) = (DF(p_0))^{-1}(F(p_0 + x) - F(p_0)),\ x \in \Omega' = \{y :\ y + p_0 \in \Omega\}.$$

Note that Ω' is open (is a translation of Ω), $G(0) = 0$, and

$$DG(x) = (DF(p_0))^{-1} DF(p_0 + x)$$

by the chain rule, so $DG(0) = \mathrm{Id}$. Thus, G satisfies the assumptions of the theorem with p_0, q_0 resp. $DF(p_0)$ replaced by 0, 0, resp. Id.

If we show that G is invertible as stated, then

$$F^{-1}(z) = p_0 + G^{-1}((DF)(p_0)^{-1}(z - q_0))$$

for z near q_0. Indeed, substituting in $z = F(p_0 + y)$ gives $F^{-1}(F(p_0 + y)) = p_0 + G^{-1}(G(y)) = p_0 + y$, and $F(F^{-1}(q_0 + z)) = q_0 + z$ is also easy to check.

So from now on we assume that $F(0) = 0$, $DF(0) = \mathrm{Id}$. Let $\delta_0 > 0$ be such that

$$\overline{B(0, \delta_0)} = \{u :\ \|u\| \leq \delta_0\} \subset \Omega.$$

Then $F(u) = u + R(u)$ with $\lim_{\|u\| \to 0} \frac{\|Ru\|}{\|u\|} = 0$. We are assuming also that $DR(u) = DF(u) - \mathrm{Id}$ is continuous and $DR(0) = 0$. Combining these two we see that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\delta \leq \delta_0$ and

$$\|u\| \leq \delta \Rightarrow \|R(u)\| \leq \frac{\epsilon}{2}\|u\|,\ \|DR(u)\| \leq \frac{\epsilon}{2}.$$

Let $\delta_1 = \delta(1)$, i.e. the δ corresponding to $\epsilon = 1$.

Let

$$X = \overline{B(0, \delta_1)} = \{u :\ \|u\| \leq \delta_1\},$$

so $X \subset \Omega$, and X is a complete metric space (with the induced metric from $\mathbb{R}^n$). We first remark that for $u, w \in X$,

$$\begin{aligned} \|R(u) - R(w)\| &= \|\int_0^1 (DR)(w + t(u - w))(u - w)\, dt\| \\ &\leq \int_0^1 \|(DR)(w + t(u - w))(u - w)\|\, dt \\ &\leq \int_0^1 \|(DR)(w + t(u - w))\|\, \|u - w\|\, dt \leq \frac{1}{2}\|u - w\| \end{aligned} \tag{1}$$

since for u, w in X, the line segment connecting u and w is also in X (i.e. X is convex):

$$\|w + t(u-w)\| = \|(1-t)w + tu\| \leq (1-t)\|w\| + t\|u\| \leq \delta_1.$$

Thus,

$$\|F(u)-F(w)\| = \|(u-w)+(R(u)-R(w))\| \geq \|u-w\| - \|R(u)-R(w)\| \geq \frac{\|u-w\|}{2},$$

so for $u, w \in X$, $F(u) = F(w)$ implies $u = w$, hence F is injective.

For $v \in \overline{B(0, \frac{\delta_1}{2})}$, i.e. with $\|v\| \leq \frac{\delta_1}{2}$, let

$$T_v : X \to \mathbb{R}^n,\ T_v(u) = v - R(u).$$

Note that $T_v(u) = u$ means $v = u + R(u) = F(u)$, so we want to find a fixed point of T_v. First,

$$\|T_v u\| \leq \|v\| + \|R(u)\| \leq \|v\| + \frac{\|u\|}{2} \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1, \tag{2}$$

so $T_v : X \to X$. Moreover, for $u, w \in X$, by (1),

$$\|T_v u - T_v w\| = \|R(u) - R(w)\| \leq \frac{1}{2}\|u - w\|, \tag{3}$$

so T_v is a contraction mapping. Correspondingly, T_v has a unique fixed point in X, i.e. there is a unique $u \in X$ such that $T_v u = u$, i.e. $F(u) = v$.

We *define* $\Phi : \overline{B(0, \frac{\delta_1}{2})} \to X$ by letting $\Phi(v)$ be the unique fixed point of T_v, so $F \circ \Phi$ is the identity map on $\overline{B(0, \frac{\delta_1}{2})}$.

Note that $F : \overline{B(0, \frac{\delta_1}{4})} \to \overline{B(0, \frac{\delta_1}{2})}$ since for $u \in \overline{B(0, \frac{\delta_1}{4})}$,

$$\|F(u)\| \leq \|u\| + \|R(u)\| \leq \frac{3}{2}\|u\| \leq \frac{\delta_1}{2}.$$

Correspondingly, $\Phi \circ F$ is well-defined on $\overline{B(0, \frac{\delta_1}{4})}$, and $F \circ (\Phi \circ F) = (F \circ \Phi) \circ F = F$. Since F is injective on $\overline{B(0, \delta_1)}$, this gives $\Phi \circ F = \mathrm{Id}$ on $\overline{B(0, \frac{\delta_1}{4})}$.

This argument can also be repeated on a function space:

$$Y = C^0(\overline{B(0, \frac{\delta_1}{2})}, X);$$

this is again complete with the metric induced by the sup norm. Let

$$T : Y \to C^0(\overline{B(0, \frac{\delta_1}{2})}, \mathbb{R}^n)$$

for $G \in Y$ by

$$(TG)(v) = v - R(G(v)),\ v \in \overline{B(0, \frac{\delta_1}{2})}.$$

In fact, $T : Y \to Y$ as for $G \in Y$ and $v \in \overline{B(0, \frac{\delta_1}{2})}$

$$\|TG(v)\| \leq \|v\| + \|R(G(v))\| \leq \delta_1$$

by (2), and T is a contraction mapping as for $G, H \in Y$ and $v \in \overline{B(0, \frac{\delta_1}{2})}$

$$\|TG(v) - TH(v)\| = \|R(G(v)) - R(H(v))\| \leq \frac{1}{2}\|G(v) - H(v)\|,$$

so

$$\|TG-TH\|_Y = \sup_{v\in\overline{B(0,\frac{\delta_1}{2})}} \|TG(v)-TH(v)\| \le \frac{1}{2} \sup_{v\in\overline{B(0,\frac{\delta_1}{2})}} \|G(v)-H(v)\| = \|G-H\|_Y.$$

Correspondingly, T has a unique fixed point G, and then for $v \in \overline{B(0,\frac{\delta_1}{2})}$, $v - R(G(v)) = G(v)$, that is $F(G(v)) = v$. But the unique $u \in X$ with $F(u) = v$ is $u = \Phi(v)$, so $\Phi = G$, and hence Φ is in fact continuous.

This also gives that $V = F(B(0,\frac{\delta_1}{4}))$ is open, for it is $\Phi^{-1}(B(0,\frac{\delta_1}{4}))$ and Φ is continuous. Thus, $F : U \to V$, $U = B(0,\frac{\delta_1}{4})$ is a homeomorphism (an invertible continuous map with a continuous inverse).

We next prove the differentiability of Φ at 0, with $D\Phi(0) = \mathrm{Id}$. That is, given $\epsilon > 0$ (may assume $\epsilon \le 1$) we need $\delta > 0$ such that $\|v\| \le \delta$ implies $\|\Phi(v) - v\| \le \epsilon\|v\|$.

So let $\delta = \delta(\epsilon)$ given in the first paragraph, and for $\|u\| \le \delta(\epsilon)$, $\|R(u)\| \le \frac{\epsilon}{2}\|u\|$ and $\|DR(u)\| \le \frac{\epsilon}{2}$. For $v \in \overline{B(0,\frac{\delta(\epsilon)}{2})}$, let

$$X_{v,\epsilon} = \{u : \ \|u - v\| \le \epsilon\|v\|\}.$$

Again, $X_{v,\epsilon}$ is complete. For $u \in X_{v,\epsilon}$,

$$\|u\| \le \|v\| + \epsilon\|v\| \le 2\|v\| \le \delta(\epsilon),$$

so $\|R(u)\| \le \frac{\epsilon}{2}\|u\|$. Correspondingly,

$$\|T_v u - v\| = \|R(u)\| \le \frac{\epsilon}{2}\|u\| \le \epsilon\|v\|$$

so in fact $T_v : X_{v,\epsilon} \to X_{v,\epsilon}$. Thus, T_v is actually a contraction mapping on $X_{v,\epsilon}$ (note that $X_{v,\epsilon} \subset X$), hence has a unique fixed point in it. Correspondingly, the fixed point of T_v on X, namely $\Phi(v)$, actually lies in $X_{v,\epsilon}$, in other words, $\|\Phi(v) - v\| \le \epsilon\|v\|$ provided that $\|v\| \le \frac{\delta(\epsilon)}{2}$. This proves that Φ is differentiable at 0 with $D\Phi(0) = \mathrm{Id}$.

Returning to the original problem, we have proved that p_0 and q_0 have neighborhoods U_0 and V_0, and there is a continuous map $G : V_0 \to U_0$ such that $F : U_0 \to V_0$ is a bijection with inverse G, and G is differentiable at q_0 with derivative $(DF(p_0))^{-1}$.

But DF is also invertible near p_0, and our results so far show that p near p_0 has a neighborhood on which F is invertible, with inverse continuous, differentiable at $F(p)$, with derivative $((DF)(p))^{-1}$. As the inverse is unique, it is the restriction of G to this neighborhood, proving that G is differentiable, with derivative $(DF(p))^{-1}$, which is thus continuous.

If F is C^k, $k > 1$, then DF is C^{k-1}, hence $(DF)^{-1}$ is C^{k-1}, hence F^{-1} is C^k. $\square$

Note that we never used any special properties of $\mathbb{R}^n$, so the proof goes through equally well if it is replaced by a Banach space V (we need that closed balls are complete, which is true in a Banach space).

Also, if V, W are complex vector spaces, e.g. $V = \mathbb{C}^n$, and $\Omega \subset V$ is open, $p_0 \in \Omega$, we say that $F : \Omega \to V$ is differentiable (as a complex map) at p_0 if there exists $L \in \mathcal{L}(V, W)$ (so L complex-linear and bounded) such that $R(y) = F(p_0+y)-F(p_0)-Ly$ satisfies $\lim_{\|y\|\to 0} \frac{\|R(y)\|}{\|y\|} = 0$. Of course, if F is differentiable as a complex map, it is also differentiable as a real map, i.e. regarding V and W as real vector spaces, so differentiability in the complex sense is a stronger criterion. If F is differentiable as a complex map, with continuous derivative, one says that F is *holomorphic*. Applying our inverse function theorem we deduce that not only is F, with $DF(p_0)$ invertible, locally a diffeomorphism, but $DF^{-1}(q)$ is complex-linear (as it is the inverse of $DF(F^{-1}(q))$, which is complex linear), so F^{-1} is also holomorphic.