COHOMOLOGY OF ARITHMETIC GROUPS AND PERIODS OF AUTOMORPHIC FORMS

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1. INTRODUCTION

These notes are based on my Takagi lectures that were delivered November 15, 2014. The content of the lectures, and the corresponding sections in the notes, are thus:

(1) Firstly, I gave a brief introduction to the cohomology of arithmetic groups, in particular the fact that the "tempered" (informally: "near the middle dimension"') part of the cohomology looks like the cohomology of a torus. This corresponds to §2, §3, §4 of these notes.

For the reader totally unfamiliar with the cohomology of arithmetic groups, §3 might be the best thing to focus on.

(2) Next, I draw attention to the problem arising from the first part (§5): can one produce extra endomorphisms of cohomology that "explain" the torus?

(3) Finally, I outline what we understand at present about this problem:

One can construct this with $\mathbb{Q}_p$ or $\mathbb{C}$ coefficients, as we explain in §6, and finally the main conjecture – formulated in §7 – is that one can understand exactly how these actions interact with the rational structures.

This conjecture is unfortunately technically very heavy to formulate. It involves a motivic cohomology group of that motive which (conjecturally, by the Langlands program) underlies the adjoint L-function of the associated automorphic representation. It is interesting because, at present, there seems to be no known algebraic mechanism that explains it. It also is related to "derived" structures in the Langlands program.

Despite the fact that the conjecture may seem on very shaky ground, resting as it does on a large web of other conjectures and involving almost uncomputable groups, nonetheless it produces testable predictions about cohomology (§7.6), and we have verified these in some cases, using ideas from the theory of periods of automorphic forms and also ideas related to analytic torsion.

Many of the ideas presented here represent joint work (in progress) with K. Prasanna [19]. I mention also some work on the derived deformation ring, which is also ongoing work (in progress) with S. Galatius [12].

Finally the current notes, although somewhat expanded, reflect the lectures relatively faithfully. I have tried to maintain the spirit of general accessibility, and I have not added much extra material (partly because of my own constraints of time). In particular, a lot of emphasis is put on a single example case (§3.2), where the symmetric space is the nine-dimensional product of three hyperbolic spaces. At some points the material is (in the interest of accessibility) presented in a non-standard way. In these cases I have tried to briefly outline how it is related to the standard presentation.

There are no proofs of any of the new results, nor, in many cases, even complete formulations. These will appear in the papers [19, 22, 12]. I hope primarily that the present notes will serve to show that the underlying problem itself is interesting.

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CONTENTS

2. ARITHMETIC GROUPS AND THEIR COHOMOLOGY

Our primary concern in these notes is with the cohomology of arithmetic groups, which can also be understood as the cohomology of an associated manifold or orbifold. Unfortunately, the basic definitions referring to arithmetic groups and their symmetric spaces are, at first, very difficult to absorb in a precise form. Thus we largely give examples. For definitions, see [7] and [15].

2.1. **Arithmetic groups.**

An *arithmetic group* is a group such as

$$\mathrm{SL}_n(\mathbb{Z}), \mathrm{Sp}_{2n}(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}]), \cdots$$

obtained, roughly speaking, by taking the "$\mathbb{Z}$-points" of a classical group of matrices.

Now $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}])$ looks different to the others at first, but it can be presented similarly: one can regard 2×2 matrices over $\mathbb{Z}[i]$ as 4×4 matrices over $\mathbb{Z}$:

$$\begin{pmatrix} a+bi & c+di \\ e+fi & g+hi \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{pmatrix}$$

An precise defintion, adequate for our purposes, is given as follows: A "congruence arithmetic group" is a group Γ obtained by taking a semisimple $\mathbb{Q}$-group $\mathbf{G} \subset \mathrm{SL}_N$, and taking

$$\Gamma = \{g \in \mathbf{G}(\mathbb{Q}) : g \text{ has integral entries } \}$$

Each such group Γ is contained in an ambient Lie group, namely the real points of $\mathbf{G}$:

$$G = \mathbf{G}(\mathbb{R}).$$

2.2. **Symmetric spaces.** Each such group Γ acts discontinuously on a canonically associated Riemannian manifold S (the "symmetric space" for the ambient Lie group G). In general S is, as a manifold, the quotient of G by a maximal compact subgroup $K \subset G$; it is known that all such K are conjugate inside G. It's easy to verify that G preserves a Riemannian metric on S.

For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we have $G = \mathrm{SL}_2(\mathbb{R})$ and can take $K = \mathrm{SO}_2$; the associated geometry $S = G/K$ can be identified with the Poincaré upper-half plane

$$S = \{z \in \mathbb{H}^2 : \mathrm{Im}(z) > 0\}$$

and the action of G is by fractional linear transformations; it preserves the standard hyperbolic metric $|dz|^2/\mathrm{Im}(z)^2$.

If $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ we have $G = \mathrm{SL}_2(\mathbb{C})$ and can take $K = \mathrm{SU}_2$; the associated geometry S can be identified with the three-dimensional hyperbolic space $\mathbb{H}^3$.

Finally, for $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ the situation is less familiar: $G = \mathrm{SL}_n(\mathbb{R})$, we can take $K = \mathrm{SO}_n$, and the geometry S can be identified with the space of positive definite, symmetric, real-valued $n \times n$ matrices A with $\det(A) = 1$ and with metric given by $\mathrm{trace}\left(A^{-1}dA\right)^2$.

2.3. **Cohomology.** Our primary concern here is with the group cohomology of such Γ. We can identify the cohomology of Γ with the cohomology of the quotient manifold S/Γ:

$$H^*(\Gamma, \mathbb{C}) \simeq H^*(S/\Gamma, \mathbb{C})$$

In fact, for our purposes, this can be taken as a definition of $H^*(\Gamma, \mathbb{C})$. In this document we will use the notation $H^*(\Gamma, -)$ and $H^*(S/\Gamma, -)$ interchangeably.

2.4. **Context.** Why would one study the cohomology of arithmetic groups at all?

From the point of view of number theory, one can understand this is as offering a generalization of the notion of modular form. It was observed by Eichler and Shimura that one can fruitfully "embed" the theory of weight 2 modular forms into the cohomology of subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Namely, a classical weight 2 modular form [1] $f(z)$ for $\mathbb{H}^2/\Gamma$ gives

$$f(z)dz \in \ \Gamma\text{-invariant forms on } \mathbb{H}^2$$

and thus to

$$[f(z)dz] \in H^1(\mathbb{H}^2/\Gamma, \mathbb{C}) = H^1(\Gamma, \mathbb{C})$$

To be explicit, such an f gives rise to a homomorphism $\Gamma \to \mathbb{C}$, defined by the rule $\gamma \in \Gamma \mapsto \int_{z_0}^{\gamma z_0} f(z)dz$; here z_0 is an arbitrary point of $\mathbb{H}$ (the integral does not depend on its choice).

In this way we can reformulate the theory of holomorphic modular forms (at least for weight 2, but indeed any weight ≥ 2 works similarly) in terms of cohomology of subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

In general, then, the cohomology of arithmetic groups gives a generalization of the theory of modular forms from SL_2 to any group. There are many such generalizations, most of which are subsumed in the notion of "automorphic form" – see [5] for the modern definition thereof. However, studying the cohomology of arithmetic groups has some advantages: it can be defined for all groups, it has a natural integral structure, and that integral structure detects torsion. (The general notion of automorphic form, as defined in [5], does not have the last two properties.)

Note that the case of $\mathrm{SL}_2(\mathbb{Z})$ is misleading for one reason: the quotient S/Γ, in that case, carries a complex structure. Indeed, one can identify $\mathbb{H}^2/\mathrm{SL}_2(\mathbb{Z})$ with the moduli space of complex tori, i.e. of complex elliptic curves. In general S/Γ has no complex structure; it is often odd-dimensional. The S/Γ that admit a natural complex structure are "Shimura varieties." They play a special role and and they are much better understood than the general case. The structures that we will study in these notes are, however, only of interest when S/Γ *does not* have a complex structure.

Another important way in which cohomology of arithmetic groups enters mathematics is through Borel's computation [8] of the algebraic K-theory of the integers

$$\mathrm{rank}\ K_i(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 1, & i = 4n+1, n > 0 \\ 0, & \text{else} \end{cases}$$

Borel's proof is based on a study of $H^*(\Gamma)$ for $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and uses the Hodge-theory that we will discuss in subsequent sections.

Finally, on a more speculative note: why study only cohomology and not other homology theories, e.g. why not discuss the K-theory of $B\Gamma$? I don't see a compelling reason why K-theory is less interesting than homology from the point of view of number theory; the emphasis on homology and cohomology may just be a matter of tradition. While the two carry the same rational information (assuming that Γ is torsion-free) the integral phenomena at the level of K-theory might well be interesting at small primes; I don't know.

3. HODGE THEORY

For simplicity, in the rest of this document, I will assume that S/Γ is a compact manifold. Although this excludes, literally speaking, cases such as $\mathrm{SL}_n(\mathbb{Z})$, modified versions of the discussion still apply.

Let notation be as before: Γ an arithmetic group, S the associated symmetric space. Being a compact manifold the cohomology of S/Γ satisfies Poincaré duality; but in fact it usually satisfies many more (less apparent) constraints; see e.g. (3.1) below. In general, these constraints are rather complicated and can be precisely described using the theory of $(\mathfrak{g}, K)$ cohomology: see [6].

Since in general the setup of $(\mathfrak{g}, K)$ cohomology is rather forbidding, what we do instead is discuss a rather special situation where one can get to the main points without too much formalism. We try to emphasize the analogy of this situation with Kähler manifolds: special algebraic structure of the metric gives rise to extra algebraic structures on cohomology. I hope this conveys the flavor of the general story.

The main point to focus on is the fact (lacking any really simple explanation) that there is always a "piece" of the cohomology of Γ which looks like the cohomology of a torus.

3.1. **Hodge theory; comparison with Kähler manifolds.** The Riemannian structure gives us a way to analyze the cohomology of S/Γ.

Namely, each class in $H^*(S/\Gamma, \mathbb{C})$ has a "harmonic representative": a representative of minimal L^2 norm, or equivalently a form ω annihilated by the Hodge Laplacian Δ. (See e.g. [14] for discussion of this story). In this way one obtains a *canonical* differential form representing the class. Indeed, this story makes sense for any Riemannian manifold M: one gets

$$\underbrace{\text{harmonic } i\text{-forms on } M}_{:=\mathcal{H}^i(M)} \xrightarrow{\simeq} H^i(M, \mathbb{C}).$$

But the Riemannian metric on S is of a very special type – its holonomy group is very small. In other words, there are invariant tensors on the tangent space that are preserved by parallel transport. Hodge theory can be used to promote these local algebraic structures to structures that exist on $H^i(M, \mathbb{C})$.

This situation is very similar to what happens for Kähler manifolds. In that case, the holonomy is a unitary group, and correspondingly there is an action of $\mathbb{C}^*$ on each tangent space, preserved by parallel transport. This $\mathbb{C}^*$ then acts on differential forms, and (nontrivially) it preserves harmonic forms - thus $\mathbb{C}^*$ acts on cohomology – giving rise to the Hodge decomposition.

The analogy between the S/Γ and Kähler manifolds is a fruitful one in other contexts, see, for example, [4].

3.2. **A worked example.** As an example, suppose we take the case where Γ is an arithmetic group whose associated symmetric space is $S = \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$. We will also assume that Γ acts irreducibly on S, i.e., the projection of Γ to automorphisms of each $\mathbb{H}^3$ factor has dense image inside $\mathrm{PSL}_2(\mathbb{C})$.

An example of this situation is given by $\Gamma = \mathrm{SL}_2(\mathfrak{o})$ where $\mathfrak{o}$ is the ring of integers in a number field F such that $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^3$, i.e. $F = \mathbb{Q}(\alpha)$ where α has degree 6 and its minimal polynomial has no real roots. For example, $\alpha = \sqrt[6]{-2}$ and $\mathfrak{o} = \mathbb{Z}[\sqrt[6]{-2}]$ will do. As stated, however, this example does not have S/Γ compact; to get Γ compact, one can replace SL_2 by the units in a quaternion algebra.

The Betti numbers of the compact 9-manifold S/Γ satisfy the relation $b_i = b_{9-i}$ because of Poincaré duality. But much more is true: the vector $(b_0, \ldots, b_9)$ of Betti numbers of S/Γ is actually always a linear combination of the vectors

$$(1,0,0,3,0,0,3,0,0,1)+k\,(0,0,0,1,3,3,1,0,0,0) \tag{3.1}$$

for some integer $k \geq 1$.

Let us explain why this is true. Our proof is roughly a "translation" of the proof via $(\mathfrak{g},K)$ cohomology into elementary terms.

The key point is that *locally* S/Γ splits as a product: the universal cover S splits as

$$S \simeq \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3,$$

and although this splitting doesn't descend to S/Γ, nonetheless it induces a splitting of tangent spaces

$$T_xS \simeq T_xS^{(1)} \oplus T_xS^{(2)} \oplus T_xS^{(3)}, \tag{3.2}$$

and therefore also a splitting at the level of differential forms:

$$\Omega^q = \bigoplus_{a+b+c=q} \Omega^{a,b,c}$$

where $\Omega^{a,b,c}$ is the space of differential forms which, everywhere locally, can be written $F(\pi_1^*\omega_a)\wedge(\pi_2^*\omega_b)\wedge(\pi_3^*\omega_c)$ where F is a smooth function, π_i are the three projections $(\mathbb{H}^3)^3 \to \mathbb{H}^3$, and ω_q is a q-form on $\mathbb{H}^3$. These splittings are Γ-invariant and descend to S/Γ.

(Such a splitting does not exist for general Γ, but the splitting – invariant by parallel transport – is just a particularly concrete way of remembering that the holonomy of S/Γ is unusually small.)

The key point is that this splitting preserves harmonic forms. In other words:

if $\omega \in \Omega^q$ is harmonic, so too are all its components ω^{abc}.

Proof. (Sketch) The splitting (3.2) is preserved by the action of G on S, and gives rise to decompositions of everything else in sight:

The differential operator d (from q-forms to $q+1$ forms) splits as

$$d = d_1 + d_2 + d_3$$

where $d_1 : \Omega^{a,b,c} \to \Omega^{a+1,b,c}$, etc.

One verifies that the Laplace operator, i.e. $dd^* + d^*d$, splits as a sum

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3,$$

where $\Delta_i := d_i d_i^* + d_i^* d_i$, and d_i is the formal adjoint of d_i. In particular each Δ_i preserves each of the Ω^{abc}, whence the conclusion. $\square$

Thus the splitting above yields a cohomology splitting

$$H^q(S/\Gamma,\mathbb{C}) = \bigoplus_{a+b+c=q} H^{a,b,c}.$$

To go further, note that (just as with the d operator) that the factorization (3.2) induces a factorization of Hodge $*$: writing

$$\wedge^* T_xS \simeq \wedge^* T_xS^{(1)} \otimes \wedge^* T_xS^{(2)} \oplus \wedge^* T_xS^{(3)}, \tag{3.3}$$

we define $*_i$ simply as the tensor product of Hodge $*$ on $\wedge^* T_x S^{(i)}$ and the identity operator on the other components. Thus, for example, $*_1 : \Omega^{a,b,c} \to \Omega^{3-a,b,c}$ is "Hodge star in the first copy of $\mathbb{H}^3$," and so on. Then $*$ factorizes on each Ω^{abc} as $*_1 *_2 *_3$, up to sign.

The key point is that $*_i$ preserves harmonic forms. Write for short $h^{a,b,c} = \dim H^{a,b,c}$. We get

$$h^{a,b,c} = h^{3-a,b,c} = h^{a,3-b,c} = h^{a,b,3-c}.$$

At this point, we see that

$$1 = \dim H^0(S/\Gamma, \mathbb{C}) = h^{0,0,0} = h^{3,0,0} = h^{0,3,0} = h^{0,0,3} = h^{3,3,3}$$
$$h^{1,1,1} = h^{2,1,1} = h^{1,2,1} = h^{1,1,2} = h^{2,2,1} = h^{1,2,2} = h^{2,1,2} = h^{2,2,2}$$

It remains to check the other numbers are zero, e.g. $h^{1,0,0} = 0$.

Proof. (Sketch) Suppose for example that $\omega \in \mathcal{H}^{1,0,0}$. Explicitly, ω is of the form $f dx_1 + g dy_1 + h dz_1$, where (x_1, y_1, z_1) are local coordinates on the first copy of $\mathbb{H}^3$. From the prior argument we see that the harmonicity of ω forces $d_2\omega = d_3\omega = 0$, i.e. $d_2 f = d_3 f = 0$ and similarly for g, h. This means that the pullback Ω of ω to $(\mathbb{H}^3)^3$ is both Γ-invariant and pulled back from the first factor $\mathbb{H}^3$. The irreducibility of Γ (see start of §3.2 for definition) means that the resulting 1-form on $\mathbb{H}^3$ is invariant by $\mathrm{SL}_2(\mathbb{C})$, and thus is equal to 0. $\square$

This concludes the proof of (3.1).

3.3. **Concluding remarks.** In fact, we can split the cohomology in a fashion that corresponds to the numerical decomposition (3.1)

$$H^*(S/\Gamma, \mathbb{C}) = \underbrace{H^*(\Gamma, \mathbb{C})_{\mathrm{inv}}}_{(1,0,0,3,0,0,3,0,0,1)} \oplus \underbrace{H^*(\Gamma, \mathbb{C})_{\mathrm{temp}}}_{k(0,0,0,1,3,3,1,0,0,0)} . \tag{3.4}$$

where we wrote below each summand its dimension in each degree.

The first summand, corresponding to all the $\mathcal{H}^{0,0,0}, \mathcal{H}^{3,0,0}, \ldots$ summands, can be alternately characterized as the harmonic forms that are actually invariant under parallel transport; thus the notation H^*_{inv}, for "invariant."

The second term is the orthogonal complement of the first (when we identify everything to harmonic forms). The subscript in H^*_{temp} is short for "tempered," which has its origin in the notion of "tempered representation" from the representation theory of semisimple Lie groups.

What if we want to correspondingly split cohomology with *rational* coefficients $H^*(S/\Gamma, \mathbb{Q})$? In general, studying cohomology by means of differential forms loses contact with the natural rational structure on cohomology, and this is no exception:

Harmonic representatives for H^*_{inv} are easy to describe. For example, a harmonic representative for the one-dimensional space $\mathcal{H}^{3,0,0}$ is obtained by taking the volume form ν on $\mathbb{H}$, pulling back via the first coordinate projection $\pi_1 : S \to \mathbb{H}$, and then (since $\pi_1^*\nu$ is Γ-invariant) descending to S/Γ. It is not at all clear which combinations of $\pi_1^*\nu, \pi_2^*, \pi_3^*\nu$ are rational. (This can be answered: it has to do with the Borel regulator on K_3 of the field F).

It is in fact true that the splitting (3.4) is actually defined over $\mathbb{Q}$; that is to say, it descends to a splitting

$$H^*(S/\Gamma, \mathbb{Q}) = H^*(\Gamma, \mathbb{Q})_{\mathrm{inv}} \oplus H^*(\Gamma, \mathbb{Q})_{\mathrm{temp}}.$$

and we will discuss why this is so in §4. We will need to replace, e.g., the characterization of H^*_{inv} as "invariant by parallel transport" by a characterization that works better with $\mathbb{Q}$ coefficients. For this we will use Hecke operators as a kind of substitute for parallel transport.

3.4. **Period matrices.** This subsection will only be used in the discussion at the very end of the paper and can be skipped for the moment:

Using the Riemannian structure, we can extract some numerical invariants from cohomology, which we call "period matrices."

Notation as above, choose an integral basis $\{\gamma_1, \dots, \gamma_d\}$ for $H_j(S/\Gamma, \mathbb{Z})$ modulo torsion, and an orthonormal basis $\{\omega_1, \dots, \omega_d\}$ for the harmonic j-forms on S/Γ. I will call the matrix

$$M_{ab} = \int_{\gamma_a} \omega_b, \quad 1 \leq a, b, \leq d, \tag{3.5}$$

"the period matrix of H^j." It is well-defined up to multiplication on one side by $\mathrm{GL}_d(\mathbb{Z})$ and on the other side by an orthogonal matrix.

Intrinsically, the information in this matrix corresponds to the fact that $H^j(\Gamma, \mathbb{Z}) \simeq H^j(S/\Gamma, \mathbb{Z})$ modulo torsion is equipped with a positive definite quadratic form coming from the Riemannian structure on S/Γ.

Notational warning: The notion of "period of automorphic forms, " as mentioned here or in the title, is not the same as that of (3.5)! However, the two notions overlap in some cases, which will be important for us.

4. ENDOMORPHISMS OF COHOMOLOGY: HECKE OPERATORS AND LEFSCHETZ OPERATORS

In the previous section we discussed cohomology of Γ in a specific case (see in particular (3.4)). In the present section, we identify *in the general case* two natural summands ("invariant cohomology" and "tempered cohomology")

$$H^*(\Gamma, \mathbb{Q})_{\mathrm{inv}}, H^*(\Gamma, \mathbb{Q})_{\mathrm{temp}} \subset H^*(\Gamma, \mathbb{Q})$$

which not only recover the splitting of (3.4), but show that it is defined over $\mathbb{Q}$.

In general, $H^*(\Gamma, \mathbb{Q})_{\mathrm{temp}}$ will look like a sum of copies of the cohomology of a δ-dimensional torus, where δ is an invariant depending only on the ambient Lie group G.

To describe this situation we will have to use Hecke operators.

4.1. **Hecke operators.** These are extra endomorphisms of $H^*(\Gamma)$ arising from "almost-automorphisms" of Γ. The definition is somewhat bewildering if one hasn't seen it before; on the other hand, the exact definitions don't matter so much for our purposes.

Given $\Gamma_1, \Gamma_2 \subset \Gamma$ of finite index and an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$, we get an endomorphism of $H^j(\Gamma)$,

$$H^j(\Gamma) \xrightarrow{\text{res}} H^j(\Gamma_2) \xrightarrow{\varphi^*} H^j(\Gamma_1) \xrightarrow{\text{cores}} H^j(\Gamma). \tag{4.1}$$

Such an operator is called a *Hecke operator*. Here the "restriction" map is just pull-back of cohomology classes under the covering $S/\Gamma_2 \to S/\Gamma$, and the "corestriction" is the push-forward under the covering $S/\Gamma_1 \to S/\Gamma$.

The prototypical example, for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, is the endomorphism (usually denoted as T_p) arising from the following data:

$$\Gamma_1 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \ \ (p)\}, \ \ \Gamma_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : b \equiv 0 \ \ (p)\} \tag{4.2}$$

$$\varphi = \text{conjugation by} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

Arithmetic groups admit many such "almost-automorphisms"; for example, for $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, taking φ to be conjugation by any element of $\mathrm{SL}_n(\mathbb{Q})$ works, for suitably chosen Γ_1, Γ_2. Remarkably, all the resulting endomorphisms commute with one another.

In general, one wants to slightly restrict the $(\Gamma_1, \Gamma_2, \varphi)$ in order to obtain a commutative algebra, namely, we consider only Hecke operators that are prime to the level of Γ.

In this way one obtains a commuting algebra $\mathbb{T}$ of endomorphisms of $H^j(\Gamma, \mathbb{Q})$, or for that matter $H^*(\Gamma, S)$ for any coefficient ring S (or indeed even other homology theories, applied to $B\Gamma$, cf. last paragraph of §2.4).

4.2. **Invariant and tempered cohomology.** Now consider how a Hecke operator T acts on the trivial class in $H^0(\Gamma, \mathbb{C})$. The restriction map carries it to the same constant class in $H^0(\Gamma_2)$, and then corestriction multiplies it by the degree $[\Gamma : \Gamma_2]$. In our setting, we always have $[\Gamma : \Gamma_1] = [\Gamma : \Gamma_2]$; this common number is called the *degree* of T. Thus

$$T \cdot \text{ trivial class} = (\deg T) \text{ trivial class.}$$

We may now define the "invariant" classes

$$H^*(\Gamma, \mathbb{Q})_{\text{inv}} := \{s \in H^*(\Gamma, \mathbb{Q}) : Ts = \deg(T)s \text{ for all } T\}.$$

the set of classes such that T acts on s the same way that T acts on the trivial cohomology class. One can think of this as a substitute for "constant under parallel transport." Indeed, any harmonic representative for an element of $H^*(\Gamma, \mathbb{Q})_{\text{inv}}$ *is* parallel constant; the proof is similar to the proof on page 7. However, the way it has been just defined, it makes sense for cohomology with $\mathbb{Q}$-coefficients as well as cohomology with $\mathbb{C}$-coefficients.

Remark: It is known, for example, that if $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ then all cohomology $H^j(\Gamma, \mathbb{C})$ with $j < n$ belongs to H^*_{inv} (this follows from Borel's computation [8]).

The definition of "tempered cohomology" is intended as the "opposite" to invariant cohomology. Just taking the orthogonal complement of invariant classes is not a strong enough requirement. Rather, we will ask that the Hecke operators act with eigenvalues that are "as small as possible:"

> The tempered subspace $H^*(\Gamma, \mathbb{Q})_{\mathrm{temp}}$ is the largest $\mathbb{T}$-stable subspace $W \subset H^*(\Gamma, \mathbb{Q})$ with the property that, for all $\varepsilon > 0$ and all $T \in \mathbb{T}$, every eigenvalue of T on $W \otimes \mathbb{C}$ is bounded by $c(\varepsilon)\deg(T)^{1/2+\varepsilon}$.

For the reader familiar with the theory of automorphic forms: this *ad hoc* condition amounts to asking that for almost every prime the associated local representation π_p of the local p-adic group is tempered.

Note that the number $1/2$ is as small as possible: if we replaced it with any smaller number, the resulting space would be zero. It is motivated from the idea of "square root cancellation": When we compute $T\omega$ for a differential form ω and evaluate it at a suitable tangent vector $v \in \wedge^* T_x$, the computation amounts to adding up values of ω at $\deg(T)$ different inputs $v_1, \dots, v_{\deg(T)}$. Absent any conspiracies, one expects the size of this sum to be around $\sqrt{\deg(T)}$.

4.3. **The tempered cohomology looks like a cohomology of a torus.** Assuming standard conjectures in the theory of automorphic forms[1] one can always describe H^*_{temp} rather explicitly, at a numerical level. It is supported in a narrow range of dimensions and is symmetric around the middle degree $\frac{\dim(S/\Gamma)}{2}$. The vector of dimensions $\dim H^j_{\mathrm{temp}}$ is always a sequence of binomial coefficients, just as in (3.1), i.e.

$$\dim H^{j_0+j}_{\mathrm{temp}}(\Gamma, \mathbb{C}) = k \binom{\delta}{j} \tag{4.3}$$

where we interpret $\binom{\delta}{j}$ as zero if $j < 0$ or $j > \delta$; and

- The integer-valued invariants δ, j_0 depend only on the associated symmetric space S. We will describe δ in more detail below; and given δ, the invariant j_0 is determined by symmetry around the middle degree: we have $2j_0 + \delta = \dim(S)$.
- $k = k(\Gamma) \geq 0$ is an integer depending on Γ, not just on S.

To say differently, the cohomology of Γ (or at least the "piece" H^*_{temp}) looks like k copies of the cohomology of the δ-dimensional torus $(S^1)^\delta$, except shifted in such a way that the middle dimension is $\frac{\dim S}{2}$. It is certainly possible that $\delta = 0$ – this happens for "Shimura varieties", i.e. when S/Γ admits a Kähler structure, and in that case the conjectures that I will formulate are vacuous.

[1] Namely, if an automorphic representation is tempered at almost all places, it is tempered at *all* places. This can probably be avoided by varying the definitions slightly.

4.4. **The invariant δ.** The important invariant here is δ, which tells the dimension of the torus. It is defined to equal the rank of G minus the rank of a maximal compact subgroup K.

For example, if $\Gamma = \mathrm{SL}_5(\mathbb{Z})$, then $G = \mathrm{SL}_5(\mathbb{R})$ has rank 4 and $K = \mathrm{SO}_5(\mathbb{R})$ has rank 2; so $\delta = 4 - 2 = 2$.

δ is, in fact, the dimension of a canonical vector space $\mathfrak{a}$ attached to G, the split part of a fundamental Cartan subalgebra: see §6.3.

Now, two more informal ways of understanding δ:

- δ equals the "smallest dimension of any family of tempered representations of G."

  A unitary representation of G is called *tempered* if it occurs weakly inside $L^2(G)$ cf. [10]. Langlands classification [18] shows that tempered representations occur in families indexed by pairs (M, σ) of Levi subgroups of G and discrete-series representations σ of M. Then δ is the smallest dimension of any such family; for example, if $\delta = 0$, then G has a natural discretely parameterized family of tempered unitary representations –the "discrete series" constructed by Harish–Chandra.
- δ measures the obstructedness for Galois representations:

  In the Langlands program, one is interested in a "moduli space of Galois representations." More precisely, if one studies deformations of crystalline Galois representations with targets in the Langlands dual group to $\mathbf{G}$, the virtual dimension of the moduli space, at least in characteristic 0, should be equal to $-\delta$.

  Thus $\delta > 0$ corresponds to obstructed deformation theory. This is an important observation: The conjectures that we will formulate are closely related to "derived" structures in the Langlands program, that is to say, to a derived version of the moduli space of the Galois representations that takes account the obstructedness of the deformation problems. See §7.5 for some brief further remarks.

4.5. **Some context.** Let us describe the more standard presentation of the foregoing theory, and how our discussion relates to it:

We have described two canonical summands of cohomology. In general, one can extend this to a splitting

$$H^*(\Gamma, \mathbb{C}) = \bigoplus_{\mathfrak{p}} H^*(\Gamma, \mathbb{C})_{\mathfrak{p}} \tag{4.4}$$

where the summands are indexed by θ-stable parabolic subgroups $\mathfrak{p}$ [23] of G – the invariant cohomology corresponds to maximal $\mathfrak{p}$ and the tempered cohomology to minimal $\mathfrak{p}$.

In general, one obtains this splitting as follows: by Matsushima's formula [6] we have $H^*(\Gamma, \mathbb{C}) = \bigoplus_\pi H^*(\mathfrak{g}, K, \pi)$ where the sum is over automorphic representations π, taken with multiplicity, that appear in $L^2(G/\Gamma)$. Now we split according to the isomorphism class of π. Vogan and Zuckerman [24] have classified the representations with nonvanishing $(\mathfrak{g}, K)$ cohomology; they are indexed by θ-stable parabolic subgroups. This gives rise to the splitting (4.4).

This discussion is for $\mathbb{C}$ coefficients; in order to obtain a splitting over $\mathbb{Q}$ one should group together some classes of θ-stable parabolics.

5. The fundamental problem

We have, at this point, isolated a summand $H^*_{\text{temp}}(\Gamma, \mathbb{Q})$ of the cohomology of any arithmetic group Γ. It looks like the shifted cohomology of a δ-dimensional torus, where δ was the invariant described in §4.4, and we have

$$\dim H^{j_0+\delta}_{\text{temp}} = \binom{\delta}{j} \dim H^{j_0}_{\text{temp}}. \tag{5.1}$$

In fact, the same equality continues to hold even if we break into Hecke eigenspaces. In other words, if we pick a Hecke operator T and a complex number λ and look only at the λ-eigenspaces of T, the equality (5.1) remains valid.

Thus our situation is that the "spectrum" of the Hecke algebra acting on $H^*_{\text{temp}}(\Gamma, \mathbb{Q})$ is degenerate: the same eigenvalues occur in many different degrees. In general, in such a situation, it is natural to look for extra symmetries that explain the degeneracy, just as degeneracy of the energy spectrum of a physical system is often explained by symmetries of that system.

Thus:

> Basic problem (naive formulation): produce enough endomorphisms $H^a_{\text{temp}}(\Gamma, \mathbb{Q}) \to H^b_{\text{temp}}(\Gamma, \mathbb{Q})$ to explain (5.1).

These extra endomorphisms should commute with the Hecke operators (see remark after (5.1)). As for "enough": we would like sufficiently many endomorphisms to produce all of H^*_{temp} starting from the lowest degree, for instance. So we can formulate a more refined version:

> Basic problem (refined formulation): Construct a "natural" $\mathbb{Q}$-vector space V of dimension δ (see (4.3)) and a "natural" action of $\wedge^*\mathsf{V}$ on $H^*_{\text{temp}}(\Gamma, \mathbb{Q})$, over which H^*_{temp} is freely generated in dimension j_0.

One might think that one could do this simply by thinking more carefully about the proof of (5.1). However, that proof is intrinsically transcendental; it uses differential forms and really works with $\mathbb{C}$ coefficients. It seems to be much more difficult to produce endomorphisms of *rational* cohomology $H^*(\Gamma, \mathbb{Q})$ that change cohomological degree and yet are natural enough to commute with all the natural symmetries (the Hecke operators). For example, a natural candidate to shift cohomological degree is to take cup product with a class $\beta \in H^i(\Gamma, \mathbb{Q})$. This does not work, or at the very least is inadequate : very often $H^1(\Gamma, \mathbb{Q}) = 0$, so one cannot produce endomorphisms that shift degree by $+1$ this way.[2]

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[2] Note that, in the Shimura case, endomorphisms of this type play an important role, and were one of the motivations for Arthur's SL_2, see [2, p 60]. However, even in that case, these endomorphisms are trivial on $H^*_{\text{temp}}(\Gamma, \mathbb{Q})$.

The main goal of the talks was to discuss the following proposal:

> *Proposal:* V is the $\mathbb{Q}$-linear dual of the motivic cohomology group of a certain $\mathbb{Q}$-motive, the motive attached to the adjoint L-function (see §7.2 for more). Moreover, one can explicitly construct the action of $\mathsf{V} \otimes \mathbb{Q}_p$ and $\mathsf{V} \otimes \mathbb{C}$ on cohomology with $\mathbb{Q}_p$- or $\mathbb{C}$- coefficients.

Motivic cohomology belongs *a priori* to an entirely different world to $H^*(\Gamma, \mathbb{Q})$ and it seems to me that any mechanism to construct the action must be very interesting.

We will flesh out this proposal in the next two sections.

6. Constructing endomorphisms of cohomology with $\mathbb{Q}_p$ or $\mathbb{C}$ coefficients

We now outline a first step towards the problem formulated in §5. Namely, we have asked there to construct extra endomorphisms of cohomology of an arithmetic group, which shift the degree. We will sketch here how to produce extra endomorphisms with $\mathbb{Q}_p$ coefficients or with $\mathbb{C}$ coefficients.

In the next section §7, we will examine the deeper question of how these actions interact with the rational structure on cohomology.

6.1. **The derived Hecke algebra.** Return to the idea of using cup product to produce degree shifts. I mentioned earlier that there are, in general, not enough classes in $H^1(\Gamma, \mathbb{Q})$ and thus one cannot hope to produce any endomorphisms that shift degree by 1.

However, there can be, in general, many more *torsion classes* $\alpha \in H^1(\Gamma, \mathbb{Z}/q)$ for various q. In particular, we can always arrange the existence of such classes if we are willing to replace Γ by a finite index subgroup Γ'.

For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and Γ' is the subgroup of matrices of the form

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \ \ (p)\}$$

then, for any q dividing $p-1$, the homomorphism

$$\alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \ a \bmod p \ \in (\mathbb{Z}/p)^* \to (\mathbb{Z}/q) \tag{6.1}$$

(where the latter map is any nontrivial homomorphism) gives a nontrivial class $\alpha \in H^1(\Gamma', \mathbb{Z}/q)$.

To define derived Hecke operators, return to the setting of (4.1) above. Suppose given a torsion class $\alpha \in H^1(\Gamma_1, \mathbb{Z}/q)$. (We suppress the $\mathbb{Z}/q$ coefficients in the notation that follows, for simplicity.) Then we can cup with α, modifying the previous operation.

$$H^k(\Gamma) \xrightarrow{\text{res}} H^k(\Gamma_2) \xrightarrow{\varphi^*} H^k(\Gamma_1) \xrightarrow{\cup\alpha} H^{k+1}(\Gamma_1) \xrightarrow{\text{cores}} H^{k+1}(\Gamma).$$

In other words, a derived Hecke operator is produced from a usual Hecke operator by "inserting" a cohomology class α during the process.

In this way we produce an endomorphism of cohomology that shifts degree; and one can check that it commutes with all Hecke operators.

One can similarly use α in higher cohomological degree, but anyway α must be a torsion class. One produces operations on $\mathbb{Z}_q$ cohomology only as a "limit" of such torsion operations: by using torsion classes with mod q^n coefficients as $n \to \infty$.

Thus derived Hecke operators are parameterized by pairs $(\Gamma_1 \xrightarrow{\phi} \Gamma_2, \alpha \in H^*(\Gamma_1))$. These operators seem rather arbitrary, as I have defined them. Let us describe two ways in which they are in fact very natural.

Re-indexing: In fact they can be indexed in a much more canonical way; let us restrict to SL_n for concreteness. Set $G = \mathrm{SL}_n(\mathbb{Q}_p)$ and $K_p = \mathrm{SL}_n(\mathbb{Z}_p)$. Let S be a coefficient ring (e.g. $\mathbb{Z}/q$, to match with the discussion above) . Then

$$\mathbb{T}_p = \mathrm{End}_{SG}(S[G/K], S[G/K])$$

is visibly an algebra under composition (here $S[G/K]$ is just the free abelian group on G/K). In fact it is commutative, and a basic fact in the theory of automorphic forms is that $\mathbb{T}$ indexes Hecke operators: For each $\tau \in \mathbb{T}$ one can write down a Hecke operator acting on $H^*(\Gamma, S)$, in the sense of (4.1), such that $\tau \mapsto T(\tau)$ is a homomorphism.

This story has an analog for the derived Hecke algebra. If one replaces $\mathbb{T}_p$ by the corresponding Ext-algebra

$$\widetilde{\mathbb{T}}_p = \bigoplus \mathrm{Ext}^*_{SG}(S[G/K], S[G/K])$$

one gets a graded algebra, and again to each $\tau \in \widetilde{\mathbb{T}_p}$ we can associated a derived Hecke operator so that $\tau \mapsto T_\tau$ is a homomorphism; the degree of τ in the grading on $\widetilde{\mathbb{T}}_p$ corresponds to the degree by which it shifts cohomology.

Relationship to Taylor-Wiles method. The second interesting point concerning derived Hecke operators are their interaction with the Taylor–Wiles method. Examine a typical derived Hecke operator in the setting of (4.2).

For example, for SL_2, we can construct[3] a derived Hecke operator by starting with the usual Hecke operator from (4.2) and "inserting" the cohomology class α described in (6.1). In this way we get a derived Hecke operator from a prime p and a homomorphism $(\mathbb{Z}/p)^* \to \mathbb{Z}/q$.

At a superficial level, this resembles the Taylor–Wiles method in the fact that it also uses "auxiliary" primes p with the property that $p \equiv 1 \bmod q$ (where q is a power of the characteristic of the residual Galois representation being studied). But the resemblance goes much deeper.

[3]In fact, derived Hecke operators are no real interest for subgroups of $\mathrm{SL}_2(\mathbb{Z})$; there we have $\delta = 0$, and the action of derived Hecke operators will usually be zero. But the derived Hecke algebra already becomes interesting for subgroups of $\mathrm{SL}_2(\mathbb{Z}[i])$.

Write, for short, $\Delta = (\mathbb{Z}/p)^*$. Let Γ_1' be the kernel of the natural map (6.1) $\Gamma_1 \to \Delta$. From $\Gamma_1' \subset \Gamma$ we get a corresponding covering of spaces

$$\underbrace{S/\Gamma_1'}_{X} \to \underbrace{S/\Gamma_1}_{Y}$$

with Galois group Δ.

If we examine the above definitions, we see that the derived Hecke algebra is related to studying the action of $H^*(\Delta)$ on $H^*(\Gamma_1) = H^*(Y)$. (In general, given a quotient $Y = X/\Delta$, there is an action of $H^*(\Delta)$ on $H^*(Y)$, coming from pulling back cohomology classes via $Y \mapsto B\Delta$).

On the other hand, in the Taylor–Wiles method, a starring role is played by the action of Δ on the cohomology of the covering space X.

The two actions (of Δ on $H^*(X)$ and of $H^*(\Delta)$ on $H^*(Y)$) are closely related. If we replace Δ by a torus, they would be "Koszul dual" (cf. [13]).

Using these general ideas, I have checked [22] that (after passing to $H^*(\Gamma, \mathbb{Z}_p)_{\mathrm{temp}}$ by a limit process, tensoring with $\mathbb{Q}_p$, and then localizing at a single Hecke eigensystem) that the algebra of endomorphisms generated by these operations is isomorphic to $\wedge^* \mathbb{Q}_p^\delta$, *modulo*

- The existence of Galois representations attached to (torsion) cohomology classes. This has been checked by Scholze for many Γ; but I also need
- a local-global compatability, not known at present, and
- Various technical conditions at p, which are satisfied for large enough p.

Thus, on the assumptions above, the derived Hecke algebra indeed seems to be the correct way of constructing extra endomorphisms with $\mathbb{Q}_p$-coefficients.

6.2. **Differential forms.** We now indicate how to produce extra endomorphisms of *complex* cohomology. Unfortunately, to do this in general involves a lot of structure theory of semisimple real Lie groups and background on $(\mathfrak{g}, K)$ cohomology.

So we will carry out only in the case already discussed in §3.2. More precisely, we will produce an action of $\wedge^* \mathbb{C}^3$ on $H^*(\Gamma, \mathbb{C})_{\mathrm{temp}}$. This will appear rather *ad hoc* but it a specialization of a definition that makes sense in general.

To produce an action of $\wedge^* \mathbb{C}^3$, we must simply produce three endomorphisms e_i of H^*_{temp} which shift degree by $+1$ and satisfy $e_i^2 = 0, e_i e_j = -e_j e_i$. Recall that

$$H^*_{\mathrm{temp}} = \bigoplus_{a,b,c \in \{1,2\}} \mathcal{H}^{abc}$$

with the notation of §3.2.

On $\mathcal{H}^{a_1 a_2 a_3}$, with $a_i \in \{1,2\}$ we take

$$e_i = \begin{cases} (-1)^{\sum_{k<i} a_k} *_i, & a_i = 1 \\ 0, a_i = 2 \end{cases}$$

so that on $\mathcal{H}^{111}$, these endomorphism are $*_1, -*_2, *_3$ respectively, and on $\mathcal{H}^{211}$, these endomorphisms are $0, *_2, -*_3$, and so on. The signs have been chosen to make the e_i anti-commute.

Then the endomorphisms e_1, e_2, e_3 give rise to an action of $\wedge^* \mathbb{C}^3$ on cohomology. Note that, more intrinsically, the $\mathbb{C}^3$ is a 3-dimensional vector space with a basis given by the embeddings of F into $\mathbb{C}$.

Of course, the $*_i$ operators are completely transcendental; to make this of any interest, we must understand how it interacts with rational cohomology. In the next section we propose a conjecture that pins this down. Namely, we will describe a $\mathbb{Q}$-vector space V equipped (in this case) with a map $\mathsf{V} \to \mathbb{C}^3$ (more intrinsically, a map $\mathsf{V} \to \mathbb{C}$ for each complex place). Conjecturally, V is 3-dimensional, the map is injective and $\iota(\mathsf{V})$ preserves $H^*(\Gamma, \mathbb{Q})$.

6.3. **Remarks on the general case.** The general situation is as follows: Let G be the Lie group containing Γ. Strictly speaking, in what follows, we want to think of G as a real reductive algebraic group. Now, we can construct a δ-dimensional complex vector space $\mathfrak{a}_G$ canonically attached to G, and we produce (in a less *ad hoc* way than above, but again using explicit constructions with differential forms) an action of $\wedge^* \mathfrak{a}_G^*$ on $H^*_{\text{temp}}(\Gamma, \mathbb{C})$.

To define $\mathfrak{a}_G$, we recall that amongst the various maximal tori of a real semisimple algebraic group G there are two distinguished conjugacy classes:

If T is the real points of any algebraic torus $\mathbf{T}$ over $\mathbf{R}$, the connected component is a product of copies of $(\mathbb{R}_+)$ and copies of (S^1):

$$T^\circ \sim (\mathbb{R}_+)^a \times (S^1)^b.$$

It turns out there is a single conjugacy class of $\mathbf{T}$ with a maximal (the maximally split tori) and a single conjugacy class with b maximal (the fundamental Cartan subgroups; [6, III §4.1]). We are interested in the fundamental Cartan subgroups.

For example, if $G = \mathrm{SL}_n$, then

$$\mathbf{T}_f = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & 0 & \dots \\ -b_1 & a_1 & 0 & 0 & \\ 0 & 0 & a_2 & b_2 & \dots \\ 0 & 0 & -b_2 & a_2 & \dots \end{pmatrix} : \prod (a_i^2 + b_i^2) = 1 \right\}$$

is a fundamental Cartan subgroup; it is isomorphic to $(\mathbb{R}_+)^{[n/2]} \times (S^1)^{[n/2]}$.

On the other hand, if $G = \mathrm{SL}_2(\mathbb{C})^3$, then $D \times D \times D$ is a Cartan subgroup, where $D \subset \mathrm{SL}_2(\mathbb{C})$ is the subgroup of diagonal matrices; and in fact here all Cartan subgroups are conjugate.

The subgroup $\mathfrak{a}$ is then defined as the Lie algebra of the $(\mathbf{R}_+)^a$ part of a fundamental Cartan subgroup:

$$\mathfrak{a} = (\text{Lie algebra of the maximal split subtorus of fundamental Cartan } \mathbf{T}_f) \otimes_{\mathbb{R}} \mathbb{C}$$

In the examples above, the maximal split subtorus of $\mathbf{T}_f$ is given (for $G = \mathrm{SL}_n(\mathbb{R})$) by setting all $b_i = 0$ and (for $G = \mathrm{SL}_2(\mathbb{C})$) by restricting z to be real.

This definition can be made more canonical: Although different choices $\mathbf{T}_f, \mathbf{T}'_f$ are all conjugate, there are in general different conjugacies between $\mathbf{T}_f, \mathbf{T}'_f$. However, by choosing some extra data, one can rigidify the situation. Once this is done, the action of $\wedge^* \mathfrak{a}_G^*$ is not hard to define, but it requires a computation of the $(\mathfrak{g}, K)$ cohomology of a tempered representation. For this see [24, 22].

7. Rational structures and motivic cohomology

We now come to the most subtle part of the story.

With reference to the question formulated in §5, we have produced extra endomorphisms on cohomology with $\mathbb{Q}_p$ coefficients and with $\mathbb{C}$ coefficients; but how do we understand which of these preserve $\mathbb{Q}$-coefficients?

We will only sketch this in general (because the general version is notationally heavy). There are two difficulties of exposition:

- Unfortunately, it is almost impossible even to sketch it without assuming the reader has a substantial background in the Langalnds program. We apologize that we have not been able to avoid this.
- To formulate our conjectures requires both the full strength of conjectures in the Langlands program, and certain conjectures about motivic cohomology. On the other hand (see discussion at the start of §7.6) the results about these conjectures can be interpreted unconditionally, and are interesting even in this sense.

7.1. **Setup: the representation π and the Galois representation ρ_π.** As above Γ is an arithmetic group and $H^*_{\text{temp}}(\Gamma, \mathbb{Q}) \subset H^*(\Gamma, \mathbb{Q})$ is the tempered summand of cohomology discussed above (it would be great to extend our story to the whole cohomology, see final section). For simplicity, let us assume the integer k of (4.3) is actually equal to 1. In other words we have

$$\dim H^{q+j}(\Gamma, \mathbb{Q})_{\text{temp}} = \binom{\delta}{j}.$$

Here the Hecke algebra acts on all of $H^*(\Gamma, \mathbb{Q})_{\text{temp}}$ by a scalar; and there is a single automorphic representation π associated to this Hecke eigensystem. In general, one can reduce to a similar situation by decomposing H^*_{temp} over the Hecke algebra.

To avoid some technical complications (see the general discussion of [9]) we shall assume that the group $\mathbf{G}$ is simply connected, and for simplicity of phrasing we suppose it to be split. It has an attached Langlands dual group $\widehat{G}$, which we regard as as a split algebraic group over $\mathbb{Q}$. For example, if $\mathbf{G} = \mathrm{SL}_n$ then $\widehat{G} = \mathrm{PGL}_n$.

If we fix an algebraic closure $\overline{\mathbb{Q}_\ell}$ of the ℓ-adic numbers, it is conjectured that there exists a Galois representation

$$\rho_\pi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

characterized by a compatibility [9] between the values of ρ_π at Frobenius elements, and the eigenvalues of Hecke operators.

Now, we would like to get a usual (linear, i.e. with targets in GL_m) representation. To do this, we can compose with a representation of $\widehat{G}$ which exists for all $\widehat{G}$, namely, the coadjoint representation: the conjugation action of $\widehat{G}$ on its dual Lie algebra

$$\mathrm{Ad} : \widehat{G} \to \mathrm{GL}(\widehat{\mathfrak{g}}^*).$$

Since $\widehat{G}$ is semisimple, the Killing form identifies $\mathfrak{g}$ and its dual; but we prefer to maintain the distinction.

The composite

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \stackrel{\rho_\pi}{\to} \widehat{G}(\overline{\mathbb{Q}}_\ell) \stackrel{\mathrm{Ad}}{\to} \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell) \quad (d = \dim \widehat{G} = \dim \widehat{\mathfrak{g}})$$

will be an object of primary interest. We call it the *co-adjoint Galois representation attached to* π and denote it by $\mathrm{Ad}\ \rho_\pi$.

Example: To clarify what we are talking about, we discuss the example of $\mathrm{SL}_2(\mathbb{Z})$: if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, it is known [20] that to a weight 2 modular form for Γ there is associated a Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$. It has the property that the trace of ρ_f on the Frobenius element at p coincides with the scalar by which the Hecke operator T_p (see (4.2) for definition) acts on f. Then the adjoint representation that we are considering is not ρ_f, but rather the three-dimensional representation

$$\mathrm{Ad} \circ \rho_f$$

where $\mathrm{Ad} : \mathrm{GL}_2 \to \mathrm{GL}_3$ is obtained by the conjugation action of GL_2 on 2×2 matrices of determinant zero.

Returning now to the general case, the representation ρ_π is known to exist in many cases of interest to us, in particular the case of $\mathbf{G} = \mathrm{SL}_n$, because of the work of P. Scholze [21]. In any case, even if not known, the standard conjectures (again see [9]) are sufficiently precise to pin down ρ_π up to isomorphism.

Unfortuately, we want even more:

7.2. **The adjoint motive.** In the situation above, the Langlands program predicts that the representation ρ_π is actually *motivic*: in the best situations, there exists a motive [4] M_π of weight 0 and dimension $\dim(\mathbf{G})$ so that the action of the Galois group on the étale cohomology of M gives a representation isomorphic to $\mathrm{Ad}\ \rho_\pi$.

If it exists, this motive has the property that

$$L(M, s) = L(\mathrm{Ad}, \pi, s).$$

the L-function of M coincides with the L-function of the *adjoint L-function of* π. Note that M has weight 0, and the functional equation for this L function switches s and $1 - s$. In particular, $s = 1$ is at the edge of the critical strip. We will examine this L-value next.

The adjoint L-function is one of substantial interest in the theory of automorphic forms. It exists for automorphic forms on any group, rather than being attached to a specific low-dimensional "standard" representation of the dual group. The special value $L(\mathrm{Ad}, \pi, 1)$ is of particular interest: it shows up in practically all computations of automorphic periods, in the deformation theory of Galois representations, and also in local representation theory where it is closely related to Plancherel measure [16].

[4] We have written "in the best situation" because there are some subtleties related to descent of coefficient field, in general.

Unfortunately, the situation as regards the existence of M_π is much worse than for the Galois representation ρ_π. The methods used to construct ρ_π (in the cases when $\delta > 0$) really seem very far from giving a motive: they construct a ℓ-adic Galois representation by a system of congruences modulo higher and higher powers of ℓ, a process that has no apparent analog at the level of motives. On the other hand, there is substantial experimental evidence at least in the simplest cases (see e.g. [11] and subsequent papers) that such motives should exist.

7.3. **The motivic cohomology group** V**.** Now that we have produced a motive M_π we can consider the motivic cohomology

$$\mathsf{V} := H^1_{\mathrm{mot}}(M_\pi, \mathbb{Q}(1)). \tag{7.1}$$

Since we are only interested, at least for now, in this group with rational coefficients, we could also construct it by means of algebraic K-theory.

Beilinson's conjecture relates V to the L-function $L(\mathrm{Ad}, \pi, 1)$ above: roughly speaking, it relates $L(\mathrm{Ad}, \pi, 1)$ to the "volume" of a lattice inside the $\mathbb{Q}$-vector space V. The notion of volume arises by means of an explicit map (conjecturally an isomorphism) between $\mathsf{V} \otimes \mathbb{C}$ and an explicit finite-dimensional $\mathbb{C}$-vector space.

I now try to explain in a little more detail the conjectural answer to the problem above.

7.4. p**-adic and complex regulators.** In general, although very difficult to directly compute, motivic cohomology admits regulators to a finite-dimensional complex vector space constructed via the Hodge theory (the Beilinson regulator, [3]) and a finite-dimensional p-adic vector space constructed from the étale cohomology with $\mathbb{Q}_p$-coefficients (see discussion in [17]).

In our case, conditional on the Langlands conjectures which predict the exact Hodge structure for the motive M_π, the Beilinson regulator amounts to a morphism

$$\mathsf{V} \otimes \mathbb{C} \to \mathfrak{a} \tag{7.2}$$

where $\mathfrak{a}$ is the canonical vector space that we mentioned in §6.3 and the p-adic regulator gives a map

$$\mathsf{V} \to H^1_f(\mathrm{Ad}\rho_\pi(1)) \tag{7.3}$$

where H^1 is the Galois cohomology in the sense of Bloch–Kato.

We can now formulate the conjecture: Let $\mathfrak{a}^*_{\mathbb{Q}}$ be the elements of $\mathfrak{a}^*$ which map to V^* under the dual to (7.2). If, as is conjectured, the regulator is an isomorphism, this is actually a rational structure on $\mathfrak{a}^*$. Then:

> *Main conjecture:* For the action of $\wedge^*\mathfrak{a}^*$ on $H^*(\Gamma, \mathbb{C})$ defined as in §6.3, the elements of $\wedge^*\mathfrak{a}^*_{\mathbb{Q}}$ preserve $H^*(\Gamma, \mathbb{Q})$.

i.e. "the rational structures of motivic cohomology and automorphic cohomology line up."

Let us again emphasize the reason this is interesting: the motivic cohomology group V and the cohomology $H^*(\Gamma, \mathbb{Q})$ belong to wholly different worlds, and there is no known mechanism from the Langlands program why they should "know" about each other.

7.5. **Derived Hecke algebra and derived deformation ring.** There is a similar conjecture concerning the action of the derived Hecke algebra, but we do not formulate it carefully here.

It is worth noting at this point that, along with the derived Hecke algebra, there also exists [12] a derived version of the Galois deformation ring. This is a pro-simplicial ring $\mathcal{R}$ which classifies deformations of a fixed Galois representations but with coefficients in simplicial rings; the cohomology groups of the cotangent complex of $\mathbb{Z}_p \to \mathcal{R}$ are closely related to the groups $H^*_f(\mathrm{Ad}\rho(1))$ mentioned above (*a priori*, these cohomology groups are related to $H^*(\mathrm{Ad}\rho)$, and one obtains the Tate twist and the coadjoint representation after applying Tate global duality).

The paper [12] studies this ring and its relationship with the Taylor–Wiles method further. It seems that the graded ring $\pi_*\mathcal{R}$ should act on the homology of Γ. Currently the relationship between the derived Hecke ring and the derived Galois deformation ring is not as clear as the relationship between the usual Hecke ring and the usual Galois deformation ring.

7.6. **Testing the conjecture.** The paper [19] gives evidence for the conjecture mentioned in §7.4. Before describing it, let us describe more precisely what it means to give "evidence" when basic properties of motivic cohomology, such as finite dimensionality, are not proven:

What we actually do (unconditionally) is compute certain parts of the period matrix (see §3.4) for $H^*(S/\Gamma, \mathbb{C})$. To see that these computations actually relate to the Conjecture of §7.4, we use Beilinson's conjectures; it is Beilinson's conjectures that allow us to "access" various invariants of motivic cohomology. Thus our evidence is, strictly speaking, compatibility between our conjecture and Beilinson's conjecture. However, the main point is that the computation of the period matrix is of interest in its own right, even if one doesn't want to assume Beilinson's conjectures.

Let us return, again, to the situation of (3.2). Choose a differential form $\omega \in \mathcal{H}^{111}$ that has the property that ω is rational, i.e. the class of ω belongs to $H^3(S/\Gamma, \mathbb{Q})$, or equivalently every period $\int_Y \omega$ over a homology 3-cycle $Y \in H_3(S/\Gamma, \mathbb{Z})$ is actually rational.

Now the conjecture predicts exactly which combinations of the partial Hodge operators (see after (3.3)) $*_1, *_2, *_3$ map ω into rational cohomology. In particular, recalling that $* = *_1 *_2 *_3$ up to sign, the conjecture predicts a complex number α for which

$$\alpha\,(*\omega) \text{ is a rational class inside } H^6(S/\Gamma, \mathbb{C})$$

Take cup products and evaluate on the fundamental class:

$$\alpha \int_{S/\Gamma} \langle \omega, \omega \rangle d\nu \in \mathbf{Q}^*$$

where $d\nu$ is the Riemannian volume form. Said differently,

$$\langle \omega, \omega \rangle_{S/\Gamma} \in \mathbf{Q}^* \alpha^{-1}.$$

So a simple consequence of the conjecture is that it predicts $\langle \omega, \omega \rangle$ up to rational factors, in terms of V. (Even in this case it predicts much more, but let us focus on this for the moment).

In order to test the conjecture, then, we need to be able to compute $\langle \omega, \omega \rangle_{S/\Gamma}$ up to rational quantities. The sublety here comes from the fact that ω has been normalized by means of the requirement that $[\omega] \in H^3(S/\Gamma, \mathbb{Q})$.

Fortunately, the theory of periods of automorphic forms allows us to do just this: The key point is that one can *explicitly* compute $\int_Y \omega$ for certain 3-cycles in terms of L-values, by using Waldspurger's formulas [25]. The cycles Y in question arise from embeddings of tori into SL_2. This allows us to compute $\langle \omega, \omega \rangle$ in terms of the L-functions that appear in Waldspurger's formula (in particular, the adjoint L-function $L(1, \mathrm{Ad}, \pi)$ plays a prominent role, and in this way V appears.). Then we are able to verify that Beilinson's conjectures for these L-functions implies our conjecture.

There is something curious about this discussion. The usual theory of automorphic periods gives a large and rich supply of similar situations: one can explicitly evaluate the integral of a harmonic differential form on some S/Γ in terms of L-functions. In general, many different L-functions appear. It is striking that nonetheless these evaluations appear to be, in all cases we have checked, compatible with our conjecture, which *only* involves the vector space V associated to the adjoint L-function. The explanation, in each case, is a minor miracle of "Hodge linear algebra," which causes the influence of the auxiliary L-functions to cancel in the end.

However, there is also a much more interesting check on the conjecture carried out in [19]:

Although not obvious from our formulation, the conjecture of §7.4 says enough about H^*_{temp} *to predict the entire period matrix (in the sense of* (3.5)*) of* H^*_{temp} *(at least up to* $\mathbf{Q}^*$*).* The computation we did above amounts to the case of H^3_{temp} where the period matrix amounts only to the scalar $\langle \omega, \omega \rangle$. In [19] we are able to check some of these predictions about the period matrix in an indirect way: combining the theory of periods of automorphic forms and the theory of analytic torsion for Riemannian manifolds. This is the most direct evidence we have for the conjecture at present.

8. Concluding remark

I have discussed only the ‘ tempered part of the cohomology, i.e. $H^*_{\mathrm{temp}}(\Gamma)$. It would be very interesting to extend the conjecture to the *whole cohomology*. Ideally such a conjecture, when specialized to the low degree homology of $\mathrm{GL}_n(\mathbf{Z})$, would recover its relationship with the algebraic K-theory of $\mathbf{Z}$. In this way, some features of the algebraic K-theory of integers would become "degenerate limits" of phenomenona in the Langlands program.

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