BEYOND ENDOSCOPY AND SPECIAL FORMS ON GL(2)

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Abstract. We carry out (with technical modifications) some cases of a procedure proposed by R. Langlands in *Beyond Endoscopy*. This gives a new proof of the classification of "dihedral forms" on GL(2), avoiding endoscopic methods: instead, it uses an infinite limiting process in the trace formula. We also explain the relationship of Langlands' idea to the results of [10] that count automorphic forms associated to Galois representations. We prove results of a similar nature over an arbitrary number field. The main tool is a version of the relative trace formula, namely a slight variant (for PGL(2)) of the Kuznetsov-Bruggeman-Miatello formula.

MSC: 11F72 (11F70).

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1. Introduction

1.1. **Remarks on *Beyond Endoscopy.*** In his recent paper *Beyond Endoscopy* [9], Langlands proposes a new approach to the trace formula. The aim of this paper is to demonstrate how this approach, with some modifications, can be carried out for GL(2). We briefly review the main ideas.

Let $\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. Let π range over all automorphic, cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. If ρ is a representation of the dual group $\mathrm{GL}(2, \mathbb{C})$, we denote by $m(\pi, \rho)$ the order of the pole at $s = 1$ of $L(s, \pi, \rho)$, when defined. Let f be a nice function on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$; we shall denote by $\mathrm{tr}(\pi)(f)$ the trace of the operator defined by f on the representation π.

In *Beyond Endoscopy* [9], Langlands suggests a means to develop a formula of the form:

$$\sum_{\pi} m(\pi, \rho)\mathrm{tr}(\pi)(f) = \sum \dots \tag{1}$$

where the right hand sum ranges over a "geometric" contribution (that is, something resembling a sum over conjugacy classes.) This would therefore "isolate" the π for which $m(\pi, \rho) > 0$. Such π are expected to be functorial transfers from other groups; therefore, one might hope to be able to match the resulting formula with the trace formulae for these groups, and thereby prove these functorial lifts. (One hopes, of course, to do this for more general groups than GL(2).)

In concrete terms, the idea of Langlands can be expressed as follows. From the trace formula we will be able to evaluate:

$$\sum_{\pi} \lambda(n, \pi, \rho) \mathrm{tr}(\pi)(f) = \dots$$

where $\lambda(n, \pi, \rho)$ is the coefficient of n^{-s} in the Dirichlet series $L(s, \pi, \rho)$. At least, this is true for n coprime with those primes where f ramifies – we can then express the summand as the trace of a new function related to f. This is quite enough for our purposes.

The right hand side is a geometric contribution, a sum over conjugacy classes. In concrete terms, the most complicated part of the sum (the "elliptic" term) is a sum over all quadratic orders and involves their class numbers.

We take $n = p$ to be prime, and sum over all $p < X$, weighted by $\log(p)$. For each π, the quantity $\lim_{X \to \infty} \frac{1}{X} \sum_{p<X} \lambda(p, \pi, \rho) \log(p)$ is equal to $m(\pi, \rho)$. We therefore evaluate:

$$\sum_{\pi} \frac{1}{X} \mathrm{tr}(\pi)(f) \sum_{p<X} \log(p) \lambda(p, \pi, \rho)$$

by means of the trace formula. We will then obtain a sum over primes and conjugacy classes on the right hand side, and one can hope to evaluate the resulting limit as $X \to \infty$ by techniques of analytic number theory; we will then obtain a sum just over those forms for which $L(s, \pi, \rho)$ has a pole, and we will obtain an expression for (1), as desired.

Implicit in this is the hope of being able to identify the multiplicity of the pole of the L-function purely from the trace formula, without recourse to integral representations; this is of interest in itself. Unfortunately the technical details in inverting the spectral sum and limit are rather formidable.

Relation to other trace formulae: The "endoscopic" form of the trace formula, roughly speaking, seeks to classify representations that are distinguished by being fixed by an automorphism. The "relative trace formula" of Jacquet seeks to classify representations distinguished by period integrals. In this setting, Langlands' idea is to classify – by an infinite limiting process in the usual trace formula – representations that are distinguished by the poles of certain L-functions.

(It should be noted that results of a similar flavor may also be proven by an entirely different method: the technique of *backward lifting*; see [4].)

1.2. **Discussion of Results.** We now explain the contents of this paper.

In the first part of the paper – Sections 3 and 4 – we carry out (with certain modifications that we shall explain) the idea of Langlands in the case $G = \mathrm{GL}(2), \rho = \mathrm{Sym}^2$ — in other words, the classification of forms whose symmetric square has a pole. We shall term these "dihedral" forms; one expects them to correspond to Grössencharacters of quadratic field extensions. A proof of this classification was given by Langlands and Labesse;

their proof is endoscopic, while this proof has the potential to be applied to non-endoscopic situations, as we shall see.

Over $\mathbb{Q}$, the proof we shall give amounts to directly computing

$$\sum_{\pi} \operatorname{tr}(\pi)(f) \lim_{X\to\infty} \frac{1}{X} \sum_{p<X} \lambda(p, \pi, \mathrm{Sym}^2) \log(p) \tag{2}$$

Since the limit will only be non-zero for those π for which $L(s, \pi, \mathrm{Sym}^2)$ has a pole, this will construct the desired trace formula.

The modifications are as follows. Rather than use the usual trace formula, we use a relative trace formula: to be precise, we use a variant of the Kuznetsov-Bruggeman-Miatello formula. Further, we replace a sum over primes (or prime ideals) by a sum over all ideals. These are both technical modifications, but the second seems essential. Over $\mathbb{Q}$, our procedure amounts to considering in place of (2) the sum:

$$\sum_{f} h(t_f)\overline{a_m(f)} \lim_{X\to\infty} \frac{1}{X} \sum_{n<X} a_{n^2}(f) \tag{3}$$

where the sum is taken over some space of automorphic forms, and $a_n(f)$ denotes the nth Fourier coefficient; $1/4 + t_f^2$ is the Laplacian eigenvalue of f, and h an appropriate test function.

Since the symmetric square L-function of f coincides with (up to some harmless factors) the Dirichlet series $\sum_n a_{n^2}(f)n^{-s}$, the inner limit is still only nonvanishing when $L(s, \mathrm{Sym}^2 f)$ has a pole. Thus (3) serves the same purpose as (2) – it picks out the dihedral forms.

The final Theorem is stated as Theorem 1, Section 4.7. It is somewhat involved, so we do not reproduce it here. This result is originally due (in a slightly different form) to Langlands and Labesse, [8]; the proof offered here is entirely different, and can be regarded as the realization, in the first cases, of the method suggested by Langlands in [9].

We give a brief sketch of the method over $\mathbb{Q}$ in Section 3 before proceeding to the complete treatment in Section 4; we strongly suggest reading Section 3 prior to Section 4.

The second part of the paper — Section 5 — considers the case where $\rho = \mathrm{Sym}^k$, for $k > 2$. We explain in Section 5.1 why the analogue of the first part is difficult to carry through. Nevertheless, partial results may be obtained. Rather than a classification of those π for which $L(s, \mathrm{Sym}^k \pi)$ has a pole, we must content ourselves with *bounds* on the number of such π; we explain how this type of result fits into the scheme of *Beyond Endoscopy.* The final result is Theorem 2, stated in Section 5.3.

Over $\mathbb{Q}$, such bounds were given in [10]; over a general number field, two new features emerge: the necessity of dealing with the units, and the dependence on the discriminant of the number field. We will also achieve a modest saving in this "discriminant aspect."

Simplifying assumption: Throughout this paper, we deal with forms spherical at ∞; there are considerable notational complications in dealing with the general case. These notational complications would not arise in a purely adelic approach, but we are not entirely sure how such an approach would proceed. Although the proof is complete from a global perspective, we do not analyze the structure of dihedral forms at ramified places; it will be clear, however, that this is possible by the same technique.

Unproved local hypothesis at complex places: At present, the results of Section 4 are only proved for totally real fields; the extension to complex places would only require verifying the validity of a certain explicit integral transform involving I-Bessel functions; see Section 6.7 for the transform over $\mathbb{R}$. This amounts to the local computation over $\mathbb{C}$– a "local fundamental lemma." The author hopes to establish the validity of this transform in the near future.

Remark 1. The meaning of "dihedral form", for us, is one whose symmetric square L-function has a pole. This definition is unusual; this is a strictly smaller class than the set of forms associated to a representation of the Weil group with dihedral image. The methods of this paper generalize easily to the latter class by including an appropriate twist: namely, rather than classifying forms f so that $L(s, \mathrm{Sym}^2 f)$ has a pole, one may equally easily classify f so that $L(s, \mathrm{Sym}^2 f \times \xi)$ has a pole, for any fixed GL(1)-form ξ.

1.3. **Generalizations and Related Work.** There are other applications of this circle of ideas; it is possible, for example, to give a proof of a converse theorem starting from the trace formula. One may also (in the notation of the Introduction) carry out a similar procedure in the case $G = \mathrm{GL}(2)$, ρ the standard 2-dimensional representation. These are both closely related to questions posed by Langlands in [9], but we will not discuss it here; the case of ρ the standard 2-dimensional representation was already outlined by Sarnak in a letter to Langlands. Further details are contained in the PhD thesis of the author, [13].

More generally, this method is potentially applicable in non-endoscopic situations, and can yield partial results even if it cannot be carried out in full. On the other hand, we still face the "orbital integral" matching of the trace formula, though in a disguised form; and finally, there is a fundamental analytic difficulty which may be encountered (see Section 5).

This method seems very likely to yield at least partial results in higher dimensions: a typical and concrete starting question is giving a quantitative upper bound, when counting by conductor, on the number of $\mathrm{GL}(n)$ forms that are not *primitive* in the sense of J. Arthur. This would constitute a broad generalization of Duke's theorem, [3]. Considerable progress on this question has been made by the author.

In terms of functorial application, the most promising case seems to be the lifting from a classical group to $\mathrm{GL}(n)$; this, of course, can also be treated with the trace formula, and it remains to be seen if the analytic questions and the questions in local harmonic analysis that arise will be tractable.

Finally, a related paper is that of Mizumoto, [11]; I thank Bill Duke for bringing it to my attention. Although the focus of that paper is quite different, there is a commonality of technique in that essentially "spectral" methods are used to conclude results about specific L-functions.

1.4. **Trace formula and relative trace formula.** For the applications presented here, the "Kuznetsov-type" formulas are far more useful than the trace formula. An important difference is that these "Kuznetsov-type" formula include only representations possessing Whittaker models; this excludes immediately the identity representation. It therefore avoids the regularization that might be needed to handle the identity representation, mentioned in *Beyond Endoscopy.* A second difference is that the Kloosterman sums of the Kuznetsov-formula are more easily manipulable than the elliptic part of the trace formula.

From the point of view of functorial applications, an adelic form of the Kuznetsov formula might be optimal (it can be regarded as a relative trace formula, and presented as such). Nevertheless, since the Kuznetsov formula and the Bruggeman-Miatello formula are phrased classically, and almost all applications of the Kuznetsov formula have been "classical," we have chosen to retain this language. We will also have to deal with relatively delicate analytic questions; although in the end it is immaterial whether they are dealt with in an adelic or classical context, it seems to the author more transparent in the phrasing we have chosen.

This leads to greater concreteness, but also to more complicated notation to deal with class number and units. We will work on GL(2) and not on SL(2).

We give in the Appendix (Section 6) a derivation of the GL(2) version of the Bruggeman-Miatello formula. The most delicate points, which are convergence and inversion questions for the archimedean transforms ("Bessel transforms"), however, are identical, and we assume those results, referring to [2] and [7] for a detailed discussion.

1.5. **Structure of Paper.** We work over a number field F; in Section 4 F will always be totally real. The notational conventions are set down in Section 2; the bulk of the section is devoted to a statement and discussion of the Kuznetsov-Bruggeman-Miatello formula, appropriately modified for $\mathrm{PGL}_2(F_\infty)$ (rather than $\mathrm{SL}_2(F_\infty)$). In Section 4 the procedure outlined earlier in this introduction is carried out; a brief sketch over $\mathbb{Q}$ is given in Section 3. In Section 5 we discuss higher symmetric powers, and how one may obtain *bounds* but not a precise classification in this context.

The reader might also like to pretend $F = \mathbb{Q}$ in a first reading, and begin with Section 3.

There are two main reasons that we have presented the argument over a number field: firstly, the PGL(2) version of the Bruggeman-Miatello formula may be of interest to others, and secondly, it is important to demonstrate that the method works in the presence of infinitely many units. This latter point is not clear from the argument over $\mathbb{Q}$.

Finally, owing to the notational complexity, we have been quite liberal in cross-referencing.

1.6. **Acknowledgments.** The material here forms part of the author's PhD thesis, [13]. The author would like to express his gratitude to his thesis advisor, Professor P. Sarnak, and to Professor R. Langlands; for their kindness in sharing their ideas, many of which are to be found in the following pages, and for their support and encouragement during the preparation of this work.

2. Notations and formulas

Most of the notation is standard. This section is therefore included for reference. The text usually refers to the appropriate point in this section when we use any unusual notation. Other sections that introduce significant new notation are Sections 4.2 and 4.7.

2.1. **Notational Conventions.** *Characters:* The word *character* refers to a unitary continuous one-dimensional representation of a topological group. Thus a character of the group G is a map of G into the circle group. If G is an abelian group, we denote its dual group by $\widehat{G}$ (the same notation will be used for Fourier transform, but this should not cause confusion.)

Number fields: We will work over a number field F of degree $[F:\mathbb{Q}]$ over $\mathbb{Q}$. $d_{\mathbb{R}}$ and $d_{\mathbb{C}}$ will be the number of (equivalence classes of) real and complex embeddings, respectively, of F; then $[F:\mathbb{Q}] = d_{\mathbb{R}} + 2d_{\mathbb{C}}$. (See comment at the end of Section 1.2, however.) The ring of integers of F will be denoted $\mathfrak{o}_F$, and the class group of F denoted C_F; the units in $\mathfrak{o}_F$ will be denoted $\mathfrak{o}_F^*$. The number of roots of unity in F will be denoted ω_F. $\mathfrak{d}$ will be the different of F.

In general, we will use gothic letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, and so on, to denote fractional ideals of F. We will use the notation $\mathfrak{a} \sim \mathfrak{b}$ to mean that $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class.

Places and completions: The symbol v will be reserved for a place of F. F_v will denote the completion of F at v, $\mathfrak{o}_v$ denotes the integral subring of F_v, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{o}_v$, and $\mathfrak{o}_v^*$ the units of $\mathfrak{o}_v$. F_∞ is defined to be $F \otimes \mathbb{R} = \prod_{v|\infty} F_v$.

It is convenient to set, for $x \in F_\infty$, the "norm" $||x||_{F_\infty} = \sup_{v \,\text{infinite}} |x_v|$. Since F injects into F_∞, we can also speak of $||x||_{F_\infty}$ if $x \in F$.

We will use $\mathrm{Norm}(\mathfrak{a})$ to denote the norm of $\mathfrak{a}$; for $x \in F$ or $x \in F_\infty$, $\mathrm{Norm}(x)$ is the factor by which multiplication by x affects additive Haar measure on F_∞; in particular $\mathrm{Norm}(x) > 0$. If K/F is a field extension, $\mathrm{Norm}_{K/F}$ is the (usual) relative norm from K to F.

Adele ring: We denote the adele ring of F by $\mathbb{A}_F$, and we denote the ring of *finite* adeles by $\mathbb{A}_{F,f}$ (i.e. the restricted product of the F_v, where v varies only over finite places). We denote by $\hat{\mathfrak{o}}$ the *maximal compact subring* of $\mathbb{A}_{F,f}$; it is the topological product of $\mathfrak{o}_v$, for v finite, or equivalently the profinite completion of $\mathfrak{o}_F$. Similarly, we denote by $\mathbb{A}_F^\times$, $\mathbb{A}_{F,f}^\times$ and $\hat{\mathfrak{o}}^\times$ the group of units in these rings; these three groups are topologized via the idelic topology. In particular, $\hat{\mathfrak{o}}^\times = \{x \in \mathbb{A}_{F,f}^\times : v(x) = 0 \text{ for all finite places } v\}$.

Ideals and the class group, and representatives: If $x \in F^\times$, we denote by (x) the corresponding ideal. If $\mathfrak{a}$ is a fractional ideal, we will often just write $x\mathfrak{a}$ for the product $(x)\mathfrak{a}$.

We now make certain choices of representatives for ideals. These choices are admittedly clumsy and inelegant, and not strictly necessary. Making them, however, will ease life later on.

Firstly, we choose a set of $\mathfrak{o}_F$-ideals $\tilde{C}_F$ that represent the class group. We denote by $\tilde{C}_F(2) \subset \tilde{C}_F$ representatives for the 2-torsion in the class group.

If $\mathfrak{J}$ is a fractional ideal of F, and v is a finite place of F, we denote by $v(\mathfrak{J})$ the valuation of $\mathfrak{J}$ at v, i.e. the maximal power of the local maximal ideal in F_v that divides $\mathfrak{J}$. Further, we say that $x \in \mathbb{A}_{F,f}^\times$ is a "corresponding" idele if $v(x) = v(\mathfrak{J})$ for all finite places v.

For each principal fractional ideal $\mathfrak{I}$, we choose once and for all a generator which we denote $[\mathfrak{I}]$. Finally, for each fractional ideal $\mathfrak{J}$, we fix a corresponding idele that we denote $\pi_{\mathfrak{J}}$.

2.2. **Automorphic Forms: Normalizations and Conventions.** $\mathrm{GL}_2(F)$ and $\mathrm{GL}_2(\mathbb{A}_F)$ are the F- and $\mathbb{A}_F$- valued points of the group GL_2. We denote by Z the center of GL_2, that is, the diagonal matrices. N will be the subgroup of GL_2 consisting of *unipotent upper triangular matrices.*

For R any ring (usually $R = F_\infty$ or $R = \mathbb{A}_F$), we will define some useful elements of $\mathrm{GL}_2(R)$ as follows: For $x \in R$, we set $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and for $x, y \in R^\times$ we set $\mathrm{diag}(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

Open compact subgroups; congruence subgroups: Let v be a place of F. Let $\mathfrak{I}_v$, $\mathfrak{a}_v$ be fractional ideals of $\mathfrak{o}_v$. We set $K_{0,v}(\mathfrak{I}_v, \mathfrak{a}_v)$ to be the subgroup of $\mathrm{GL}_2(F_v)$ consisting of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $ad - bc \in \mathfrak{o}_v^\times$, $c \in \mathfrak{I}_v \mathfrak{a}_v^{-1}$ and $b \in \mathfrak{a}_v$. By $K_{0,v}(\mathfrak{I}_v)$ we mean $K_{0,v}(\mathfrak{I}_v, \mathfrak{o}_v)$. (Over $\mathbb{Q}$, one only needs to consider $\mathfrak{a}_v$ trivial; in general, the inclusion of $\mathfrak{a}_v$ is necessary to deal with multiple ideal classes.)

Let $\mathfrak{I}$ be an integral ideal of F and $\mathfrak{a}$ a fractional ideal; they define fractional ideals $\mathfrak{I}_v$, $\mathfrak{a}_v$ for each completion $\mathfrak{o}_v$, and we set $K_{0,v}(\mathfrak{I}, \mathfrak{a}) = K_{0,v}(\mathfrak{I}_v, \mathfrak{a}_v)$. Finally, we set $K_0(\mathfrak{I}, \mathfrak{a}) = \prod_{v \text{ finite}} K_{0,v}(\mathfrak{I}, \mathfrak{a})$. It is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,f})$. If $\mathfrak{I}$ is trivial, then it is maximal. In particular:

$$K_0(\mathfrak{I}) = \prod_v K_0(\mathfrak{I}_v, \mathfrak{o}_v) =$$
$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_{F,f}) : a_v, b_v, d_v \in \mathfrak{o}_v, c_v \in \mathfrak{I}_v, ad - bc \in \mathfrak{o}_v^\times\}$$

If χ_v is a ramified character of $F_v^\times$ with conductor dividing $\mathfrak{I}_v$, it induces a character (that we shall also denote χ_v) of $K_0(\mathfrak{I}_v, \mathfrak{a}_v)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \chi_v(a_v) \tag{4}$$

If χ_f is a character of $\mathbb{A}_{F,f}^\times$ with (global) conductor dividing $\mathfrak{I}$, we denote by χ_f the character of $K_0(\mathfrak{I}, \mathfrak{a})$ defined by:

$$\chi_f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \prod_{v:\chi_v \text{ ramified}} \chi_v(a_v) \tag{5}$$

We set $\Gamma_0(\mathfrak{I}, \mathfrak{a}) = \mathrm{GL}_2(F) \cap K_0(\mathfrak{I}, \mathfrak{a})$, the intersection being taken in $\mathrm{GL}_2(\mathbb{A}_{F,f})$. It is a congruence subgroup of $\mathrm{GL}_2(F)$. We set $\Gamma_N(\mathfrak{I}, \mathfrak{a}) = N(F) \cap \Gamma_0(\mathfrak{I}, \mathfrak{a})$. Note that $\Gamma_0(\mathfrak{I}, \mathfrak{a}), \Gamma_N(\mathfrak{I}, \mathfrak{a})$ are subgroups of $\mathrm{GL}_2(F)$; they can therefore also be regarded as discrete subgroups of $\mathrm{GL}_2(F_\infty)$ or as subgroups of $K_0(\mathfrak{I}, \mathfrak{a})$. We finally set $Z_\Gamma = \Gamma \cap Z(F)$.

Parameterizations of archimedean representations: In this paper, we will be concerned only with spherical representations of $\mathrm{GL}_2(F_\infty)$.

Let $\mathfrak{a}$ be the vector space $\mathbb{R}^{d_\mathbb{R}+d_\mathbb{C}}$; we will think of an element of $\mathfrak{a}$ as giving a real number for each infinite place of F. Accordingly, we denote a typical element by $\nu = (\nu_v)_{v|\infty}$. Let $\mathfrak{a}_\mathbb{C}$ be its complexification.

One can associate to $\nu = (\nu_v)_{v|\infty} \in \mathfrak{a}_\mathbb{C}$ a unique spherical representation of $\mathrm{PGL}_2(F_\infty)$. Namely, ν determines a character of the diagonal torus via:

$$\mathrm{diag}(x, y) \to \prod_v |xy^{-1}|_v^{i\nu_v}, \quad x, y \in F_\infty^\times$$

and we let $\pi(\nu)$ be the unique spherical constituent of the representation unitarily induced from this character. Note that changing the sign of any coordinate of ν does not change the isomorphism class of $\pi(\nu)$.

Let $Y \subset \mathfrak{a}_\mathbb{C}$ be the closure of the set of ν for which $\pi(\nu)$ is irreducible unitary; let Y_{temp} be the set of tempered parameters, i.e. $Y_{temp} = \mathfrak{a}$. Let $d\nu$ be the usual Lebesgue measure on $\mathfrak{a}$. Following [2], we also equip Y_{temp} (and thus $\mathfrak{a}$) with the positive measure $d\mu_\nu = \frac{1}{2} \prod_v \nu_v \sinh(\pi\nu_v) d\nu_v$.

Let $\underline{\cosh} : \mathfrak{a}_\mathbb{C} \to \mathbb{C}$ be the function

$$\underline{\cosh}(\nu) = \prod_{v|\infty} \cosh(\pi\nu_v) \tag{6}$$

Maximal compact subgroups at archimedean places: We will fix a maximal compact subgroup of $\mathrm{GL}_2(F_v)$ for each infinite place v; we denote by K_v this fixed choice. If v is real, then K_v is isomorphic to $O(2,\mathbb{R})$. If v is complex, then K_v is isomorphic to $U(2)$. We set $K_\infty = \prod_{v|\infty} K_v$.

Conductors: Let π_v be a representation of $\mathrm{GL}_2(F_v)$ on a space V_{π_v} with central character χ_v. For $\mathfrak{I}_v$ an integral $\mathfrak{o}_v$-ideal, divisible by the conductor of χ_v, let

$$V^{\mathfrak{I}_v} = \{v \in V_{\pi_v} : \pi(k)v = \chi_v(k)v \;\; \forall\, k \in K_0(\mathfrak{I}_v)\}$$

Then there is a maximal (i.e. minimally divisible) $\mathfrak{I}_v$ for which $V^{\mathfrak{I}_v}$ is nonzero; for this choice of $\mathfrak{I}_v$, the space $V^{\mathfrak{I}_v}$ is one-dimensional. This $\mathfrak{I}_v$ is the *conductor of* π_v.

Let π be a representation of $\mathrm{GL}_2(\mathbb{A}_F)$. The conductor of π is defined as the product of the local conductors – that is, the product of the conductors of π_v, for v finite. It is an integral $\mathfrak{o}_F$-ideal.

Logarithms and Exponentials: We introduce two "logarithms" that will be convenient.

Given $x \in F_\infty^\times$, we define $\log_F(x) = (\log|x_v|)_{v|\infty} \in \mathfrak{a}$.

If K is a quadratic extension of F, split at ∞, we define the "relative logarithm" $\log_{K/F}(x) : K_\infty^\times/F_\infty^\times \to \mathfrak{a}$ as follows: For each infinite place v of F, let $\tilde{v}_1, \tilde{v}_2$ be the places above v in K, and set

$$(\log_{K/F}(x))_v = \frac{1}{2}(\log|x_{\tilde{v}_1}/x_{\tilde{v}_2}|) \tag{7}$$

This is trivial on F_∞. Each coordinate of $\log_{K/F}(x)$ depends on the choice of $\tilde{v}_1$, but only up to a sign, and this will be harmless.

Additive characters: We will fix once and for all a nontrivial character $\psi_{\mathbb{A}}$ of $\mathbb{A}_F/F$, namely $\psi_{\mathbb{A}}(a) = \psi_{\mathbb{A}_\mathbb{Q}}(\mathrm{tr}_{F/\mathbb{Q}}(a))$, where $\psi_{\mathbb{A}_\mathbb{Q}}$ is the unique unramified character of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ with restriction to $\mathbb{R}$ of the form $x \mapsto e^{2\pi i x}$.

$\psi_{\mathbb{A}}$ defines, for each place v, a nontrivial additive character ψ_v of F_v; in particular, it defines a character $\psi_\infty : F_\infty \to \mathbb{C}$, namely $\psi_\infty(x) = e^{2\pi i \mathrm{tr}_{F_\infty/\mathbb{R}}(x)}$. It also induces a character $\psi_f : \mathbb{A}_{F,f} \to \mathbb{C}$.

Haar measure: Identifying F_∞ with $\mathbb{R}^{d_\mathbb{R}} \times \mathbb{C}^{d_\mathbb{C}}$, we fix the Haar measure corresponding to the product of dx at real places and $|dx \wedge \overline{dx}|$ at complex places. We shall also require a measure on $\mathrm{PGL}_2(F_\infty)$; this is prescribed (in terms of the measure in [2]) in Section 6.2.

Automorphic forms: We denote by $\mathcal{A}(\mathrm{GL}_2)$ the space of automorphic forms on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ – for definition, see [1]; we denote by $\mathcal{C}(\mathrm{GL}_2) \subset \mathcal{A}(\mathrm{GL}_2)$ the cuspidal subspace.

Continuous spectrum: Many of the spectral formulas will involve the contribution of the continuous spectrum. We will generally not write out these explicitly, but generically will denote it by CSC, for "Continuous spectrum contribution." In each case where we use this, it will (hopefully) be clear what the precise formula would be, by analogy with the cuspidal case.

2.3. **The space of automorphic forms with specified conductor and central character.** We will, rather than working with all automorphic forms, fix a conductor and work on a particular real symmetric space. Fix

χ, a Grössencharacter of $\mathbb{A}_F^\times/F^\times$. Fix an ideal $\mathfrak{I}$ divisible by the conductor of χ. We denote by $\chi_v, \chi_f, \chi_\infty$ the restriction of χ to $F_v, \mathbb{A}_{F,f}^\times, F_\infty^\times$ respectively. We will assume that $\chi_\infty \equiv 1$. As remarked previously, χ_f induces a character of $K_0(\mathfrak{I}, \mathfrak{a})$ (see (5)) and therefore (by restriction) a character of $\Gamma_0(\mathfrak{I}, \mathfrak{a})$.

We will be interested in the space of automorphic forms with *central character* χ *and conductor* $\mathfrak{I}$: that is, functions on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ that are (right) K_∞-fixed, transform under the right action of $K_0(\mathfrak{I})$ by the character χ_f and transform under $Z(\mathbb{A}_F)$ by the character χ. We denote the corresponding L^2-space by FS_χ, i.e.:

$$(8)\ \mathrm{FS}_\chi = \{f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C} : \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/Z(\mathbb{A}_F)} |f(g)|^2 dg < \infty,$$
$$f(\gamma g \mathrm{diag}(z,z) k_1 k_2) = f(g)\chi_f(k_1)\chi(z),$$
$$\gamma \in \mathrm{GL}_2(F), z \in \mathbb{A}_F^\times, k_1 \in K_0(\mathfrak{I}), k_2 \in K_\infty\}$$

It is also convenient to introduce a slightly larger space, containing FS_χ; this will only be referred to in the appendix and can be safely skipped. Namely, we consider functions on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ that are (right) K_∞-fixed, transform under the right action of $K_0(\mathfrak{I})$ by χ_f and transform under $Z(F_\infty)$ by χ_∞; we denote the corresponding L^2-space of functions by FS; thus $\mathrm{FS}_\chi \subset \mathrm{FS}$. FS amounts to considering automorphic forms of conductor $\mathfrak{I}$, whose central character is *a twist of* χ *by a class group character.*

$$\mathrm{FS} = \{f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C} : \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/Z(\mathbb{A}_F)} |f(g)|^2 dg < \infty,$$
$$f(\gamma g \mathrm{diag}(z,z) k_1 k_2) = f(g)\chi_f(k_1)\chi(z),$$
$$\gamma \in \mathrm{GL}_2(F), z \in F_\infty^\times, k_1 \in K_0(\mathfrak{I}), k_2 \in K_\infty\}$$

We will now identify these space more concretely. Let $\mathfrak{a}$ be a fractional ideal. We set $X_\mathfrak{a} = \Gamma_0(\mathfrak{I}, \mathfrak{a})\backslash\mathrm{PGL}_2(F_\infty)/K_\infty$. Let $\mathrm{Fun}_\chi(X_\mathfrak{a})$ be the vector space of functions f on $\mathrm{PGL}_2(F_\infty)$ so that $f(\gamma g k) = \chi_f(\gamma)^{-1} f(g)$, for $\gamma \in \Gamma_0(\mathfrak{I}, \mathfrak{a})$ and $k \in K_\infty$, and which are square-integrable on $X_\mathfrak{a}$; these are L^2-sections of a line bundle over $X_\mathfrak{a}$.

Given a fractional ideal $\mathfrak{a}$, one has a natural map

$$\mathrm{FS}_\chi \subset \mathrm{FS} \to \mathrm{Fun}_\chi(X_\mathfrak{a})$$

which associates to each $\phi \in \mathrm{FS}$ the following member of $\mathrm{Fun}_\chi(X_\mathfrak{a})$:

$$g_\infty \to \overline{\chi(\pi_\mathfrak{a})}\phi(\begin{pmatrix} \pi_\mathfrak{a} & 0 \\ 0 & 1 \end{pmatrix} g_\infty) \tag{9}$$

This has the pleasant property that it is independent of the choice of $\pi_\mathfrak{a}$, in view of the transformation property of ϕ under $K_0(\mathfrak{I})$; the right-hand side therefore depends only on the ideal $\mathfrak{a}$.

In particular, this gives an isomorphism $FS \xrightarrow{\sim} \oplus_{\mathfrak{a}\in\tilde{C}_F} \mathrm{Fun}_\chi(X_\mathfrak{a})$. We therefore will denote an element of FS (and thus FS_χ) by $\mathbf{f} = (\mathbf{f}_\mathfrak{a})_{\mathfrak{a}\in\tilde{C}_F}$: a collection of functions $\mathbf{f}_\mathfrak{a} \in \mathrm{Fun}_\chi(X_\mathfrak{a})$, one for each ideal class representative

$\mathfrak{a}$. Note that, given $\mathbf{f} \in \mathrm{FS}$, one can speak of $\mathbf{f}_{\mathfrak{a}} \in \mathrm{Fun}_{\chi}(X_{\mathfrak{a}})$ for any fractional ideal $\mathfrak{a}$, i.e. $\mathfrak{a}$ does not have to belong to the set of representatives $\tilde{C}_F$.

We equip each $\mathrm{Fun}_{\chi}(X_{\mathfrak{a}})$ with the L^2-norm corresponding to the chosen measure on $\mathrm{PGL}_2(F_\infty)$. Correspondingly one has an L^2 norm on FS and FS_χ.

We fix bases $\mathbf{B}(\mathrm{FS}_\chi), \mathbf{B}(\mathrm{FS})$ for FS_χ and FS, respectively, that are orthonormal, and consist of eigenforms for the Hecke algebra as well as invariant differential operators. In the case of FS, we also require them to be eigenfunctions of the $Z(\mathbb{A}_F)$-action. In particular, each $\mathbf{f} \in \mathbf{B}(\mathrm{FS})$ or $\mathbf{f} \in \mathbf{B}(\mathrm{FS}_\chi)$ "transforms under" a certain representation $\pi(\nu_{\mathbf{f}})$ of $\mathrm{GL}_2(F_\infty)$, for some $\nu_{\mathbf{f}} \in Y$.

The center $Z(\mathbb{A}_F)$ acts on $\mathrm{FS} = \oplus_{\mathfrak{a} \in \tilde{C}_F} \mathrm{Fun}_\chi(X_{\mathfrak{a}})$. One can identify FS_χ as the subspace of FS on which $Z(\mathbb{A}_F)$ acts by χ. For the derivation of the PGL_2-Bruggeman-Miatello formula, it will be necessary to explicate this action; see the Appendix for details.

2.4. **Eisenstein Series.** We must also discuss the continuous part of the spectrum. In the context of this paper, Eisenstein series form the "easy" part of the spectrum. We will generally only sketch the role they play, since their treatment is straightforward but (it seems to the author) they would add too much length and notation to an already burdened paper!

In our setting, it will be preferable to index the the continuous spectrum using adelic data, rather than in terms of cusps, since we will eventually want to understand their relation to Grössencharacter in any case.

We briefly reprise standard facts to fix notation. Given Grossencharacters ω_1, ω_2 of $\mathbb{A}_F^\times / F^\times$, let $I(\omega_1, \omega_2, s_1, s_2)$ be the representation of $\mathrm{GL}_2(\mathbb{A}_F)$ obtained by (normalized) parabolic induction of the character of the diagonal torus $\mathbb{A}_F^\times \times \mathbb{A}_F^\times$ given by $\mathrm{diag}(x, y) \to \omega_1(x)|x|_{\mathbb{A}}^{s_1} \omega_2(y)|y|_{\mathbb{A}}^{s_2}$ Then then one has an intertwiner

$$E(\omega_1, \omega_2, s_1, s_2) : I(\omega_1, \omega_2, s_1, s_2) \xrightarrow{E} \mathcal{A}(\mathrm{GL}_2)$$

which is defined for $\Re(s_1 - s_2) \gg 0$ as an absolutely convergent series and by a process of meromorphic continuation in general. It does not have any poles on $\Re(s_1 - s_2) = 0$.

Now set $I(\omega_1, \omega_2, s_1, s_2)^{K_\infty \times (K_0(\mathfrak{I}), \chi)}$ to be the (finite dimensional) space of vectors in $I(\omega_1, \omega_2, s_1, s_2)$ that are K_∞-fixed and transform under $K_0(\mathfrak{I})$ by χ_f. Denoting by $\mathrm{cond}(\omega)$ the conductor of ω, one notes that this space is nonempty only if $\mathrm{cond}(\omega_1) \cdot \mathrm{cond}(\omega_2)$ divides $\mathfrak{I}$. In any case, we may form the map, for $t \in \mathbb{R}$:

$$E(\omega, \chi\omega^{-1}, it, -it) : I(\omega, \chi\omega^{-1}, it, -it)^{K_\infty \times (K_0(\mathfrak{I}), \chi)} \to \mathcal{A}(\mathrm{GL}_2) \tag{10}$$

The left-hand space is a finite dimensional vector space; it is equipped with a natural inner product, unique up to scaling (since it lies inside a unitary irreducible $\mathrm{GL}_2(\mathbb{A}_F)$-representation).

For each ω, then, we fix a unitary basis $\{f_j\}$ for $I(\omega, \chi\omega^{-1}, 0, 0)^{K_\infty \times (K_0(\mathfrak{I}), \chi)}$. Now our Eisenstein series are parameterized by data $\mu = (\omega, f_j)$, where ω varies over a set of representatives for Grössencharacters of $\mathbb{A}_F^\times / F^\times$ *modulo twists by* $|\cdot|_{\mathbb{A}}^s$; and, for each such ω, f_j is an element of the chosen basis for $I(\omega, \chi\omega^{-1}, 0, 0)^{K_\infty \times (K_0(\mathfrak{I}), \chi)}$. For each such μ we have a family of Eisenstein series $E(\mu, it)$ for $t \in \mathbb{R}$, arising from (10). These functions $E(\mu, it)$ on $\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)$ transform under $Z(\mathbb{A}_F)$ by χ , under $K_0(\mathfrak{I})$ by χ_f and are K_∞-fixed.

Suppose $\mu = (\omega, f_j)$ is as above. We denote by $\nu_\mu \in \mathfrak{a}$ the infinity type of $E(\mu, 0)$, i.e. so that $\pi(\nu_\mu)$ is the archimedean component of the representation $I(\omega, \chi\omega^{-1}, 0, 0)$; then $E(\mu, it)$ transforms under $\pi(\nu_\mu + t\rho)$. Here ρ is the element $(\deg(v))_{v|\infty} \in \mathfrak{a}$, where, as before, $\deg(v) = 1, 2$ according to whether v is real or complex.

As one varies over all μ, these parameterize the continuous part of the spectrum of FS_χ. Note that, if $F \neq \mathbb{Q}$, there will be infinitely many μ that occur.

2.5. **Fourier Coefficients: Definition and Normalizations.** We must fix, once and for all, a spherical Whittaker vector corresponding to each $\pi(\nu)$, for $\nu \in Y$ (see (2.2) for the definition), with respect to the character $n(x) \to \psi_\infty(x)$, $x \in F_\infty$. We fix an assignment $\nu \mapsto W_\nu$, sending each $\nu \in \mathfrak{a}$ for which $\pi(\nu)$ is generic, to a spherical vector in the Whittaker model of $\pi(\nu)$. We use the following (essentially following [2]):

$$W_\nu(\mathrm{diag}(x, y)) = \prod_{v\,\mathrm{real}} |x_v y_v^{-1}|^{1/2} K_{i\nu_v}(2\pi |x_v y_v^{-1}|) \cdot \prod_{v\,\mathrm{complex}} |x_v y_v^{-1}|^{1/2} K_{i\nu_v}(4\pi |x_v y_v^{-1}|)$$

Here K_{it} is the K-Bessel function.

This assignment being fixed, let $\mathbf{f} \in \mathrm{FS}$ be an eigenfunction, transforming under the spherical representation $\pi(\nu_{\mathbf{f}})$ of $\mathrm{GL}_2(F_\infty)$. The Fourier coefficients will be indexed by a pair consisting of an ideal class and an element of F. Let $\mathfrak{a}$ be a fractional ideal, $\alpha \in F^\times$; as we have seen we have an element $\mathbf{f}_{\mathfrak{a}} \in \mathrm{Fun}_\chi(X_{\mathfrak{a}})$ associated to $\mathbf{f}$. We define the normalized Fourier coefficients $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ so that

$$\mathbf{f}_{\mathfrak{a}}(g_\infty) = \sum_{\alpha \in F^\times} \frac{a_{\mathbf{f}}(\mathfrak{a}, \alpha)}{\sqrt{\mathrm{Norm}(\alpha \mathfrak{a} \mathfrak{d})}} W_{\nu_{\mathbf{f}}}(\mathrm{diag}(\alpha, 1) g_\infty) \tag{11}$$

$a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ vanishes unless $\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. Observe the denominator $\sqrt{\mathrm{Norm}(\alpha\mathfrak{a}\mathfrak{d})}$; this is included so that $a_{\mathbf{f}}(\mathfrak{a}, \alpha)$ remains of comparable size as $\alpha, \mathfrak{a}$ vary.

In an identical way one can define the Fourier coefficients of Eisenstein series, first removing the constant term; correspondingly, we denote by $a_{E(\mu, it)}(\mathfrak{a}, \alpha)$ the $(\mathfrak{a}, \alpha)$-Fourier coefficient of $E(\mu, it)$.

We observe that $a_{\mathbf{f}}(\mathfrak{a},\alpha)$ only depends on the ideal $\mathfrak{a}\cdot(\alpha)$. If $\mathfrak{m}=\mathfrak{a}\mathfrak{d}(\alpha)$, where $\mathfrak{d}$ is the different, we will therefore set:

$$a_{\mathbf{f}}(\mathfrak{m}) = a_{\mathbf{f}}(\mathfrak{a},\alpha) \tag{12}$$

We will, however, continue to use the notation $a_{\mathbf{f}}(\mathfrak{a},\alpha)$: the reason is that, although it depends only on the ideal $\mathfrak{a}(\alpha)$, some of our intermediate constructions (like the notion of Kloosterman sum) will depend on a particular choice of generator α. Thus, although the final result will depend only on $\mathfrak{a}(\alpha)$, it is convenient to keep track of a chosen generator until a certain stage in our working.

Relation between Fourier coefficients and Hecke eigenvalues: One defines Hecke operators in the usual fashion; we do not explicate it here. We will only deal with them at places prime to the conductor.

We denote $\lambda_{\mathbf{f}}(\mathfrak{m})$ the $\mathfrak{m}$th Hecke eigenvalue of $\mathbf{f}$, if $\mathbf{f}$ is a Hecke eigenform; we normalize in such a way that the Ramanujan conjecture corresponds to $|\lambda_{\mathbf{f}}(\mathfrak{p})| \leq 2$, if $\mathfrak{p}$ is a prime ideal. We have, as usual,

Lemma 1. *Suppose $\mathbf{f}$ is a Hecke eigenform. There is a constant $C_{\mathbf{f}}$ so that, for any ideal $\mathfrak{m}$ prime to $\mathfrak{I}$ one has:*

$$C_{\mathbf{f}}\lambda_{\mathbf{f}}(\mathfrak{m}) = a_{\mathbf{f}}(\mathfrak{m})$$

2.6. **Kuznetsov-Bruggeman-Miatello formulas for PGL_2.** We must first discuss Kuznetsov's formulas for $\mathrm{PGL}_2(F_\infty)$ over a number field. Such a formula for SL_2 was given by R. Bruggeman and R. Miatello – see, for instance, [2]. We will use their formula, with minor modifications to account for the fact that we deal with PGL_2 (and we have a Nebentypus). These modifications are outlined in the Appendix.

We must first define the Kloosterman sum. Our definition differs from that of [2]; by necessity, we must include ideal classes as parameters. We first make a preliminary definition.

Definition 1. *Let $\mathfrak{p},\mathfrak{q}$ be fractional ideals of F with $\mathfrak{p}|\mathfrak{q}$. We set $(\mathfrak{p}/\mathfrak{q})^\times$ to be those elements $x\in\mathfrak{p}/\mathfrak{q}$ which generate $\mathfrak{p}/\mathfrak{q}$ as an $\mathfrak{o}_F$-module. For $x\in(\mathfrak{p}/\mathfrak{q})^\times$, we define x^{-1} to be the unique class $y\in(\mathfrak{p}^{-1}/\mathfrak{q}\mathfrak{p}^{-2})^\times$ so that $xy\in 1+\mathfrak{q}\mathfrak{p}^{-1}$.*

Suppose that $\mathfrak{p},\mathfrak{q}$ are as above, and χ is a character of $\mathbb{A}_F^\times/F^\times$ with conductor dividing $\mathfrak{q}\mathfrak{p}^{-1}$. Then χ induces in a natural way a function $(\mathfrak{p}/\mathfrak{q})^\times\to\mathbb{C}$. (This is a generalization of the fact that a character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$ with conductor dividing m induces a character of $(\mathbb{Z}/m\mathbb{Z})^\times$. Indeed, given $x\in(\mathfrak{p}/\mathfrak{q})^\times$, let $\tilde{x}\in\mathfrak{p}$ be a representative for x. We can choose for each finite place v an element $z_v\in F_v$ so that $v(z_v)=v(\mathfrak{p})$ and $v(z_v-\tilde{x})\geq v(\mathfrak{q})$. Then set $\chi(x)\equiv\prod_{v\,\mathrm{finite}}\chi(z_v)$; it is easily checked this is independent of the choices.)

In addition to the arguments $(\mathfrak{a}_1,\alpha_1)$, $(\mathfrak{a}_2,\alpha_2)$, the *modulus* of the Kloosterman sum also has an ideal class parameter. This parameter will be denoted by $\mathfrak{c}$ and is restricted by the fact that its square must belong to the same ideal class as $\mathfrak{a}_1\mathfrak{a}_2$.

Definition 2. *(Kloosterman sum) Let $\mathfrak{a}_1, \mathfrak{a}_2$ be fractional ideals of F, and $\mathfrak{c}$ be any ideal so that $\mathfrak{c}^2 \sim \mathfrak{a}_1\mathfrak{a}_2$. Say $c \in \mathfrak{c}^{-1}\mathfrak{I}$, $\alpha_1 \in \mathfrak{a}_1^{-1}\mathfrak{d}^{-1}$, and $\alpha_2 \in \mathfrak{a}_1\mathfrak{d}^{-1}\mathfrak{c}^{-2}$. We define the* Kloosterman sum*:*

$$KS(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) = \sum_{x \in (\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1(c))^\times} \psi_\infty(\frac{\alpha_1 x + \alpha_2 x^{-1}}{c})\overline{\chi(x)}$$

Here $(\mathfrak{a}_1\mathfrak{c}^{-1}/\mathfrak{a}_1(c))^\times$ and $x^{-1} \in (\mathfrak{a}_1^{-1}\mathfrak{c}/\mathfrak{a}_1^{-1}(c)\mathfrak{c}^2)^\times$ are defined as in Definition 1; and $\chi(x)$ is as noted after Definition 1.

The parameters of this Kloosterman sum should be thought of as the *ideals* $\mathfrak{a}_1\alpha_1$ and $\mathfrak{a}_2\alpha_2$, and the "modulus" should be thought of as the *ideal* $c\mathfrak{c}$. However, KS does depend on the choice of generator; it is not invariant under the substitution $\alpha \to \alpha\epsilon$, if ϵ is a unit. To relate the definition to the usual Kloosterman sum, we give the following Example:

Example 1. Suppose $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathfrak{c} = \mathfrak{o}_F$. Then for $\alpha_1, \alpha_2 \in \mathfrak{d}^{-1}$ and $c \in \mathfrak{I}$:

$$KS(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c}) = \sum_{x \in (\mathfrak{o}/c)^\times} \psi_\infty(\frac{\alpha_1 x + \alpha_2 x^{-1}}{c})\overline{\chi(x)}$$

One also has the "Weil bound for twisted Kloosterman sums:"

(13)
$$|KS(\alpha_1, \mathfrak{a}_1; \alpha_2, \mathfrak{a}_2; c, \mathfrak{c})| \ll \mathrm{Norm}(\alpha_1\mathfrak{a}_1\mathfrak{d}, \alpha_2\mathfrak{c}^2\mathfrak{a}_1^{-1}\mathfrak{d}, c\mathfrak{c})^{1/2}\mathrm{Norm}(\mathfrak{c}c)^{1/2+\epsilon}$$

where the brackets $(\cdot, \cdot, \cdot)$ denote greatest common divisor (of ideals).

Finally, we need to define an appropriate space of test functions and an integral kernel relating the "spectral" and "geometric" test functions.

Definition 3. *(Space of test functions and measures) Let $M > 2, N > 6$. We set $\mathcal{H}(M, N)$ to be the space of functions $h : \mathfrak{a} \to \mathbb{C}$ that are of the following form:*

$$h(\nu_1, \dots, \nu_{d_\mathbb{R}+d_\mathbb{C}}) = \prod_i h_i(\nu_i)$$

where each $h_i : \mathbb{R} \to \mathbb{C}$ extends to an even holomorphic function on the strip $\{z : |\Im(z)| \le M\}$ such that, for each $|\sigma| < M$, we have uniformly:

$$|h(i\sigma + t)| \ll e^{-\pi|t|}(1 + |t|)^{-N} \tag{14}$$

For our applications we are free to choose M and N large; this simplifies some considerations of convergence. Indeed we will always assume that $M > 2, N > 6$ to guarantee convergence in the formula, but many of our other constructions will converge only "for sufficiently large M and N."

Definition 4. *(Integral kernel) We define $\mathcal{B}(x,\nu) : F_\infty \times \mathfrak{a}_\mathbb{C} \to \mathbb{C}$ as follows: for $x = (x_v) \in F_\infty \equiv \mathbb{R}^{d_\mathbb{R}} \times \mathbb{C}^{d_\mathbb{C}}$ and $\nu \in \mathfrak{a}_\mathbb{C} \equiv \mathbb{C}^{d_\mathbb{R}+d_\mathbb{C}}$,*

$$\mathcal{B}(x,\nu) = \prod_v B_v(x_v,\nu_v)$$

where

$$B_v(x_v,\nu_v) = \begin{cases} \frac{1}{\sin(\pi\nu)}(J_{-2\nu}(4\pi\sqrt{x_v}) - J_{2\nu}(4\pi\sqrt{x_v})), & v \text{ real and } x_v > 0 \\ \frac{1}{\sin(\pi\nu)}(I_{-2\nu}(4\pi\sqrt{|x_v|}) - I_{2\nu}(4\pi\sqrt{|x_v|})), & v \text{ real and } x_v < 0 \\ \frac{1}{\sin(\pi\nu)}(I_{-\nu}(4\pi\sqrt{x_v})I_{-\nu}(4\pi\overline{\sqrt{x_v}}) - I_\nu(4\pi\sqrt{x_v})I_\nu(4\pi\overline{\sqrt{x_v}})), & \text{else.} \end{cases}$$

Proposition 1. *(Bruggeman-Miatello formula for $\mathrm{GL}(2)$) Define the "Kronecker-delta" for ideals:*

$$\Delta(\mathfrak{a}_1\alpha_1;\mathfrak{a}_2.\alpha_2) = \begin{cases} 1, & \mathfrak{a}_1(\alpha_1) = \mathfrak{a}_2(\alpha_2) \\ 0, & \text{otherwise} \end{cases}$$

Let h be a test function on $\mathfrak{a}_\mathbb{C}$ belonging to $\mathcal{H}(M,N)$ (defined in Definition 3) and define $\varphi : F_\infty^\times \to \mathbb{C}$ in terms of h via

$$\varphi(x) = \int_{\nu\in\mathfrak{a}} h(\nu)\mathcal{B}(x,i\nu)d\mu_\nu$$

where $\mathcal{B}(x,\nu)$ is defined in Definition 4. Let $\mathbf{B}(\mathrm{FS}_\chi)$ be an orthonormal basis for the discrete spectrum of FS_χ, so that each $f \in \mathbf{B}(\mathrm{FS}_\chi)$ has eigenvalue $\nu_\mathbf{f} \in Y$, i.e. transforms under the $\mathrm{GL}_2(F_\infty)$-representation $\pi(\nu_\mathbf{f})$. Let $\mathfrak{a}_1, \mathfrak{a}_2$ be fractional ideals, and $\alpha_1 \in \mathfrak{a}_1^{-1}\mathfrak{d}^{-1}, \alpha_2 \in \mathfrak{a}_2^{-1}\mathfrak{d}^{-1}$. Then, if h_F and D_F are the class number and discriminant of F, respectively,

$$\mathrm{CSC} + \sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS}_\chi)} h(\nu_\mathbf{f})a_\mathbf{f}(\mathfrak{a}_1,\alpha_1)\overline{a_\mathbf{f}(\mathfrak{a}_2,\alpha_2)} = \tag{15}$$

$$c_1\Delta(\mathfrak{a}_1\alpha_1;\mathfrak{a}_2\alpha_2)\int_{\nu\in\mathfrak{a}} h(\nu)d\mu_\nu +$$

$$c_2 \sum_{\mathfrak{c}:\mathfrak{c}^2\sim\mathfrak{a}_1\mathfrak{a}_2} \sum_{\epsilon\in\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2} \sum_{c\in\mathfrak{c}^{-1}\mathfrak{J}} \varphi\left(\frac{\epsilon\alpha_1\alpha_2}{c^2[\frac{\mathfrak{c}^2}{\mathfrak{a}_1\mathfrak{a}_2}]}\right)\frac{KS(\alpha_1,\mathfrak{a}_1;\epsilon\alpha_2[\frac{\mathfrak{c}^2}{\mathfrak{a}_1\mathfrak{a}_2}]^{-1},\mathfrak{a}_2;c,\mathfrak{c})}{\mathrm{Norm}(c\mathfrak{c})}$$

where $[\frac{\mathfrak{c}^2}{\mathfrak{a}_1\mathfrak{a}_2}]$ is the chosen generator for this fractional ideal, see 2.1; and c_1, c_2 are constants given as:

$$c_1 = \frac{2^{d_\mathbb{R}+2d_\mathbb{C}}}{\pi^{d_\mathbb{R}+d_\mathbb{C}}}\frac{D_F^{1/2}}{h_F} c_2 = \frac{1}{2}\cdot 2^{d_\mathbb{R}+3d_\mathbb{C}}\pi^{d_\mathbb{C}}\frac{1}{h_F} \tag{16}$$

and CSC is the contribution of the continuous spectrum, given explicitly in the notation of Section 2.4 by

$$CSC = \sum_\mu \int_{t=-\infty}^{\infty} h(\nu_\mu + t\rho)a_{E(\mu,it)}(\mathfrak{a}_1,\alpha_1)\overline{a_{E(\mu,it)}(\mathfrak{a}_2,\alpha_2)}dt$$

Remark 2. This closely resembles the "usual" Kuznetsov formula, but there are two additional summations – over $\mathfrak{c}$ and over ϵ. These extra factors account for the translation between SL(2) and PGL(2). They are both finite sums and do not affect convergence considerations.

We also see that both the KS and φ term depend on $\alpha_1, \alpha_2, \mathfrak{a}_1, \mathfrak{a}_2$, but when the averaging over ϵ is performed the result depends only on the *ideals* $\mathfrak{a}_1 \cdot (\alpha_1)$ and $\mathfrak{a}_2 \cdot (\alpha_2)$.

Remark 3. If the ideal class of $\mathfrak{a}_1 \cdot \mathfrak{a}_2$ is not a square in C_F, the sum is *zero*. This is as it should be: let $\widehat{C_F}$ be the group of class group characters of F, and $\widehat{C_F}(2)$ the 2-torsion in this group. Then $\widehat{C_F}(2)$ acts on automorphic forms of conductor $\mathfrak{I}$ and fixed central character (by twisting), and summing over $\widehat{C_F}(2)$-orbits shows that the sum (15) should indeed vanish if $\mathfrak{a}_1 \cdot \mathfrak{a}_2$ is not a square ideal class.

3. Classification of Dihedral Forms: Special Case

We begin by sketching the method of classifying dihedral forms, over $\mathbb{Q}$. The finer details – involving behavior of the exponential sums, convergence, and a careful interpretation of the answer – we do not treat carefully, leaving it for the next Section. The purpose of this section is purely expository: to convey the main idea in a special case.

Suppose that $p \equiv 1 \bmod 4$ is a prime number, χ is the quadratic character mod p. Let $\mathcal{B}$ be an orthonormal basis for Maass forms on the upper half plane of level p and Nebentypus χ; for $f \in \mathcal{B}$, let t_f be so that the Laplacian eigenvalue of f is $1/4 + t_f^2$, and let $a_n(f)$ be the nth Fourier coefficient of f, appropriately normalized. Finally, choose $m \neq n$, and assume $m, n > 0$. The Kuznetsov formula takes the shape:

$$\sum_f h(t_f) a_n(f) \overline{a_m(f)} + \mathrm{CSC} = \sum_{p|c} \frac{1}{c} \varphi\left(\frac{4\pi\sqrt{nm}}{c}\right) KS(n,m;c) \tag{17}$$

where CSC is an Eisenstein term, h is an appropriate test function, φ is a certain Bessel transform of h, and $KS(n,m;c) = \sum_{x \in (\mathbb{Z}/c)^\times} \overline{\chi}(x) e(\frac{nx+mx^{-1}}{c})$. (Note that the normalizations for φ here are slightly different to that in Proposition 1, owing to differences in normalization between the standard Kuznetsov formula over $\mathbb{Q}$ – that we have used above – and the Bruggeman-Miatello formula that Proposition 1 is based on. We have followed in (17) the standard normalization for φ "over $\mathbb{Q}$"; the φ of (17) would look like $\varphi(x^2)$ in the notation of Proposition 1.)

Let g be a C^∞ function on $(0, \infty)$, compactly supported and with integral 1, and assume $m > 0$ is not a square. We now form:

$$\Sigma(X) = \sum_f h(t_f) \left(\sum_{n=1}^{\infty} g(n/X) a_{n^2}(f)\right) \overline{a_m(f)} + \mathrm{CSC} \tag{18}$$

As $X \to \infty$, the dominant contribution to this is from f such that $\sum_n g(n/X)a_{n^2}(f)$ is large – i.e., those forms such that $L(s, \mathrm{Sym}^2 f)$ has a pole, or the dihedral forms. We use (17) to evaluate $\Sigma(X)$. Inverting the $\sum_n$ and the $\sum_f$ and applying (17), we obtain:

$$\Sigma(X) = \sum_{n=1}^{\infty} \sum_{p|c} g(n/X)\varphi(\frac{4\pi\sqrt{n^2 m}}{c})\frac{KS(n^2, m; c)}{c} \tag{19}$$

We now switch the order of summation. Now $n \mapsto KS(n^2, m; c)$ is periodic in n, with period c; we thus split the n-sum based on the congruence class of n mod c. Thus:

$$\Sigma(X) = \sum_{p|c} \sum_{x \in \mathbb{Z}/c} \frac{KS(x^2, m; c)}{c} \sum_{n \equiv x \bmod c} g(n/X)\varphi(\frac{4\pi n\sqrt{m}}{c})$$

As usual set $e(x) = e^{2\pi i x}$. We apply Poisson summation to the inner n-sum, obtaining:

(20)

$$\Sigma(X) = \sum_{p|c} \sum_{x \in \mathbb{Z}/c} \frac{KS(x^2, m; c)}{c^2} \sum_{\nu \in \mathbb{Z}} e(-\frac{x\nu}{c}) \left(\int_0^{\infty} \varphi(\frac{4\pi u\sqrt{m}}{c}) g(u/X) e(\frac{\nu u}{c}) du \right)$$

Let $S(\nu, m, c) = \sum_{x \in \mathbb{Z}/c} KS(x^2, m, c) e(-x\nu/c)$. In terms of this,

$$\Sigma(X) = \sum_{\nu \in \mathbb{Z}} \sum_{p|c} \frac{S(\nu, m, c)}{c} \left(\int_0^{\infty} \varphi(4\pi\sqrt{m}u) g(uc/X) e(\nu u) du \right)$$

$$= \sum_{\nu \in \mathbb{Z}} \int_0^{\infty} \varphi(4\pi\sqrt{m}u) \left(\sum_{p|c} \frac{S(\nu, m, c)}{c} g(uc/X) \right) e(\nu u) du \tag{21}$$

The sum $\sum_{p|c} \frac{S(\nu,m,c)}{c} g(uc/X)$ expresses behavior of $S(\nu, m, c)$ *on average with respect to* c. Now $S(\nu, m, c)$ can essentially be evaluated in closed form. As it turns out, the sum over c exhibits cancellation except for certain special values of ν: those ν for which $\nu^2 - 4m = pY^2$, for some integer Y. Therefore, only those ν contribute "significantly" to $\Sigma(X)$ as $X \to \infty$. Quantifying this argument, we will obtain:

$$\lim_{X \to \infty} \frac{\Sigma(X)}{X} = C \sum_{\nu : \nu^2 - 4m = pY^2} \int_0^{\infty} \varphi(4\pi\sqrt{m}u) e(\nu u) \frac{du}{u} \tag{22}$$

for a certain arithmetic constant C.

This demonstrates that the behavior of $\Sigma(X)$, as $X \to \infty$, is controlled by units for the quadratic field $\mathbb{Q}(\sqrt{p})$, and hence related to Grossencharacters of $\mathbb{Q}(\sqrt{p})$. By examining the relevant integral transforms, we can convert (22) back into a statement in terms of h; when interpreted correctly, it gives the classification of dihedral forms.

We now carry out this argument, with details, over a number field. The main points of the argument that are not apparent are those immediately after (21): a justification for the interchanging of various sums, integrals and limits; a very precise analysis of the behavior of the exponential sum $S(\nu, m, c)$; and the archimedean computation relating h and φ. These are somewhat subtle issues. The most difficult is the behavior of $S(\nu, m, c)$, for which it is important to understand many "ramified" cases (e.g. Gauss sums modulo primes ramified at 2).

4. The Classification of Dihedral Forms: General Case

4.1. **Introductory Remarks.** We now carry out the limiting procedure sketched in the introduction, in the setting of a totally real number field. The only serious obstacle in the general case (involving complex places) is the validity of the a certain integral transform (proved over $\mathbb{R}$ in Section 6.7); the author does not yet know how to achieve this in the complex case. Except for this, the method, as will be clear, does not rely on this assumption in any important way; the obstacle should thus be seen as purely local.

We refer to the introduction for the motivation for this and the meaning of "dihedral."

Over a number field, the method is extremely notationally complicated, and it may be difficult to extract the ideas from the equations. We therefore recommend the reader examine Section 3, where we have described the main ideas over $\mathbb{Q}$, prior to reading this one.

The final evaluation of the limit is given in Proposition 2, Section 4.3. It is converted to a more pleasant form in Theorem 1, Section 4.7.

4.2. **Notation.** Let $\mathfrak{m}$ be an integral ideal of F. Let χ be a quadratic[1] Grossencharacter of F, unramified at ∞; it determines by class field theory a quadratic extension K of F. K is therefore a totally real extension of the totally real field F. Let $\mathfrak{D}_{K/F}$ be the relative discriminant of K over F; equivalently, the conductor of χ. We fix an ideal $\mathfrak{I}$ divisible by $\mathfrak{D}_{K/F}$, and set $\mathfrak{f}$ to be the smallest (i.e. most divisible) ideal for which $\mathfrak{D}_{K/F}\mathfrak{f}^2|\mathfrak{I}$. Set $\mathfrak{o}_{K,\mathfrak{I}}$ to be the order of K given by $\mathfrak{o}_F + \mathfrak{f}\mathfrak{o}_K$; it has relative discriminant $\mathfrak{D}_{K/F}\mathfrak{f}^2$. Let $\mathfrak{o}_{K,\mathfrak{I}}^*$ be the units of $\mathfrak{o}_{K,\mathfrak{I}}$.

Let $\zeta_F(s)$ be the Dedekind zeta-function of F. We also will need the following constant.

$$\Gamma = \frac{\mathrm{Res}_{s=1}\zeta_F(s)\sqrt{\mathrm{Norm}(\mathfrak{D}_{K/F}\mathfrak{f}^2)}}{\zeta_F(2)\mathrm{Norm}(\mathfrak{I})\prod_{\mathfrak{p}|\mathfrak{I}}(1+\mathrm{Norm}(\mathfrak{p})^{-1})} \tag{23}$$

These notions are fixed throughout this section.

[1]It is easy to see from the definitions that dihedral forms do not exist unless χ is quadratic, i.e. associated to a quadratic field extension; it also follows by the technique that we follow. Therefore, there is no loss of generality in this assumption.

4.3. **Statement of result; overview of proof.** Let g be a C^∞ function, compactly supported on $(0,\infty)$ and with integral 1. Its role is strictly auxiliary: it is used for smoothing. Let $\mathfrak{m}$ be an integral ideal of F. Let $h \in \mathcal{H}(M,N)$, and φ be the integral transform defined in Proposition 1.

Define $\Sigma(X)$ as follows:

(24)

$$\Sigma(X) = \sum_{\mathbf{f}\in \mathbf{B}(\mathrm{FS}_\chi)} h(\nu_{\mathbf{f}}) \left(\sum_{\mathfrak{j}\,\mathrm{integral}\,\mathrm{F-ideal}} a_{\mathbf{f}}(\mathfrak{j}^2) g(\mathrm{Norm}(\mathfrak{j})/X) \right) \overline{a_{\mathbf{f}}(\mathfrak{m})} + \mathrm{CSC}$$

Here CSC is the "continuous spectrum contribution"; we do not make it explicit here. We discuss it and how to evaluate it in Section 4.9 (see, in particular, (69)).

We state immediately the main result of the Section. It will be proved by a direct limiting process in a Kuznetsov-type formula, and will imply the classification of dihedral forms.

Proposition 2. *Define $\Sigma(X)$ as in (24), and assume $h \in \mathcal{H}(M,N)$ with M and N sufficiently large. The limit $\lim_{X\to\infty} \frac{\Sigma(X)}{X}$ exists, and equals:*

(25)

$$\lim_{X\to\infty} \frac{\Sigma(X)}{X} = c_2 \Gamma D_F^{-1/2} \sum_{\mathfrak{b}:\mathfrak{b}^2\sim\mathfrak{m}} \sum_{x\in X_{\mathfrak{b},\mathfrak{m}}} \sum_{\delta\in\mathfrak{o}_{K,\mathfrak{J}}^*/\mathfrak{o}_F^*} \widehat{h\cdot\underline{\cosh}}(\log_{K/F}(\frac{x^2\delta^2}{\mathrm{Norm}(x)}))$$

Here c_2 is as defined in (16), Γ is as defined in (23), and $\mathfrak{b}$ ranges over a set of representatives for ideal classes of F with square $\mathfrak{m}$.

$\widehat{h\cdot\underline{\cosh}}$ is the Fourier transform of $\nu \to h(\nu)\underline{\cosh}(\nu)$, given via:

$$\widehat{h\cdot\underline{\cosh}}(k) = \int_{\mathfrak{a}} h(t)\underline{\cosh}(t)e^{2\pi i k\cdot t}dt \tag{26}$$

where $k\cdot t = \sum_{v|\infty} k_v t_v$ for $k,t\in\mathfrak{a}$ and $\underline{\cosh}$ is as in (6). The function $\log_{K/F} : K_\infty^\times/F_\infty^\times \to \mathfrak{a}$ was defined in Section 2.2. Finally, for $\mathfrak{b}$ a fractional ideal of F,

$$X_{\mathfrak{b},\mathfrak{m}} = \{x\in\mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{J}} : (\mathrm{Norm}(x)) = \mathfrak{m}\mathfrak{b}^{-2}\}/\mathfrak{o}_{K,\mathfrak{J}}^* \tag{27}$$

The relationship of $\Sigma(X)$ to units (and so also Grossencharacters) of the quadratic extension K is already apparent from the δ-sum in (25).

Note that we will establish convergence only for M and N sufficiently large; this assumption provides convenient decay estimates.

On account of the notational complexity, we provide a summary of the role of each of the subsections that follow in the overall picture.

(1) (In Section 4.4) We switch the $\mathbf{f}$ and $\mathfrak{j}$ sums in (24), and apply the PGL_2-Kuznetsov-Bruggeman-Miatello formula. This transforms the sum above into a sum of the form (c.f. (19)):

$$\sum_{\mathfrak{j}} \left(\sum_c \text{Geometric Contribution} \right) \tag{28}$$

(2) (In Section 4.4) Poisson summation. Here, we apply Poisson summation to the $\mathfrak{j}$ sum in (28) above. (This is a little misleading: we do not use the multiplicative structure of ideals $\mathfrak{j}$, but rather the additive structure of F!)

After this stage, we see that the behavior as $X \to \infty$ is governed by certain exponential sums. (These are the number-field versions of $S(\nu, m, c)$ defined prior to (21)).

(3) (In Section 4.5 and 4.6) Analysis of local exponential sums. It is here that units of K enter into the analysis. They correspond to certain parameters for which the exponential sums *do not* exhibit cancellation (e.g. over $\mathbb{Q}$, we have seen they emerge from the solutions to a Pell equation, c.f. (22)).

In Section 4.5, we reduce the analysis of these local exponential sums to a purely local question, and in Section 4.6, which is entirely self-contained, we resolve this local question.

(4) Archimedean theory: abstractly, this is the analogue of the "local exponential sums" at the archimedean place. Practically, it involves a computation of the integral transforms that relate the function h and the function φ above. The relevant computation is contained in the Appendix, and only quoted in the main text.

(5) Conclusion: Theorem 1 in Section 4.7.

4.4. **Application of Kuznetsov-Bruggeman-Miatello sum formula; Poisson summation.** *Reading suggestions for this section:* The notation in this section is unavoidably very involved. In the course of what follows, there are many variables which intervene. The symbols $\mu, \mathfrak{b}$ are fixed once and for all. We suggest that the reader focus as much as possible on the dependence in variables α, c and (later) $\mathfrak{q}$ and ν– the other variables are essentially indexing, and do not affect the central point.

There are also a number of delicate convergence issues. We have dealt in reasonable detail with the most difficult point, the interchange of the ν- and $\mathfrak{q}$- sums; see Lemmas 2, 3 and 4. The other issues are dealt with in a similar way and we do not carefully spell out each case.

We begin our analysis of (24). The sum over $\mathfrak{j}$ can be thought as a sum over $\mathfrak{a}(\alpha)$, where $\mathfrak{a}$ ranges over a system of ideal classes and (for a fixed $\mathfrak{a}$) α ranges over $\mathfrak{a}^{-1}/\mathfrak{o}_F^*$. We may also assume (see Remark 3) that there is $\mathfrak{b}$ and μ such that $\mathfrak{m} = (\mu)\mathfrak{b}^2$, for some $\mu \in \mathfrak{b}^{-2}$. Then, utilizing (12),

$$\Sigma(X) = \mathrm{CSC}+ \left(\sum_{\mathbf{f} \in \mathbf{B}(\mathrm{FS}_X)} h(\nu_{\mathbf{f}}) \sum_{\substack{\mathfrak{a} \in \tilde{C}_F \\ \alpha \in \mathfrak{a}^{-1}/\mathfrak{o}_F^*}} g(\mathrm{Norm}(\alpha\mathfrak{a})/X) a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{a}^2, \alpha^2) \overline{a_{\mathbf{f}}(\mathfrak{d}^{-1}\mathfrak{b}^2, \mu)} \right) \tag{29}$$

Recall that $\mathfrak{d}$ was the different. The inner summation is finite, on account of the compact support of g, the sums are absolutely convergent for N sufficiently large. We now invert the $\sum_{\mathfrak{a},\alpha}$ and $\sum_{\mathfrak{f}}$, and apply the GL(2)-Bruggeman-Miatello formula (Proposition 1). Note that ideals $\mathfrak{c}$ whose square is in the ideal class of $\mathfrak{d}^{-1}\mathfrak{a}^2 \cdot \mathfrak{d}^{-1}\mathfrak{b}^2$ are all of the form $\mathfrak{d}^{-1}\mathfrak{a}\mathfrak{b} \cdot \mathfrak{z}$, where $\mathfrak{z}$ is a fractional ideal so that $\mathfrak{z}^2$ is principal. We obtain:

$$\Sigma(X) = c_1 \sum_{\substack{\mathfrak{a}\in\tilde{C}_F \\ \alpha\in\mathfrak{a}^{-1}/\mathfrak{o}_F^*}} g(\mathrm{Norm}(\alpha\mathfrak{a})/X)\Delta(\mathfrak{a}^2\alpha^2;\mathfrak{b}^2\mu)\int_{\nu\in\mathfrak{a}} h(\nu)d\mu_\nu + \tag{30}$$

$$c_2 \sum_{\substack{\mathfrak{a}\in\tilde{C}_F \\ \alpha\in\mathfrak{a}^{-1}/\mathfrak{o}_F^*}} g(\mathrm{Norm}(\alpha\mathfrak{a})/X) \sum_{\substack{\mathfrak{z}\in\tilde{C}_F(2) \\ \epsilon\in\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2}} \sum_{c\in\mathfrak{z}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}\mathfrak{I}} \varphi(\frac{\mu\epsilon\alpha^2}{[\mathfrak{z}^2]c^2})$$

$$\cdot \frac{1}{\mathrm{Norm}(c\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1})} KS(\alpha^2, \mathfrak{d}^{-1}\mathfrak{a}^2; \mu\epsilon[\mathfrak{z}^2]^{-1}, \mathfrak{d}^{-1}\mathfrak{b}^2; c, \mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1})$$

The sums in (30) are absolutely convergent so long as $M > 2$; here M is so that $h \in \mathcal{H}(M,N)$.

We will be interested only in the limit $\lim_{X\to\infty} \frac{\Sigma(X)}{X}$; the term of (30) involving $\Delta(\mathfrak{a}^2\alpha^2;\mathfrak{b}^2\mu)$ cannot contribute to this. Indeed it vanishes for X sufficiently large, on account of the compact support of g. We therefore denote by $\Sigma'(X)$ the second term of (30); then

$$\lim_{X\to\infty} \frac{\Sigma(X)}{X} = \lim_{X\to\infty} \frac{\Sigma'(X)}{X} \tag{31}$$

and we can safely confine ourselves to considering $\Sigma'(X)$.

For notational simplicity, we will henceforth suppress the ranges of summation in the variables $\mathfrak{z}, \mathfrak{a}, \epsilon$. The symbol $\mathfrak{z}$ will always range over $\tilde{C}_F(2)$ (representatives for 2-torsion in the class group); $\mathfrak{a}$ always ranges over $\tilde{C}_F$; ϵ always ranges over $\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2$. It should be noted that all three of these sums are finite and do not affect convergence.

Our eventual aim is to execute Poisson summation in the α-variable; first we must unfold it from a sum over $\mathfrak{a}^{-1}/\mathfrak{o}_F^*$ to a sum over $\mathfrak{a}^{-1}$. This is done by correspondingly "folding" the c-sum into a sum over ideals: namely, we set $\mathfrak{q} = c\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}$; thus $\mathfrak{q}$ ranges over ideals in the ideal class of $\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}$, divisible by $\mathfrak{I}$. This "folding" is justifiable by absolute convergence. For such $\mathfrak{q}$, we will then denote $c_{\mathfrak{q}} = [\mathfrak{q}\mathfrak{z}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}]$, the chosen generator of $\mathfrak{q}\mathfrak{z}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}$ (see Section 2.1). We may replace the c-sum by a $\mathfrak{q}$-sum while unfolding the α sum to $\mathfrak{a}^{-1}$.

$$\Sigma'(X) = c_2 \sum_{\mathfrak{z},\mathfrak{a},\epsilon} \sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \sum_{\alpha\in\mathfrak{a}^{-1}} g(\mathrm{Norm}(\alpha\mathfrak{a})/X)\varphi(\frac{\mu\epsilon\alpha^2}{[\mathfrak{z}]^2c_{\mathfrak{q}}^2})\frac{1}{\mathrm{Norm}(\mathfrak{q})}$$

$$\cdot KS(\alpha^2, \mathfrak{d}^{-1}\mathfrak{a}^2; \mu\epsilon[\mathfrak{z}^2]^{-1}, \mathfrak{d}^{-1}\mathfrak{b}^2; c_{\mathfrak{q}}, \mathfrak{q}c_{\mathfrak{q}}^{-1})$$

This is the analogue of Equation (19) over $\mathbb{Q}$; we now wish to mimic the passage from (19) to (20). One easily verifies that the KS term that occurs above is invariant under the substitution $\alpha \mapsto \alpha + \lambda$, if $\lambda \in \mathfrak{q}\mathfrak{a}^{-1}$. Thus we may split the sum above based on the residue class of α mod $\mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{q}$.

$$(32)\ \Sigma'(X) = c_2 \sum_{\mathfrak{z},\mathfrak{a},\epsilon} \sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{1}{\mathrm{Norm}(\mathfrak{q})} \sum_{x\in\mathfrak{a}^{-1}/\mathfrak{q}\mathfrak{a}^{-1}} KS(x^2, \mathfrak{d}^{-1}\mathfrak{a}^2; \dots) \left(\sum_{\substack{\alpha\in\mathfrak{a}^{-1} \\ \alpha\equiv x(\mathfrak{a}^{-1}\mathfrak{q})}} \varphi(\frac{\mu\epsilon\alpha^2}{[\mathfrak{z}^2]c_{\mathfrak{q}}^2}) g(\mathrm{Norm}(\alpha\mathfrak{a})/X) \right)$$

Now if T is a test function on F_∞ and $\hat{T}(z) = \int_{F_\infty} T(x)\psi_\infty(zx)dx$ is its Fourier transform, we have the Poisson summation formula (for any fractional ideal $\mathfrak{p}$)

$$\sum_{\lambda\in\mathfrak{p}} T(\lambda + x) = \frac{1}{\mathrm{vol}(F_\infty/\mathfrak{p})} \sum_{\nu\in\mathfrak{d}^{-1}\mathfrak{p}^{-1}} \hat{T}(\nu)\psi_\infty(x\nu)$$

Thus, setting $\hat{g\varphi}$ to be the Fourier transform of $x \mapsto g(\mathrm{Norm}(x\mathfrak{a})/X)\varphi(\frac{\mu\epsilon x^2}{[\mathfrak{z}^2]c_{\mathfrak{q}}^2})$, we apply Poisson summation over the fractional ideal $\mathfrak{a}^{-1}\mathfrak{q}$, and use $\mathrm{vol}(F_\infty/\mathfrak{q}\mathfrak{a}^{-1}) = D_F^{1/2}\mathrm{Norm}(\mathfrak{q})\mathrm{Norm}(\mathfrak{a})^{-1}$. Note that this application of Poisson summation must be justified, the convergence being unclear; as it follows, for M, N sufficiently large, by the same methods and estimates which we establish later (see Lemmas 2, 3 and 4), we omit the proof.

$$(33)\quad \Sigma'(X) = c_2 \sum_{\mathfrak{z},\mathfrak{a},\epsilon} \frac{\mathrm{Norm}(\mathfrak{a})}{D_F^{1/2}} \sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{1}{\mathrm{Norm}(\mathfrak{q})^2} \sum_{\nu\in\mathfrak{d}^{-1}\mathfrak{a}\mathfrak{q}^{-1}} \hat{g\varphi}(\nu) \left(\sum_{x\in\mathfrak{a}^{-1}/\mathfrak{q}\mathfrak{a}^{-1}} KS(\dots)\psi_\infty(-\nu x) \right)$$

It is now convenient to replace ν by $\nu c_{\mathfrak{q}}$; accordingly, we change the range from $\nu \in \mathfrak{d}^{-1}\mathfrak{a}\mathfrak{q}$ to $\nu \in \mathfrak{z}^{-1}\mathfrak{b}^{-1}$.

Now, the innermost sum of (33) – over x – is a finite exponential sum. We write it out in detail (after making the harmless substitution $x \mapsto -x$)

Definition 5. *Let $\mathfrak{z}, \mathfrak{a}, \mathfrak{b}$ be fractional ideals of F, so that $\mathfrak{z}^2 \sim \mathfrak{o}_F$. Let $\nu \in \mathfrak{z}^{-1}\mathfrak{b}^{-1}, \mu \in \mathfrak{b}^{-2}$ and $\mathfrak{q} \sim \mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}$ be so that $\mathfrak{I}|\mathfrak{q}$; then set*

$$(34)\quad S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu; \mathfrak{q}) = \sum_{x\in\mathfrak{a}^{-1}/\mathfrak{q}\mathfrak{a}^{-1}} \sum_{y\in(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{z}^{-1}/\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{z}^{-1}\mathfrak{q})^\times} \left(\psi_\infty(\frac{x^2y + \mu[\mathfrak{z}^2]^{-1}y^{-1}}{c_{\mathfrak{q}}})\chi(y)\psi_\infty(x\nu/c_{\mathfrak{q}}) \right)$$

where $c_{\mathfrak{q}} = [\mathfrak{q}\mathfrak{z}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}]$.

In terms of this:
(35)
$$\Sigma'(X) = c_2 \sum_{\mathfrak{z},\mathfrak{a},\epsilon} \frac{\mathrm{Norm}(\mathfrak{a})}{D_F^{1/2}} \sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{1}{\mathrm{Norm}(\mathfrak{q})^2} \sum_{\nu\in\mathfrak{z}^{-1}\mathfrak{b}^{-1}} \hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1}) S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu\epsilon;\mathfrak{q})$$

Lemmas 2, 3 and 4 will now verify necessary details of convergence. First, we bound derivatives of φ:

Lemma 2. *Suppose $h \in \mathcal{H}(M,N)$, and let $\mathcal{D}$ be any constant-coefficients differential operator on F_∞ – regarded as a real vector space – of degree m; suppose $m < 2M$. Then for $x \in F_\infty$*

$$|\mathcal{D}\varphi(x^2)| \ll \prod_v \min(|x|_v^{-1/2}, |x|_v^{2M-m}) \tag{36}$$

Proof. This follows from standard properties of Bessel functions, and we indicate only the method of argument in each case. To establish (36) in the case where $\mathcal{D}$ is trivial, one uses the holomorphic extension of h to shift the line of integration, and then uses estimates on Bessel functions derived from their power series expansions or defining integrals (c.f. the arguments in Section 6.7, especially (105)).

In the general case, one may differentiate within the integral sign in the differentiation of φ, use formulas for derivatives of Bessel functions, and shift contours. □

The function $x \to g(\mathrm{Norm}(x\mathfrak{a}))$ is constant along hyperbolas, which makes convergence details unclear. The next Lemma deals with this.

Lemma 3. *Let $k > 0$, and let f be a function on F_∞ so that $f(x) \ll \prod_{v|\infty} \min(|x|_v^k, |x_v|^{-1/2})$. Then*

$$\int_{x\in F_\infty} g(\mathrm{Norm}(x)/A)|f(x)|dx \ll_\epsilon A\min(A^{-1/2}, A^k)\max(A, A^{-1})^\epsilon$$

Proof. This may be verified by integrating along level surfaces of g (see also Lemma 8.1 of [2], where an analogous statement is proven for sums over units). □

Recall now the norm on F_∞: for $x \in F_\infty$ we set $||x||_{F_\infty} = \max_{v|\infty} |x|_v$.

Lemma 4. *(Convergence) Notations being as above, let $m < 2M-1$ be any positive integer, and set $K = \frac{m(2M-1)}{2M}$.*

$$\sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} |\hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})| \ll_\epsilon \begin{cases} X^{2+\epsilon}||\nu||_{F_\infty}^{-K+\epsilon}, & \nu \neq 0 \\ X^{2+\epsilon}, & \nu = 0 \end{cases} \tag{37}$$

In particular:

(1) *By choosing M large we can achieve K arbitrarily large;*

(2) *If $K > [F:\mathbb{Q}]$ the double sum over $\mathfrak{q}$ and ν in (35) converges absolutely – in particular, the ν and $\mathfrak{q}$ sum can be interchanged;*

(3) *If $\delta > \frac{1}{K-[F:\mathbb{Q}]}$, one can truncate the ν-sum of (35) to the range $||\nu||_{F_\infty} \leq X^{\delta}$ without affecting the computation of* $\lim_{X\to\infty} \Sigma'(X)X^{-1}$.

The bounds in this Lemma are not optimal, but they suffice for our purposes.

Proof. We emphasize that, since $\mathfrak{z}, \mathfrak{a}, \epsilon$ all range over finite sets, we can and will treat them as fixed for the purpose of establishing convergence in (35). In particular, $\mathrm{Norm}(\mathfrak{q}) \asymp \mathrm{Norm}(c_{\mathfrak{q}})$.

The "trivial bound" on $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}$ is that (see (34)) $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu;\mathfrak{q}) \leq \mathrm{Norm}(\mathfrak{q})^2$; using this, all assertions of the Lemma follow from (37).

To establish (37), we shall establish separate bounds in the "Norm($\mathfrak{q}$) is small" and "Norm($\mathfrak{q}$) is large" range. One has, by definition:

$$\widehat{g\varphi}(\nu c_{\mathfrak{q}}^{-1}) = \int_{F_\infty} g(\frac{\mathrm{Norm}(x\mathfrak{a})}{X})\varphi(\frac{x^2\mu\epsilon}{c_{\mathfrak{q}}^2[\mathfrak{z}^2]})\psi_\infty(x\nu/c_{\mathfrak{q}})dx \tag{38}$$

$$= \mathrm{Norm}(c_{\mathfrak{q}}) \int_{F_\infty} g(\frac{\mathrm{Norm}(x\mathfrak{a}c_{\mathfrak{q}})}{X})\varphi(x^2\mu\epsilon[\mathfrak{z}^2]^{-1})\psi_\infty(x\nu)dx$$

Using (36) with $m = 0$, together with Lemma 3, we see that

$$|\widehat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})| \ll X \cdot \min\left((\frac{\mathrm{Norm}(\mathfrak{q})}{X})^{1/2}, (\frac{X}{\mathrm{Norm}(\mathfrak{q})})^{2M}\right) \cdot \max(\frac{X}{\mathrm{Norm}(\mathfrak{q})}, \frac{\mathrm{Norm}(\mathfrak{q})}{X})^{\epsilon} \tag{39}$$

(Note that the definition of g forces the first integrand in (38) to vanish unless $\mathrm{Norm}(x) \asymp X$.)

If $\nu \neq 0$ and Norm($\mathfrak{q}$) is "small," one is better off bounding $\widehat{g\varphi}$ by integration by parts. Let $m < 2M - 1$ be an integer and let w be a place of F above ∞, corresponding to a coordinate on F_∞; we are assuming w is real. Note that the derivative of $g(\mathrm{Norm}(x)/X)$ "in the w direction" is essentially $\frac{1}{x_w}\frac{\mathrm{Norm}(x)}{X}g'(\frac{\mathrm{Norm}(x)}{X})$, and the function $\frac{\mathrm{Norm}(x)}{X}g'(\frac{\mathrm{Norm}(x)}{X})$ has the same general behavior as $g(\mathrm{Norm}(x)/X)$.

We integrate by parts m times in the direction of w (i.e. using the adjointness properties of an appropriate differential operator $\mathcal{D}^{(m)}$ of order m). Since the integrals are improper, the integration by parts requires justification, which can be given using the estimates from Lemma 2 together with (slight variants of) Lemma 3; we omit the details. In any case, using Lemma 2 and Lemma 3, we obtain:

$$|\widehat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})| = \mathrm{Norm}(c_{\mathfrak{q}}) \left| \int_{F_\infty} g(\frac{\mathrm{Norm}(x\mathfrak{a}c_{\mathfrak{q}})}{X})\varphi(x^2\mu\epsilon[\mathfrak{z}^2]^{-1})\psi_\infty(x\nu)dx \right|$$

$$\ll \mathrm{Norm}(\mathfrak{q})|\nu|_w^{-m} \int_{F_\infty} \left| \mathcal{D}^{(m)}(g(\frac{\mathrm{Norm}(x)}{X/\mathrm{Norm}(\mathfrak{a}c_{\mathfrak{q}})})\varphi(x^2\mu\epsilon[\mathfrak{z}^2]^{-1}) \right| dx$$

$$\ll_\epsilon |\nu|_w^{-m} X \max(\frac{X}{\mathrm{Norm}(\mathfrak{q})}, \frac{\mathrm{Norm}(\mathfrak{q})}{X})^{\epsilon} \tag{40}$$

Choosing w so as to maximize $|\nu|_w$, we see

$$|\hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})| \ll ||\nu||_{F_\infty}^{-m} X \max(\frac{X}{\mathrm{Norm}(\mathfrak{q})}, \frac{\mathrm{Norm}(\mathfrak{q})}{X})^{\epsilon} \tag{41}$$

In the case $\nu = 0$, (37) follows directly from (39). In the general case, we combine (39) and (41) – introducing a temporary variable T, we use the former for $\mathrm{Norm}(\mathfrak{q}) \geq XT$, and the latter otherwise. We will soon set T to be a fixed power of $||\nu||_{F_\infty}$; thus T^{ϵ} can be replaced by $||\nu||_{F_\infty}^{\epsilon}$. The number of $\mathfrak{q}$ with $\mathrm{Norm}(\mathfrak{q}) = n$ is $\ll_\epsilon n^{\epsilon}$.

One obtains thus:

$$\sum_{\mathfrak{q}} |\hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})|$$
$$\ll_\epsilon X^{1+\epsilon} ||\nu||_{F_\infty}^{\epsilon} (||\nu||_{F_\infty}^{-m} \sum_{\mathfrak{q}:\mathrm{Norm}(\mathfrak{q}) \leq XT} 1 + \sum_{\mathfrak{q}:\mathrm{Norm}(\mathfrak{q}) \geq XT} (\frac{X}{\mathrm{Norm}(\mathfrak{q})})^{2M})$$
$$\ll X^{1+\epsilon} ||\nu||_{F_\infty}^{\epsilon} (XT ||\nu||_{F_\infty}^{-m} + \frac{X}{T^{2M-1}})$$

To finish, we set $T = ||\nu||_{F_\infty}^{m/2M}$; this gives $|\hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1})| \ll X^{2+\epsilon} ||\nu||_{F_\infty}^{-(2M-1)m/2M+\epsilon}$. □

We are now free (for M, N sufficiently large) to interchange ν and $\mathfrak{q}$ summation in (35). We further wish to interchange the $\mathfrak{q}$ summation with the integral defining $\hat{g\varphi}$; this may be justified (for M, N sufficiently large) by similar but easier considerations to the above, and we will do so without comment.

Observe:

$$\hat{g\varphi}(\nu c_{\mathfrak{q}}^{-1}) = \int_{F_\infty} g(\frac{\mathrm{Norm}(x\mathfrak{a})}{X}) \varphi(\frac{x^2 \mu \epsilon}{c_{\mathfrak{q}}^2 [\mathfrak{z}^2]}) \psi_\infty(x\nu / c_{\mathfrak{q}}) dx$$
$$= \mathrm{Norm}(c_{\mathfrak{q}}) \int_{F_\infty} g(\frac{\mathrm{Norm}(x)\mathrm{Norm}(\mathfrak{q})}{X \mathrm{Norm}(\mathfrak{z}\mathfrak{b}) D_F^{-1}}) \varphi(\frac{\mu \epsilon x^2}{[\mathfrak{z}^2]}) \psi_\infty(x\nu) dx \tag{42}$$

where we have made the replacement $x \leftarrow x c_{\mathfrak{q}}$, and used the fact that $\mathrm{Norm}(\mathfrak{a} c_{\mathfrak{q}}) = \mathrm{Norm}(\mathfrak{q}\mathfrak{z}^{-1}\mathfrak{b}^{-1}\mathfrak{d})$. Using (42) in combination with this fact, and interchanging $\mathfrak{q}$-sum and integral as remarked above, we may now rewrite (35) as

$$\Sigma'(X) = c_2 \sum_{\mathfrak{z}, \mathfrak{a}, \epsilon} \frac{D_F^{1/2}}{\mathrm{Norm}(\mathfrak{z}\mathfrak{b})} \sum_{\nu \in \mathfrak{z}^{-1}\mathfrak{b}^{-1}} \int_{F_\infty} dx \, \varphi(\frac{\mu \epsilon x^2}{[\mathfrak{z}^2]}) \psi_\infty(x\nu)$$
$$\cdot \left(\sum_{\substack{\mathfrak{I} | \mathfrak{q} \\ \mathfrak{q} \sim \mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} g(\mathrm{Norm}(\mathfrak{q}) \frac{\mathrm{Norm}(x) D_F}{\mathrm{Norm}(\mathfrak{z}\mathfrak{b}) X}) \frac{S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu\epsilon, \mathfrak{q})}{\mathrm{Norm}(\mathfrak{q})} \right) \tag{43}$$

The inner sum expresses the behavior of the local exponential sum $\mathfrak{q} \mapsto S(\cdot, \cdot, \mathfrak{q})$ *on average* with respect to $\mathfrak{q}$; c.f. (21) over $\mathbb{Q}$. Proposition 3 (stated and proved in the next section) analyzes this; it evaluates the asymptotic behavior of the $\mathfrak{a}$ and $\mathfrak{q}$-sums in (43). We apply it with $Y = \frac{X\mathrm{Norm}(\mathfrak{z}\mathfrak{b})}{xD_F}$.

In fact, we are now in a position to compute $\lim_{X\to\infty} \frac{\Sigma'(X)}{X}$; the formal computation is justified using Lemma 4 (showing that, for M, N sufficiently large, the ν-sum of (43) may be truncated to the range $||\nu||_{F_\infty} \le X^\delta$ without affecting the computation of $\lim_{X\to\infty} \frac{\Sigma'(X)}{X}$), the error term provided by Proposition 3, and the decay estimates on φ from Lemma 2. (All this assuming M, N are sufficiently large!) We obtain:

$$\lim_{X\to\infty} X^{-1}\Sigma'(X) = c_2\Gamma D_F^{-1/2} \sum_{\substack{\mathfrak{z},\epsilon \\ \nu\in\mathfrak{z}^{-1}\mathfrak{b}^{-1}}} \delta_{\mathfrak{z}}(\nu^2 - 4\mu\epsilon[\mathfrak{z}^2]^{-1}) \cdot \int_{F_\infty} \varphi(\frac{\mu\epsilon x^2}{[\mathfrak{z}^2]})\psi_\infty(x\nu)\frac{dx}{\mathrm{Norm}(x)} \tag{44}$$

Here $\delta_{\mathfrak{z}}$ is as defined in Proposition 3, Section 4.5. To finish, we will interpret $\delta_{\mathfrak{z}}$ in terms of $\mathfrak{o}_{K,\mathfrak{Z}}$ in the following Lemma; this is analogous (over $\mathbb{Q}$) to identifying solutions to a Pell equation with units in a quadratic order. The proof uses standard facts about orders and is omitted.

Lemma 5. *Let $\mathfrak{o}_{K,\mathfrak{Z}}$ be the $\mathfrak{o}_F$-order of K defined in Section 4.2. Let $\mathfrak{z} \in \tilde{C}_F(2), \mathfrak{b}$ be fractional ideals of F and let $\mu \in \mathfrak{b}^{-2}$; for $t \in F$, define $\delta_{\mathfrak{z}}(t)$ as in (45). Then let*

$$\mathfrak{S} = \amalg_{\mathfrak{z}\in\tilde{C}_F(2)}\{x \in K : x \in \mathfrak{z}^{-1}\mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{Z}}, \mathrm{Norm}_{K/F}(x) = \mu[\mathfrak{z}^2]^{-1}\}$$
$$\mathfrak{S}' = \{\nu \in \mathfrak{z}^{-1}\mathfrak{b}^{-1} : \delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \neq 0\}$$

Then there is a map $\mathfrak{S} \to \mathfrak{S}'$, given by $x \mapsto \mathrm{tr}_{K/F}(x)$. It is surjective, and the fiber above ν has size 2 unless $\nu^2 = 4\mu[\mathfrak{z}^2]$, in which case it has size 1.

Finally, Proposition 7 (from the Appendix) expresses the relationship between φ and h in a form suitable for application to (44). The statement of that Proposition is expressed in such a way so that it may generalize to complex places, but for our application it may be more transparent to directly apply (106) and (112). In any case, using Proposition 7 and Lemma 5, we obtain after some manipulation:

$$\lim_{X\to\infty} \frac{\Sigma'(X)}{X} = \frac{1}{2}c_2\Gamma D_F^{-1/2} \sum_{\substack{\mathfrak{z}\in\tilde{C}_F(2) \\ \epsilon\in(\mathfrak{o}_F^*)/(\mathfrak{o}_F^*)^2}} \sum_{\substack{x\in\mathfrak{z}^{-1}\mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{Z}} \\ \mathrm{Norm}_{K/F}(x)=\mu\epsilon/[\mathfrak{z}^2]}} \widehat{h\cdot\underline{\cosh}}(\log_{K/F}\left(\frac{x^2}{\mu\epsilon/[\mathfrak{z}^2]}\right))$$

Here $\widehat{h\cdot\underline{\cosh}}$ is as in (26). Now, observe that $\mathfrak{z}$ ranges over (representatives for) 2-torsion in the class group, while $\mathfrak{b}$ was chosen so $\mathfrak{b}^2 \sim \mathfrak{m}$; thus $\mathfrak{b}\cdot\mathfrak{z}$ ranges over ideal classes with square the ideal class of $\mathfrak{m}$. Finally, recall from (31) that $\lim_X \Sigma'(X)/X = \lim_X \Sigma(X)/X$.

One now easily deduces Proposition 2 (in translating to a statement about $X_{\mathfrak{b},\mathfrak{m}}$, one loses the factor of $\frac{1}{2}$ present above.)

4.5. **The local exponential sums I: reduction to local question.** We now analyze the local exponential sums. It is essential for our technique that they be understood at *all* places – not merely "unramified" ones. Over the course of this section and the next, we will prove the following:

Proposition 3. *Let notations be as in Definition 5; in particular, define* $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu,\mathfrak{q})$ *according to (34). Let* g *be, as before a* C^∞ *function of compact support on* $(0,\infty)$ *with* $\int_0^\infty g(x)dx = 1$*. Recalling the definition of* Γ *from (23), we set for* $t \in F$*:*

$$\delta_{\mathfrak{z}}(t) = \Gamma \begin{cases} 1, & F(\sqrt{t}) \equiv K, t \in (\mathfrak{D}_{K/F}\mathfrak{f}^2)\mathfrak{z}^{-2}\mathfrak{b}^{-2}, \\ \frac{1}{2}, & t = 0 \\ 0, & else. \end{cases} \tag{45}$$

There is $A > 0$ *so that*

$$\sum_{\mathfrak{a}\in\tilde{C}_F} \sum_{\substack{\mathfrak{J}|\mathfrak{q} \\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu,\mathfrak{q})}{\mathrm{Norm}(\mathfrak{q})} g(\mathrm{Norm}(\mathfrak{q})/Y) = \delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1})Y + O_\epsilon((1+||\nu||_{F_\infty})^A Y^{1/2+\epsilon}) \tag{46}$$

Here the implicit constant in the O *depends on* $\mathfrak{a},\mathfrak{b},\mathfrak{z},\mu$.

Proof. This will be proved in the forthcoming sections. In particular, Proposition 4 (next section) expresses the meromorphic properties of the "zeta-function" attached to $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu,\mathfrak{q})$. The present result follows from that one, by standard methods of integrating this zeta-function against the Mellin transform of g. $\square$

Compare also the conclusion of the Proposition to the statement after (21) about the average behavior of $S(\nu,m,c)$. Very roughly speaking, the condition $\delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \neq 0$ picks out those ν which are solutions to certain Pell-type equations: the requirement $F(\sqrt{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}}) = K$ puts $\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}$ in a particular square class.

The error term in this Proposition is important – or, it is important to have a power saving in the Y variable. The difficulty of this proposition, as one would expect, lies in dealing with ramified places. (There is also a subtle issue which amounts to quadratic reciprocity over F; it is neatly dealt with by means of ϵ-factors for Hecke L-functions.)

The reader may wish to glance first at the unramified evaluation, sketched after Lemma 6.

4.5.1. *Interpretation of* $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}$ *as an adelic integral.* The purpose of this section is to reduce the Proposition to a question of evaluating integrals over a local field. The main work will be done in Section 4.6.

In order to make the clear the reduction to a purely local question clear, we express $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}$ as an integral over a subset of the adeles. For any fractional ideal $\mathfrak{J}$, let $\overline{\mathfrak{J}}$ be the closure of $\mathfrak{J}$ embedded (diagonally) in $\mathbb{A}_{F,f}$, and let $\overline{\mathfrak{J}}^\times =$

$\{x \in \mathbb{A}_{F,f}^{\times} : v(x) = v(\mathfrak{J})$ for all finite places $v\}$. These are both naturally principal homogeneous spaces for compact groups ($\widehat{\mathfrak{o}}$ and $\widehat{\mathfrak{o}}^{\times}$ respectively, in the notation of 2.1) and as such are equipped with a unique invariant measure class; we denote by $d^{(1)}$ that measure that has total mass 1. By abuse of notation, we use $d^{(1)}$ both for the measure on $\overline{\mathfrak{J}}$ and $\overline{\mathfrak{J}^{\times}}$. On the other hand, the x- and y- sums in (34) range over sets of size $\mathrm{Norm}(\mathfrak{q})$ and $\mathrm{Norm}(\mathfrak{q}) \prod_{\mathfrak{p}|\mathfrak{q}}(1 - \mathrm{Norm}(\mathfrak{p})^{-1})$, respectively.

The definitions then show that:

$$S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q}) = \mathrm{Norm}(\mathfrak{q})^2 \prod_{\mathfrak{p}|\mathfrak{q}} (1 - \mathrm{Norm}(\mathfrak{p})^{-1}) \int_{\overline{\mathfrak{a}^{-1}}} d^{(1)}x \cdot \int_{\overline{\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{z}^{-1}}^{\times}} d^{(1)}y \; \psi_f(-\frac{x^2 y + \mu[\mathfrak{z}^2]^{-1} y^{-1} + x\nu}{c_{\mathfrak{q}}}) \chi_f(y) \tag{47}$$

The $-$ sign in front of the ψ arises from the fact that for $z \in F$, one has $\psi_{\mathbb{A}}(z) = \psi_{\infty}(z)\psi_f(z) = 1$, thus $\psi_{\infty}(z) = \psi_f(z)^{-1}$. This sign can be removed by making the substitutions $x \leftarrow -x, y \leftarrow -y$. Recalling now that $\mathfrak{q} = c_{\mathfrak{q}}\mathfrak{a}\mathfrak{b}\mathfrak{z}\mathfrak{d}^{-1}$, and make first the substitution $x \leftarrow xy^{-1}$, and then the substitution $y \leftarrow yc_{\mathfrak{q}}$. We obtain:

$$S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q}) = \mathrm{Norm}(\mathfrak{q})^2 \prod_{\mathfrak{p}|\mathfrak{q}} (1 - \mathrm{Norm}(\mathfrak{p})^{-1}) \int_{\overline{\mathfrak{z}^{-1}\mathfrak{b}^{-1}}} \int_{\overline{(\mathfrak{q}\mathfrak{d}\mathfrak{b}^{-2}\mathfrak{z}^{-2})}^{\times}} \psi_f(y^{-1}(x^2 + x\nu + \mu[\mathfrak{z}^2]^{-1}))\chi_f(y) \; d^{(1)}x \; d^{(1)}y \tag{48}$$

Finally, recalling the notation $\pi_{\mathfrak{b}}, \pi_{\mathfrak{z}}$ from Section 2.1, we make the substitutions $x \leftarrow x\pi_{\mathfrak{z}}\pi_{\mathfrak{b}}, y \leftarrow y\pi_{\mathfrak{b}}^2\pi_{\mathfrak{z}}^2$.

$$S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q}) = \mathrm{Norm}(\mathfrak{q})^2 \prod_{\mathfrak{p}|\mathfrak{q}} (1 - \mathrm{Norm}(\mathfrak{p})^{-1}) \int_{\overline{\mathfrak{o}_F}} \int_{\overline{\mathfrak{q}\mathfrak{d}}^{\times}} \psi_f(y^{-1}(x^2 + (\nu\pi_{\mathfrak{b}}\pi_{\mathfrak{z}})x + (\mu\pi_{\mathfrak{z}}^2[\mathfrak{z}^2]^{-1}\pi_{\mathfrak{b}}^2))\chi_f(y) d^{(1)}x d^{(1)}y \tag{49}$$

It is now clear that it may be factored $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q})$ may be factored into a product of local integrals.

4.5.2. *The zeta-function associated to $S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}$.* We now construct the zeta-function associated to the function on ideals given by $\mathfrak{q} \mapsto S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q})$; once this is done, the only remaining questions will be purely local. We will deduce Proposition 3 from properties of this zeta-function.

Proposition 4. *Let notations be as in Definition 5. The series*

$$Z(s) = \sum_{\mathfrak{a} \in \tilde{C}_F} \sum_{\substack{\mathfrak{I}|\mathfrak{q} \\ \mathfrak{q} \sim \mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu, \mu, \mathfrak{q})}{\mathrm{Norm}(\mathfrak{q})^{s+1}}$$

defines a meromorphic function of s. It is analytic for $\Re(s) > 1/2$, with the exception of a possible pole at $s = 1$. The residue of this pole is

$$\mathrm{Res}_{s=1} Z(s) = \delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \tag{50}$$

where $\delta_{\mathfrak{z}}$ is defined in Proposition 3. (In particular, if $\delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) = 0$, then $Z(s)$ is holomorphic for $\Re(s) > 1/2$.)

The other data ($\mathfrak{a}, \mathfrak{b}, \mathfrak{z}, \mu$) being fixed, $Z(s)$ grows slowly in vertical strips, with polynomial uniformity in ν; that is, for $\sigma > 1/2$ there are constants $A(\sigma), B(\sigma)$ so that:

$$|Z(\sigma + it)| \ll (1 + ||\nu||_{F_\infty})^{A(\sigma)} (1 + |t|)^{B(\sigma)} \tag{51}$$

Proof. This follows from two results stated below: Lemma 6 below (giving the factorization $Z(s) = \prod_v Z_v(s)$ of $Z(s)$ into an Euler product) and Proposition 5 (evaluating the local factors).

Indeed, let ω be the Grössencharacter of F associated to the quadratic field extension $F(\sqrt{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}})$ (set $\omega = 1$ if this is split). It is necessary to treat $\omega \neq 1$ and $\omega = 1$ separately; we deal here with the former, the latter being similar. Let $L_F(s, \omega)$ be the Hecke L-function of F associated to the Grossencharacter ω, and $\zeta_F(s)$ the Dedekind L-function.

Lemma 6 and Proposition 5 shows that $Z(s)$ differs from $L_F(s, \omega\chi)/\zeta_F(2s)$ by only a finite number of Euler factors. It is clear that (except for $s = 1$) $L_F(s, \omega\chi)/\zeta_F(2s)$ is holomorphic for $\Re(s) > 1/2$, has growth properties akin to (51), and has a pole at $s = 1$ if and only if $\omega\chi = 1$, i.e. if $\omega = \chi$.

To deduce the holomorphicity in $\Re(s) > 1/2, s \neq 1$, and the growth (51) of $Z(s)$, one needs some mild information about the factors of $Z(s)$ at bad places, namely: they are holomorphic for $\Re(s) > 1/2$ and they satisfy a growth property of the type (51). This may be established by the same techniques we use in the next Section, Section 4.6 (in that section, we compute precisely $Z_v(1)$ at *all* places under the assumption that $\omega = \chi$, but the same techniques give information about $Z_v(s)$ in general.) In fact, if $\nu^2 - 4\mu[\mathfrak{z}^2]^{-1} \neq 0$, each factor of $Z_v(s)$ is a finite Dirichlet series; one must treat the case $\nu^2 - 4\mu[\mathfrak{z}^2]^{-1} = 0$ separately.

We next turn to the possible pole of $Z(s)$ at $s = 1$. In this case, $\omega = \chi$, so $F(\sqrt{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}}) \equiv K$, which verifies the first condition of (45) for $\delta_{\mathfrak{z}}(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \neq 0$.

Since $Z(s) = \prod_v Z_v(s)$ agrees at almost all places with an (understood) quotient of L-functions, the residue at $s = 1$ of $Z(s)$ may be computed if we know $Z_v(1)$ at all places. For this one may use the second part of Proposition 5. This gives (50).

Note that, in the notation of Proposition 5, $\prod_{v \text{ finite}} \epsilon_v(\omega_v, \psi_v) = 1$; this is "quadratic reciprocity for F," and can be verified by comparing the ϵ-factors for $\zeta_K(s)$ and $\zeta_F(s)$. (In particular, this uses the fact that, since $\chi|_{F_\infty^\times} \equiv 1$, all infinite places of F split in K). $\square$

Lemma 6. *Let notations be as in Definition 5; introduce ideles $\pi_{\mathfrak{z}}, \pi_{\mathfrak{b}} \in \mathbb{A}_{F,f}^\times$ as in Section 2.1.*

Let v be a finite place of F; let q_v be the cardinality of the corresponding residue field. Let $(\pi_\mathfrak{z}\pi_\mathfrak{b}\nu)_v \in F_v^\times$ denote the component at v of the idele $\pi_\mathfrak{z}\pi_\mathfrak{b}\nu$, and similarly define $(\pi_\mathfrak{b}^2\mu)_v$. Denote by dx and $d^\times y$ the measures on F_v and $F_v^\times$ that assign the mass 1 to $\mathfrak{o}_v$ and $1-q_v^{-1}$ to $\mathfrak{o}_v^\times$, respectively.

For r a positive integer, set:

(52)

$$Z_{v,r} = \int_{x\in\mathfrak{o}_v}\int_{v(y)=v(\mathfrak{d})+r} \psi_v(y^{-1}(x^2+x(\pi_\mathfrak{z}\pi_\mathfrak{b}\nu)_v+(\mu[\mathfrak{z}^2]^{-1}\pi_\mathfrak{z}^2\pi_\mathfrak{b}^2)_v))\chi_v(y)dxd^\times y$$

whereas for $r=0$, set

$$Z_{v,0} = (1-q_v^{-1})^{-1}\int_{v(y)=v(\mathfrak{d})} \chi_v(y)d^\times y$$

and define the local factor

$$Z_v(s) = \sum_{r\geq v(\mathfrak{I})} Z_{v,r} q_v^{r(1-s)} \tag{53}$$

Then

$$\sum_{\mathfrak{a}\in\tilde{C}_F}\sum_{\substack{\mathfrak{I}|\mathfrak{q}\\ \mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}}} \frac{S_{\mathfrak{a},\mathfrak{b},\mathfrak{z}}(\nu,\mu,\mathfrak{q})}{\mathrm{Norm}(\mathfrak{q})^{s+1}} = \prod_{v\text{ finite}} Z_v(s) \tag{54}$$

Proof. This is an easy consequence of (49). Note in particular that there is no explicit dependence of (49) on $\mathfrak{a}$, so the $\mathfrak{a}$-sum in (54) can be removed, taking with it the restriction $\mathfrak{q}\sim\mathfrak{z}\mathfrak{a}\mathfrak{b}\mathfrak{d}^{-1}$. The peculiar factor $(1-q_v^{-1})^{-1}$ above accounts for the factor $\prod_{\mathfrak{p}|\mathfrak{q}}(1-\mathrm{Norm}(\mathfrak{q})^{-1})$ and the change of measures between $d^{(1)}x, d^{(1)}y$ and $dx, d^\times y$. □

Unramified computation: We now briefly discuss the unramified computation in very concrete terms. The methods of Section 4.6 are more powerful and more general, but we sketch the direct unramified computation, since it is perhaps more revealing – the computation is that of two Gauss sums. Suppose that v is a finite place of F so that χ_v, ψ_v are not ramified, v does not divide 2, and so that $v(\mathfrak{z}) = v(\mathfrak{b}) = 0$. We finally assume that $v(\nu^2-4\mu[\mathfrak{z}^2]^{-1}) = 0$; this is true for almost all v unless $\nu^2-4\mu[\mathfrak{z}^2]^{-1}=0$, and that case can be handled separately.

Note that the assertions of Lemma 6 are independent of the choice of $\pi_\mathfrak{z}, \pi_\mathfrak{b}$, so we may assume $(\pi_\mathfrak{z})_v = (\pi_\mathfrak{b})_v = 1$. Let $\Pi_v \in \mathfrak{p}_v$ be a uniformizer. It follows from the definitions (52), (53) that $Z_{v,0}=1$ and for $r>0$

$$Z_{v,r} = \chi(\mathfrak{p}_v)^r q_v^{-2r}\sum_{x\in\mathfrak{o}_v/\mathfrak{p}_v^r}\sum_{y\in(\mathfrak{o}_v/\mathfrak{p}_v{}^r)^\times}\psi_v(\frac{y(x^2+\nu x+\mu[\mathfrak{z}^2]^{-1})}{\Pi_v^r}) \tag{55}$$

$$= \chi(\mathfrak{p}_v)^r q_v^{-2r}\sum_{y\in(\mathfrak{o}_v/\mathfrak{p}_v{}^r)^\times}\left(\sum_{x\in\mathfrak{o}_v/\mathfrak{p}_v^r}\psi_v(\frac{y(x+\nu/2)^2}{\Pi_v^r})\right)\psi_v(\frac{y(\mu[\mathfrak{z}^2]^{-1}-\nu^2/4)}{\Pi_v^r})$$

Here we have proceeded by completing the square. The inner sum is a Gauss sum, and after evaluating it the outer sum also becomes one.

Take the case $r=1$. Set $g=\sum_{x\in\mathfrak{o}_v/\mathfrak{p}_v}\psi_v(\frac{(x+\nu/2)^2}{\Pi_v})$. For $a\in(\mathfrak{o}_v/\mathfrak{p}_v)^\times$ set $\left(\frac{a}{\mathfrak{p}_v}\right)$ to be 1 or -1 according to whether a is a square or not. Then:

$$\begin{aligned}Z_{v,1}&=\chi(\mathfrak{p}_v)q_v^{-2}g\sum_{y\in(\mathfrak{o}_v/\mathfrak{p}_v)^\times}\left(\frac{y}{\mathfrak{p}_\mathfrak{v}}\right)\psi_v(\frac{y(\mu[\mathfrak{z}^2]^{-1}-\nu^2/4)}{\Pi_v})\\&=\chi(\mathfrak{p}_v)q_v^{-2}g^2\left(\frac{\mu[\mathfrak{z}^2]^{-1}-\nu^2/4}{\mathfrak{p}_v}\right)\end{aligned}$$

Since $g^2=q_v\left(\frac{-1}{\mathfrak{p}_v}\right)$, one obtains

$$Z_{v,1}=\chi(\mathfrak{p}_v)q_v^{-1}\left(\frac{\nu^2/4-\mu[\mathfrak{z}^2]^{-1}}{\mathfrak{p}_v}\right)=\chi(\mathfrak{p}_v)q_v^{-1}\left(\frac{\nu^2-4\mu[\mathfrak{z}^2]^{-1}}{\mathfrak{p}_v}\right)$$

Similar reasoning mod $\mathfrak{p}_v^r$, using the assumption $v(\nu^2-4\mu[\mathfrak{z}^2]^{-1})=0$, shows $Z_{v,r}=0$ for $r\geq 2$, whence

$$Z_v(s)=1+\chi(\mathfrak{p}_v)\left(\frac{\nu^2-4\mu[\mathfrak{z}^2]^{-1}}{\mathfrak{p}_\mathfrak{v}}\right)q_v^{-s}$$

We now return to the general study of $Z(s)$.

Definition 6. *For v a finite place of F, let $\epsilon(\psi_v,\chi_v)$ be the "local root number" at v associated to (ψ_v,χ_v) by Tate's thesis:*

$$\epsilon(\psi_v,\chi_v)=\begin{cases}\chi_v(c)^{-1}, & \chi_v \textit{ unramified}\\ \frac{\int_{v(x)=c}\chi_v(x)^{-1}\psi_v(x)d^\times x}{|\int_{v(x)=c}\chi_v(x)^{-1}\psi_v(x)d^\times x|}, & \chi \textit{ ramified}\end{cases}$$

Here $c\in F_v$ is an element of valuation $-f-d$, where f and d are the valuations of the conductor of χ_v and the (local) different of ψ_v, respectively.

Note in our normalization we have $|\epsilon(\psi_v,\chi_v)|=1$.

Proposition 5. *Fix notations as in Definition 5 and Lemma 6; recall the definition of $\mathfrak{f}$ from Section 4.2.*

Let $Z_v(s)$ be defined by (53). Let ω be the Grossencharacter of F associated to the quadratic extension $F(\sqrt{\nu^2-4\mu[\mathfrak{z}^2]^{-1}})$; set $\omega=1$ if $\nu^2-4\mu[\mathfrak{z}^2]^{-1}$ is a square of F. Let ω_v be the restriction of ω to F_v. Let $L_v(s,\omega_v)$ (respectively $\zeta_v(s)$) denote the local factor at v for the Hecke L-function attached to ω_v (respectively Dedekind ζ-function of F), and similarly for other Grössencharacters.

For all but finitely many v, *we have:*

$$Z_v(s)=\begin{cases}\frac{L_v(s,\omega_v\chi_v)}{\zeta_v(2s)},\nu^2-4\mu[\mathfrak{z}^2]^{-1}\neq 0\\ \frac{\zeta_v(2s-1)}{\zeta_v(2s)},\nu^2-4\mu[\mathfrak{z}^2]^{-1}=0\end{cases}$$

If $\chi_v = \omega_v$, we have additionally for all v:

$$Z_v(1) = \begin{cases} 0, & v(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) < v(\mathfrak{D}_{K/F}\mathfrak{f}^2\mathfrak{z}^{-2}\mathfrak{b}^{-2}), \\ \text{else} \begin{cases} \epsilon(\chi_v, \psi_v)(1 + q_v^{-1}), & v(\mathfrak{I}) = 0 \\ \epsilon(\chi_v, \psi_v) q_v^{-v(\mathfrak{I}) + v(\mathfrak{D}_{K/F}\mathfrak{f}^2)/2}, & v(\mathfrak{I}) = f_v > 0 \end{cases} \end{cases} \tag{56}$$

where $\epsilon(\psi_v, \chi_v)$ is as in Definition 6.

Proof. The assertion about *all but finitely many* is easy; when $\nu^2 - 4\mu[\mathfrak{z}^2]^{-1} \neq 0$ it was noted after Lemma 6 that:

$$Z_v(s) = 1 + \left(\frac{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}}{\mathfrak{q}_v}\right)\chi(\mathfrak{q}_\mathfrak{v}) q_v^{-s} = \frac{1 - q_v^{-2s}}{1 - \left(\frac{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}}{\mathfrak{q}_v}\right)\chi(\mathfrak{p}_v) q_v^{-s}}$$

which is just the assertion of the Proposition. The other case is similar.

As for the other assertion, we observe that the integrals that define $Z_v(1)$ are given by the following integrals (measures being as in Lemma 6) for a particular quadratic polynomial Q:

$$Z_v(1) = \begin{cases} \int_{x \in \mathfrak{o}_v} \int_{y \geq v(\mathfrak{d}_v) + v(\mathfrak{I})} \psi_v(y^{-1}Q(x))\chi_v(y)\, dx\, d^\times y, & v(\mathfrak{I}) \geq 1 \\ (1 - q_v^{-1})^{-1} \int_{y = v(\mathfrak{d}_v)} \chi_v(y)\, d^\times y + \\ \int_{x \in \mathfrak{o}_v} \int_{y \geq v(\mathfrak{d}_v) + 1} \psi_v(y^{-1}Q(x))\chi_v(y)\, dx\, d^\times y, & v(\mathfrak{I}) = 0 \end{cases}$$

By the assumption that $\omega_v = \chi_v$ we have that χ_v is the quadratic character of F_v associated by class field theory to $F_v(\sqrt{\mathrm{disc}(Q)})$.

After making the substitution $y \mapsto y^{-1}$, we see that this is the type of integral that is discussed and evaluated in the next section, i.e. (57). From Proposition 6, we deduce the claim.

This deduction is straightforward; we comment only on the origin of $\mathfrak{D}_{K/F}\mathfrak{f}^2$ in (56), which is not entirely clear from Proposition 6.

In the notation of Proposition 6 when applied here, $\mathrm{disc}(Q) = \nu^2 - 4\mu[\mathfrak{z}^2]^{-1}$. Proposition 6 thus shows that $Z_v(1)$ vanishes unless $v(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) + 1 \geq v(\mathfrak{I}\mathfrak{z}^{-2}\mathfrak{b}^{-2})$. On the other hand, $v(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \equiv v(\mathfrak{D}_{K/F}) \bmod 2$; this follows from a local discriminant computation with the fact that $F_v(\sqrt{\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}})$ is isomorphic to $K \otimes_F F_v$ (which in turn follows from $\chi_v = \omega_v$). In particular, observe that $v(\mathfrak{D}_{K/F}\mathfrak{f}^2)$ is the largest integer that is $\leq v(\mathfrak{I})$ *and* congruent to $v(\mathfrak{D}_{K/F}) \bmod 2$. Thus $v(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) + 1 \geq v(\mathfrak{I}\mathfrak{z}^{-2}\mathfrak{b}^{-2})$ if and only if $v(\nu^2 - 4\mu[\mathfrak{z}^2]^{-1}) \geq v(\mathfrak{D}_{K/F}\mathfrak{f}^2\mathfrak{z}^{-2}\mathfrak{b}^{-2})$, whence (56).

(Also observe that one deals with the case $v(\mathfrak{I}) = 0$ separately owing to the slightly odd definition of $Z_{v,0}$.) □

4.6. **The local exponential sums II.** This section is entirely self-contained, and contains the core of the p-adic harmonic analysis (in quite a concrete form!) It should be noted that at primes not above 2 this analysis is not difficult and can be carried out "by hand"; for residue characteristic 2 the evaluation becomes more difficult, and there is also a subtle global point that amounts to quadratic reciprocity. These issues are (to some extent) hidden in what follows.

Fix a finite place v of F. Let $\mathfrak{o}_v$ be the ring of local integers, and $\mathfrak{p}_v$ its maximal ideal. We will also use v for the valuation $v : F_v^\times \to \mathbb{Z}$.

Let $Q(X) = x^2 + Ax + B$ be a quadratic form, with $A, B \in \mathfrak{o}_v$, and let $\mathrm{disc}(Q) = A^2 - 4B \in \mathfrak{o}_v$ be the discriminant of Q. Let ψ_v be a nontrivial additive character of F_v, and d_v the valuation of its different – so that $\mathfrak{o}_v$ and $\mathfrak{p}^{-d_v}$ are dual under the pairing induced by ψ. Let χ_v be the quadratic character attached to the quadratic extension $F_v(\sqrt{A^2 - 4B})$, and let $\mathfrak{p}_v^{f_v}$ be its conductor; we define χ_v to be trivial if $A^2 - 4B$ is a square. Let dx be the Haar measure on $(F_v, +)$ that assigns mass 1 to $\mathfrak{o}_v$, and let $d^\times x = dx/|x|_v$ be that Haar measure on $F_v^\times$ assigning mass $1 - q_v^{-1}$ to $\mathfrak{o}_v^\times$. Finally let R be a positive integer. We assume $R \geq f_v$.

Proposition 6. *Measures and notations being as in the previous paragraph, set*

$$I = \int_{\mathfrak{o}_v} dx \int_{v(y) \leq -R - d_v} d^\times y \, \psi_v(yQ(x)) \chi_v(y) \tag{57}$$

Then I is convergent and vanishes unless $v(\mathrm{disc}(Q)) + 1 \geq R$. In that case, it is given by;

$$I = \begin{cases} \epsilon(\psi_v, \chi_v) q^{-R/2}, & R > 0, R \equiv f_v \pmod 2 \\ \epsilon(\psi_v, \chi_v) q^{-(R+1)/2}, & R > 0, R \equiv 1 + f_v \pmod 2 \end{cases}$$

Proof. Firstly, we note that $Q(x)$ is the norm of an element of $F_v(\sqrt{A^2 - 4B})$; thus $\chi_v(Q(x)) = 1$, and we may write

$$I = \int_{\mathfrak{o}_v} dx \int_{v(y) \leq -R - d_v} d^\times y \, \psi_v(yQ(x)) \chi_v(yQ(x)) \tag{58}$$

Set

$$\mathrm{Meas}_Q(\geq t) = \int_{x \in \mathfrak{o}_v : v(Q(x)) \geq t} dx$$

the mass of $x \in \mathfrak{o}_v$ for which $v(Q(x)) \geq t$. Then, from (58), we see that:

$$I = \sum_{s \in \mathbb{Z}} \mathrm{Meas}_Q(\geq s + R) \int_{v(y) = s - d_v} \psi_v(y) \chi_v(y) d^\times y$$

The inner integral is a Gauss sum, which is easily evaluated. It now remains to evaluate $\mathrm{Meas}_Q(\geq t)$ for each t. Using the two Lemmas that follow, we verify the truth of the Proposition; We will make implicit use of the fact that $v(\mathrm{disc}(Q)) \equiv f_v$ modulo 2; this follows easily from the fact that $\mathrm{disc}(Q)$ is the (local) discriminant of the ring $\mathfrak{o}_v[x]/Q(x)$, which is a subring of the maximal order in $F_v(\sqrt{A^2 - 4B})$.

The following Lemma is standard:

Lemma 7.

$$\int_{v(y)=r-d_v} \psi_v(y) d^\times y = \begin{cases} 1-q^{-1}, & r \geq 0 \\ -q^{-1}, & r=-1 \\ 0, & \text{otherwise} \end{cases}$$

Suppose χ_v is nontrivial and has conductor $\mathfrak{p}_v^{f_v}$, with $f_v \geq 1$.

$$\int_{v(y)=r-d_v} \psi_v(y)\chi_v(y) d^\times y = \begin{cases} q_v^{f_v/2}\epsilon(\psi_v, \chi_v), & r=-f_v \\ 0, & \text{otherwise} \end{cases}$$

Let $\alpha \in \mathbb{R}$. Define, as usual $\lceil \alpha \rceil$ to be the least integer n such that $n \geq \alpha$.

Lemma 8. *Let $t \geq 0$.*

If $\operatorname{disc}(Q)$ *is a square of* F_v*, then* $\operatorname{Meas}_Q(\geq t) = q_v^{-\lceil t/2 \rceil}$ *for* $t \leq v(\operatorname{disc}(Q))$*, and* $\operatorname{Meas}_Q(\geq t) = 2q_v^{\lceil t/2 \rceil}$ *otherwise.*

If $\operatorname{disc}(Q)$ *is not a square of* F_v*, let* f_v *be the local conductor of* χ_v*, and* e_v *the ramification degree of the quadratic extension corresponding to* χ_v*. Then* $\operatorname{Meas}_Q(\geq t) = q_v^{-\lceil t/2 \rceil}$ *for* $t \leq v(\operatorname{disc}(Q)) - f_v + (e_v - 1)$*, and* $\operatorname{Meas}_Q(\geq t) = 0$ *otherwise.*

Proof. We consider only the second case; the first is similar in technique.

Thus assume that $\operatorname{disc}(Q)$ is not a square; let $E = F_v(\sqrt{A^2 - 4B})$, associated by class field theory to χ_v. Let $\mathfrak{p}_v$ and $\mathfrak{p}_E$ be the maximal ideals of F_v and E respectively. Let e_v be the ramification degree. Then Q factors as $Q(x) = (x - \beta_1)(x - \beta_2)$, with $\beta_1, \beta_2 \in E$. We denote also by v the extension of the valuation v to E; it will take values in $e_v^{-1}\mathbb{Z}$. Then

$$v(Q(x)) = 2v(x - \beta_1) \tag{59}$$

Note that $\beta_1 \in \mathfrak{o}_v + \mathfrak{p}_v^l \mathfrak{o}_E$ if and only if the ring $\mathfrak{o}_v[\beta_1]$ is contained in $\mathfrak{o}_v + \mathfrak{p}_v^l \mathfrak{o}_E$. Comparing discriminants shows that this occurs exactly when $v(\operatorname{disc}(Q)) \geq f_v + 2l$.

If E is unramified over F_v, this suffices to determine $\max_{x \in F_v} v(\beta_1 - x)$. If E ramifies over F_v, one notes additionally that $\mathfrak{o}_{F_v} + \mathfrak{p}_E^{2l}\mathfrak{o}_E = \mathfrak{o}_{F_v} + \mathfrak{p}_E^{2l+1}\mathfrak{o}_E$. It follows from these considerations that:

$$\max_{x \in F_v} v(\beta_1 - x) = \frac{(e_v - 1) + v(\operatorname{disc}(Q)) - f_v}{2}$$

Note that this is half-integral when $e_v = 2$, since $v(\operatorname{disc}(Q)) - f_v$ is always even.

Let x_0 be such that $v(x_0 - \beta_1)$ attains this maximum value. Then, for any $x \in \mathfrak{o}_v$, one checks

$$v(x - \beta) = \min(v(x - x_0), v(x_0 - \beta)) \tag{60}$$

The Lemma now follows from (59) and (60) after some computation. $\square$

4.7. **Statement of final theorem; conclusion.** We now restate Proposition 2 in a manner that makes the classification of dihedral forms a little more transparent, by showing that it equals (essentially) the contribution of Grossencharacters of K with certain specified ramification.

We need to describe exactly which Grossencharacters of K occur. We will use the same notation for the adele ring of K, units of K, and so on, that we use for F, except that we replace F by K as appropriate: for example $\mathbb{A}_K$ is the adele ring of K, and $K_\infty = K \otimes \mathbb{R}$.

We will have need of a particular compact subgroup of $\mathbb{A}_{K,f}^\times$, which we denote $U(\mathfrak{f})$. Let $\overline{\mathfrak{o}_{K,\mathfrak{I}}}$ be the closure of $\mathfrak{o}_{K,\mathfrak{I}}$ in $\mathbb{A}_{K,f}$, and let $U_f(\mathfrak{f})$ be the subgroup of $\mathbb{A}_{K,f}^\times$ consisting of elements in $\overline{\mathfrak{o}_{K,\mathfrak{I}}}$ that are everywhere local units.

It should be noted that $U_f(\mathfrak{f})$ does not decompose into a product over finite places of K; however, it *does* decompose into a product over finite places v of F, as one sees from the fact that $\mathfrak{o}_{K,\mathfrak{I}}$ is an $\mathfrak{o}_F$-submodule of $\mathfrak{o}_K$. Indeed, the factor at a place v (of F) is the unit group of the closure $(\mathfrak{o}_{K,\mathfrak{I}})_v$ of $\mathfrak{o}_{K,\mathfrak{I}}$ inside $K \otimes_F F_v$. Thus one may (essentially) compute locally in $U_f(\mathfrak{f})$, and we will often do so without explicit mention.

$U_f(\mathfrak{f})$ is then an open compact subgroup of $\mathbb{A}_{K,f}^\times$. On the other hand, let U_∞ be the maximal compact subgroup of $K_\infty^\times$ – it is a possibly disconnected torus – and set $U(\mathfrak{f}) = U_\infty \cdot U_f(\mathfrak{f})$. Finally we set

$$C_{K/F}(\mathfrak{f}) = \mathbb{A}_K^\times/(K^\times \mathbb{A}_F^\times U(\mathfrak{f})), \qquad C_{K/F}^{finite}(\mathfrak{f}) = \mathbb{A}_{K,f}^\times/(K^\times \mathbb{A}_{F,f}^\times U_f(\mathfrak{f}))$$

where $U(\mathfrak{f})$ is as above. Unless the dependence on $\mathfrak{f}$ needs be made explicit, we will henceforth refer to $C_{K/F}(\mathfrak{f})$, $C_{K/F}^{finite}(\mathfrak{f})$ and $U(\mathfrak{f})$ as $C_{K/F}$, $C_{K/F}^{finite}$ and U respectively.

Suppose $\omega \in \widehat{C_{K/F}}$ is a character of $C_{K/F}$. We will define a corresponding element $\nu_\omega \in \mathfrak{a}_\mathbb{C}$. For any infinite place of K, $\omega|_{K_w^\times}$ has the shape $x \to |x|_w^{i\nu_{\omega,w}}$ for some $\nu_{\omega,w} \in \mathbb{R}$. We then set $\nu_\omega = (\nu_{\tilde{v}})_{v|\infty}$, where v runs through archimedean places of F, and for each v we set $\tilde{v}$ to be any place of K lying above the place v of F. Changing the choice of $\tilde{v}$ will change the sign of some coordinates of ν_ω; this will not matter.

Theorem 1. *Notations are as in Sections 4.2 and 4.3. Let $\mathfrak{m}$ be an integral ideal, coprime to $\mathfrak{I}$. For all test functions $h \in \mathcal{H}(M,N)$, with M,N sufficiently large, one has:*

$$\lim_{X\to\infty} \Sigma(X)/X = C \sum_{\omega \in \widehat{C_{K/F}(\mathfrak{f})}} h(\nu_\omega)\underline{\cosh}(\nu_\omega) \sum_{\mathcal{B}:\mathrm{Norm}(\mathcal{B})=\mathfrak{m}} \omega(\mathfrak{B}) \tag{61}$$

and the cuspidal and Eisenstein contributions to $\Sigma(X)$ are as follows:

$$\lim_{X\to\infty}\frac{1}{X}\sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS}_\chi)} h(\nu_{\mathbf{f}})|C_{\mathbf{f}}|^2\overline{\lambda_{\mathbf{f}}(\mathfrak{m})}\left(\sum_{\mathrm{Norm}(\mathfrak{j})<X}\lambda_{\mathbf{f}}(\mathfrak{j}^2)\right)= C\sum_{\substack{\omega\in\widehat{C_{K/F}}(\mathfrak{f})\\ \omega\neq\omega^{-1}}} h(\nu_\omega)\underline{\cosh}(\nu_\omega)\sum_{\mathcal{B}:\mathrm{Norm}(\mathcal{B})=\mathfrak{m}}\omega(\mathfrak{B}) \tag{62}$$

$$\lim_{X\to\infty}\frac{CSC(X)}{X}=C\sum_{\substack{\omega\in\widehat{C_{K/F}}(\mathfrak{f})\\ \omega=\omega^{-1}}} h(\nu_\omega)\underline{\cosh}(\nu_\omega)\sum_{\mathcal{B}:\mathrm{Norm}(\mathcal{B})=\mathfrak{m}}\omega(\mathfrak{B}) \tag{63}$$

where the notations on the right-hand side of (62) are defined in the paragraph prior to the Theorem, and the notations on the left-hand side are as in Section 2 and Section 4.2; and C is the following constant:

$$C=2^{2d_{\mathbb{R}}-1}h_F^{-1}\frac{\mathrm{Res}_{s=1}\zeta_F(1)}{L_{F;\mathfrak{I}}(\chi,1)}\frac{1}{D_F\mathrm{Norm}(\mathfrak{I})\prod_{\mathfrak{p}|\mathfrak{I}}(1+\frac{1}{\mathrm{Norm}(\mathfrak{p})})\zeta_F(2)}$$

where $L_{F;\mathfrak{I}}(\chi,s)$ denotes the partial L-function obtained by removing factors at primes dividing $\mathfrak{I}$ from the Hecke L-function $L_F(\chi,s)$. In particular, one deduces the classification of dihedral forms, viz

1. *One has a bijection between the set of cuspidal Hecke newforms $\mathbf{f}$, of conductor dividing $\mathfrak{I}$ and so that $\lim_{X\to\infty}\frac{1}{X}\sum_{\mathrm{Norm}(\mathfrak{j})<X}\lambda_{\mathbf{f}}(\mathfrak{j}^2)\neq 0$; and the set of pairs $\{(\omega,\omega^{-1}):\omega\in\widehat{C_{K/F}}(\mathfrak{f}),\omega\neq\omega^{-1}\}$. We denote the form corresponding to $\omega\in\widehat{C_{K/F}}(\mathfrak{f})$ as $\mathbf{f}(\omega)$.*
2. *The representation of $\mathrm{GL}_2(F_\infty)$ underlying $\mathbf{f}(\omega)$ is $\pi(\nu_\omega)$ (notation of Section 2.2)*
3. *For $\mathfrak{m}$ coprime to $\mathfrak{I}$ we have:*
$$\lambda_{\mathbf{f}(\omega)}(\mathfrak{m})=\sum_{\mathcal{B}:\mathrm{Norm}(\mathcal{B})=\mathfrak{m}}\omega(\mathcal{B})$$
4. *For each ω, we have*
$$|C_{\mathbf{f}(\omega)}|^2\lim_{X\to\infty}\frac{\sum_{\mathrm{Norm}(\mathfrak{j})<X}\lambda_{\mathbf{f}(\omega)}(\mathfrak{j}^2)}{X}=2C\underline{\cosh}(\nu_\omega)$$
(here the "2" arises from the fact that $\mathbf{f}(\omega)=\mathbf{f}(\omega^{-1})$.)

In other words, we have deduced the parameterization of dihedral forms by Grössencharacters of K, their Hecke eigenvalues, and (what amounts to) the residue of the symmetric square L-function.

Proof. (of Theorem – sketch). The main work in this, of course, is Proposition 2, which has already been shown. The passage between the Proposition and the Theorem is essentially "book-keeping"; we shall only sketch it.

(61) will follow from Proposition 2 once it is shown that

$$C \sum_{\omega \in \widehat{C_{K/F}(\mathfrak{f})}} h(\nu_\omega)\underline{\cosh}(\nu_\omega) \sum_{\mathcal{B}:\mathrm{Norm}(\mathcal{B})=\mathfrak{m}} \omega(\mathfrak{B})$$

agrees with the right-hand side of (25). This is verified in Section 4.8. The validity of (63) is sketched in Section 4.9. Finally (62) follows from (61) and (63).

To deduce the rest of the Theorem from (62), one proceeds as follows:

It is necessary to verify that the sum and limit may be interchanged on the left-hand side of (62); this may be verified from properties of symmetric square or even Rankin-Selberg L-functions. (There is no circularity involved in invoking the theory of these L-functions, and indeed with some more effort the author believes one can avoid it altogether.)

Once this is done, the rest of the Theorem follows from the fact that we have a matching of trace formulae, and that $\mathcal{H}(M,N)$ contains "sufficiently many" test functions to draw conclusions about individual forms. Some Atkin-Lehner theory allows us to convert the assertion into an assertion about newforms. (This would be more tedious, but in principle still straightforward, when $\mathfrak{m}$ and $\mathfrak{I}$ were not coprime.) □

4.8. **Proof of Theorem, I – Grossencharacters of K.** Our aim is now to explain the passage from (25) to (61), i.e. to show the right-hand sides of these equations are equal. This section is, again, essentially "book-keeping."

Continue with the notation of the previous section. Additionally, recall that C_F and ω_F are the class group and number of roots of unity of F, respectively. Since we are working with a totally real F, we have $\omega_F = 2$, but what follows works in general. Set $C_{K,\mathfrak{I}} = \mathbb{A}_{K,f}^\times / K^\times U_f(\mathfrak{f})$. Let $R_{K,\mathfrak{I}}$ be the regulator of $\mathfrak{o}_{K,\mathfrak{I}}^*$, and $\omega_{K,\mathfrak{I}}$ the number of roots of unity inside $\mathfrak{o}_{K,\mathfrak{I}}^*$.

Let $h \in \mathcal{H}(M,N)$. We set (for compactness) $H(\nu) = h(\nu)\underline{\cosh}(\nu)$ and define $\widehat{H}$ as in (26). Finally define

$$RHS(\mathfrak{m}) = \sum_{\omega \in \widehat{C_{K/F}}} H(\nu_\omega) \left(\sum_{\mathfrak{B}:\mathrm{Norm}_{K/F}(\mathfrak{B})=\mathfrak{m}} \omega(\mathfrak{B}) \right)$$

i.e. the right-hand side of (61). We will show that, for some constant Const, we have:

$$RHS(\mathfrak{m}) = \mathrm{Const} \cdot \sum_{\mathfrak{b} \in \tilde{C}_F, x \in X_{\mathfrak{b},\mathfrak{m}}} \sum_{\delta \in \mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^*} \widehat{H}(\log_{K/F}\left(\frac{x^2\delta^2}{\mathrm{Norm}_{K/F}(x)}\right)) \tag{64}$$

Now $K_\infty^\times / F_\infty^\times U_\infty$ is isomorphic to the product of $[F:\mathbb{Q}]$ copies of $\mathbb{R}_{>0}$; we denote it by V and equip it with the measure $\prod_v dx_v/x_v$. The image of $\mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^*$ (where $\mathfrak{o}_{K,\mathfrak{I}}^*$ consists of the units in $\mathfrak{o}_{K,\mathfrak{I}}$) in V is a lattice (cocompact discrete subgroup) that we denote by Λ. The map $\mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^* \to \Lambda$ has a finite kernel; we denote this kernel T, so that $\Lambda \approx \mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^* T$. Note that T contains the (image of the) roots of unity belonging to $\mathfrak{o}_{K,\mathfrak{I}}$.

There is a natural map $V \to C_{K,F}(\mathfrak{f})$, induced from the natural injection $K_\infty^\times \hookrightarrow \mathbb{A}_K^\times$; let Λ' be its kernel; note $\Lambda \subset \Lambda'$ (since an element of $\mathfrak{o}_{K,\mathfrak{I}}^*$, thought of as belonging to $\mathbb{A}_{K,f}^\times$, in fact belongs to $U_f(\mathfrak{f})$.)

We then have a natural (exact) sequence:

$$V/\Lambda' \hookrightarrow C_{K/F} \twoheadrightarrow C_{K/F}^{finite} = \mathbb{A}_{K,f}^\times/(K^\times \mathbb{A}_{F,f}^\times U_f(\mathfrak{f})) \tag{65}$$

This exhibits $C_{K,F}$ as a disconnected group whose connected component is a torus, and whose component group is a relative class group.

The absolute convergence of the sums of (64) is clear (c.f. (14)) and formal manipulations are justified.

Lemma 9. *Let $\mathfrak{B}$ be an ideal of K, prime to $\mathfrak{I}$. The sum*

$$\sum_{\omega \in \widehat{C_{K/F}(\mathfrak{f})}} H(\nu_\omega)\omega(\mathfrak{B}) \tag{66}$$

vanishes unless, for some integral ideal $\mathfrak{b}$ of F, the ideal $\mathfrak{b}^{-1}\mathfrak{B}$ is generated by an element of $\mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{I}}$. (In particular, it vanishes unless $\mathfrak{B}$ is principal modulo F-ideals.)

Proof. For $\mathfrak{B}$ an ideal of K, let $x_{\mathfrak{B}} \in \mathbb{A}_{K,f}^\times$ be an idele so that $(x_{\mathfrak{B}})_v = 1$ for finite places v such that the corresponding prime ideal $\mathfrak{p}_v$ does not divide $\mathfrak{B}$; and $(x_{\mathfrak{B}})_v$ has valuation matching that of of $\mathfrak{B}$ at places v such that $\mathfrak{p}_v|\mathfrak{B}$. By definition $\omega(\mathfrak{B}) = \omega(x_{\mathfrak{B}})$.

Fix $\nu_0 \in \mathfrak{a}$, and consider the sum just over $\omega \in \widehat{C_{K/F}}$ with $\nu_\omega = \nu_0$. Let ω_0 be *one such* ω; then all such ω are of the form $\omega_0\omega'$, with $\omega' \in \widehat{C_{K/F}^{finite}}$. Then:

$$\sum_{\omega \in \widehat{C_{K/F}}:\nu_\omega = \nu_0} \omega(\mathfrak{B}) = \omega_0(x_{\mathfrak{B}}) \sum_{\omega' \in \widehat{C_{K/F}^{finite}}} \omega'(x_{\mathfrak{B}}) \tag{67}$$

(67) vanishes unless $x_{\mathfrak{B}} \in K_\infty^\times K^\times \mathbb{A}_{F,f}^\times U(\mathfrak{f})$ (this follows from the fact $C_{K/F}^{finite} = \mathbb{A}_K^\times/(K_\infty^\times K^\times \mathbb{A}_F^\times U_f(\mathfrak{f}))$.

Suppose that one has such an inclusion $x_{\mathfrak{B}} \in K_\infty^\times K^\times \mathbb{A}_{F,f}^\times U(\mathfrak{f})$. Let λ be the corresponding element of $K^\times$, $x_{\mathfrak{b}}$ the element of $\mathbb{A}_{F,f}^\times$, and $\mathfrak{b}$ the F-ideal corresponding to $x_{\mathfrak{b}}$. One has then, in $\mathbb{A}_{K,f}^\times$, the inclusion $x_{\mathfrak{B}} \in x_{\mathfrak{b}}\lambda U_f(\mathfrak{f})$. In particular, this implies an equality of K-ideals $\mathfrak{B} = (\lambda)(\mathfrak{b})$; further, one sees $\lambda \in x_{\mathfrak{B}} x_{\mathfrak{b}}^{-1} U_f(\mathfrak{f})^{-1} = x_{\mathfrak{B}} x_{\mathfrak{b}}^{-1} U_f(\mathfrak{f}) \subset x_{\mathfrak{b}}^{-1}\overline{\mathfrak{o}_{K,\mathfrak{I}}}$. The last inclusion, which follows from $x_{\mathfrak{B}}\overline{\mathfrak{o}_{K,\mathfrak{I}}} \subset \overline{\mathfrak{o}_{K,\mathfrak{I}}}$, uses the coprimality of $\mathfrak{B}$ and $\mathfrak{I}$. In any case, this implies $\lambda \in \mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{I}}$, as required. $\square$

Let $X_{\mathfrak{b},\mathfrak{m}}$ be defined as in (27). Then, using Lemma 9, every $\mathfrak{B}$ contributing to $RHS(\mathfrak{m})$ is of the form $x\mathfrak{b}$, for some $\mathfrak{b} \in \tilde{C}_F, x \in X_{\mathfrak{b},\mathfrak{m}}$. In fact, $\mathfrak{B}$ can be expressed in such a form $x\mathfrak{b}$ ($\mathfrak{b} \in \tilde{C}_F, x \in X_{\mathfrak{b},\mathfrak{m}}$) in precisely $|\mathbf{K}|$ ways, where $\mathbf{K} = \ker(C_F \to \mathbb{A}_{K,f}^{\times}/K^{\times}U_f(\mathfrak{f}))$. (To see this, note that $\mathfrak{B} = x\mathfrak{b}$, for some $x \in \mathfrak{b}^{-1}\mathfrak{o}_{K,\mathfrak{I}}$, if and only if $x_{\mathfrak{B}} \in x.x_{\mathfrak{b}}U_f(\mathfrak{f})$; the "if" was established in the Lemma, and the "only if" is similar. Thus if $\mathfrak{B} = x_1\mathfrak{b}_1 = x_2\mathfrak{b}_2$, one has $x_1x_{\mathfrak{b}_1}U_f(\mathfrak{f}) = x_2x_{\mathfrak{b}_2}U_f(\mathfrak{f})$; this can occur, for some x_1, x_2, if and only if $x_{\mathfrak{b}_1}x_{\mathfrak{b}_2}^{-1}$ belongs to $K^{\times}U_f(\mathfrak{f})$; this amounts to saying that $\mathfrak{b}_1\mathfrak{b}_2^{-1}$ represents an ideal class belonging to $\mathbf{K}$.) Finally, if $\mathfrak{B} = x\mathfrak{b}$, we see from the above $\omega(\mathfrak{B}) = \omega_\infty(x_\infty)^{-1}$ for $\omega \in \widehat{C_{K/F}(\mathfrak{f})}$; consequently, using the fact that H is even:

$$\begin{aligned} RHS(\mathfrak{m}) &= \frac{1}{|\mathbf{K}|}\sum_{\omega\in\widehat{C_{K/F}}} H(\nu_\omega)\sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\omega_\infty(x) \\ &= |C_{K,\mathfrak{I}}||C_F|^{-1}\sum_{\omega_\infty\in\widehat{V/\Lambda'}} H(\nu_\omega)\sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\omega_\infty(x) \end{aligned} \tag{68}$$

where we have used (65) and an elementary computation of orders for the last equality. However, one computes that $\sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\omega_\infty(x)$ vanishes for $\omega_\infty \in \widehat{V/\Lambda} - \widehat{V/\Lambda'}$. (To see this, note for example that the finite group Λ'/Λ acts on $\amalg_{\mathfrak{b}}X_{\mathfrak{b},\mathfrak{m}}$.) Thus we may enlarge the range of summation to $\widehat{V/\Lambda}$, obtaining:

$$RHS(\mathfrak{m}) = \frac{|C_{K,\mathfrak{I}}|}{|C_F|}\sum_{\omega_\infty\in\widehat{V/\Lambda}} H(\nu_\omega)\sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\omega_\infty(x)$$

We may now apply Poisson summation to the inner sum, since the ν_ω fill out a union of certain lattices; computing the dual lattices is straightforward.

$$RHS(\mathfrak{m}) = \sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\frac{|C_{K,\mathfrak{I}}|}{|C_F|}\mathrm{vol}(V/\Lambda)\sum_{\delta\in\Lambda}\widehat{H}(\log_{K/F}(x^2\delta^2))$$

One computes $\mathrm{vol}(V/\Lambda) = |T|\frac{R_{K,\mathfrak{I}}\omega_K^{-1}}{R_F\omega_F^{-1}}$ (notation as indicated at the start of this section). Since the kernel of $\mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^* \to \Lambda$ had order $|T|$, we can remove the $|T|$ in this expression at the cost of replacing Λ by $\mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^*$; finally one has $\log_{K/F}(x^2\delta^2) = \log_{K/F}(\frac{x^2\delta^2}{\mathrm{Norm}_{K/F}(x)})$. Thus:

$$RHS(\mathfrak{m}) = \frac{|C_{K,\mathfrak{I}}|R_{K,\mathfrak{I}}\omega_K^{-1}}{|C_F|R_F\omega_F^{-1}}\sum_{\mathfrak{b}\in\tilde{C}_F, x\in X_{\mathfrak{b},\mathfrak{m}}}\sum_{\delta\in\mathfrak{o}_{K,\mathfrak{I}}^*/\mathfrak{o}_F^*}\widehat{H}(\log_{K/F}\left(\frac{x^2\delta^2}{\mathrm{Norm}_{K/F}(x)}\right))$$

Compare this to Proposition 2! This (and using the standard expression for the residue at 1 of the zeta-functions of F, K) demonstrates (61) of the Theorem. $\square$

4.9. **Proof of Theorem, II – Eisenstein contribution.** In this section, we sketch how (63) may be proven. In fact, we sketch how to show that $\lim_{X\to\infty} \frac{CSC(X)}{X}$ is an (explicitly evaluable) multiple of $h(0)$. Explicating the various arithmetic issues proves (63); observe that all ω contributing to the right-hand side of (63) have $\nu_\omega = 0$, so a multiple of $h(0)$ is expected.

As has been discussed in Section 2.4 – and we will use without comment the notation of that section – the continuous spectrum is parameterized by data $\mu = (\omega, f_j)$. **Note:** Here, and in the rest of this section, ω is a Grossencharacter of F, *not* of K.

Then the continuous spectrum contribution to (24) takes the form:
(69)

$$\mathrm{CSC}(X) = \sum_\mu \int_0^\infty dt h(\nu_\mu + t\rho)\overline{a_{E(\mu,it)}(\mathfrak{m})}\left(\sum_{\mathfrak{j}\,\mathrm{integral\,F-ideal}} a_{E(\mu,it)}(\mathfrak{j}^2) g(\mathrm{Norm}(\mathfrak{j})/X)\right)$$

We will evaluate the contribution of a particular $\mu = (\omega, f_j)$ to (69); we will denote this by $\mathrm{CSC}_\mu(X)$. (It can be verified that the μ-sum and the limit in X can be interchanged, and thus $\lim_{X\to\infty} \mathrm{CSC}(X)/X$ equals $\sum_\mu \lim_{X\to\infty} \mathrm{CSC}_\mu(X)/X$.) We discuss the special case when f_j is the new vector and $\omega = \omega^{-1}$; the other cases are handled similarly – the method will make clear that if ω is not a twist of ω^{-1}, then $\lim_{X\to\infty} \mathrm{CSC}_\mu(X)/X = 0$.

Note that, since $\omega = \omega^{-1}$, one has $\nu_\mu = 0$ (notation of Section 2.4). For $\mathfrak{m}$ an integral ideal of F, let $\lambda_{E(\mu,it)}(\mathfrak{m})$ denote the $\mathfrak{m}$th Hecke eigenvalue of $E(\mu, it)$. Then, as in the cuspidal case, the $\mathfrak{m}$th Fourier coefficient $a_{E(\mu,it)}(\mathfrak{m})$ is a multiple of $\lambda_{E(\mu,it)}(\mathfrak{m})$. One obtains:

$$\mathrm{CSC}_\mu(X) = \int_0^\infty dt H(t) \left(\sum_{\mathfrak{j}} \lambda_{E(\mu,it)}(\mathfrak{j}^2) g(\mathrm{Norm}(\mathfrak{j})/X)\right) \tag{70}$$

Here $H(t) = h(t\rho)\overline{\lambda_{E(\mu,it)}}(\mathfrak{m})|a_{E(\mu,it)}(\mathfrak{o}_F)/\lambda_{E(\mu,it)}(\mathfrak{o}_F)|^2$ incorporates difference between "Fourier" and "Hecke" normalizations. By choosing N large enough, we can obtain any desired rate of decay for $H(t)$ at ∞.

We can analyze the inner sum of (70) as follows, in terms of Hecke L-functions for F:
(71)

$$\sum_{\mathfrak{j}} \mathrm{Norm}(\mathfrak{j})^{-s} \lambda_{E(\mu,it)}(\mathfrak{j}^2) = \zeta_F(2s)^{-1} L_F(\omega^2, s+2it) L_F(\chi, s) L_F(\omega^{-2}, s-2it)$$

In particular, since we are assuming ω^2 is trivial, this has poles at $s = 1 \pm 2it$ (and a double pole if $t = 0$). Standard methods involving Mellin transforms and contour integration show that for every $\epsilon > 0$ there exists A such that:

$$(72) \quad \frac{1}{X}\sum_{\mathfrak{j}} g(\mathrm{Norm}(\mathfrak{j})/X)\lambda_{E(\mu,it)}(\mathfrak{j}^2)$$
$$= \zeta_F(2(1+2it))^{-1}L_F(\chi,1+2it)\zeta_F(1+4it)X^{2it}+$$
$$\zeta_F(2(1-2it))^{-1}L_F(\chi,1-2it)\zeta_F(1-4it)X^{-2it}+O((1+|t|)^A X^{-1/2+\epsilon})$$

$H(t)$ has reasonable decay as $t\to\infty$, and the error term of (72) does not contribute to $\lim_{X\to\infty}\mathrm{CSC}_\mu(X)/X$; we therefore have to analyze a contribution of the form

$$(73) \lim_{X\to\infty}\int_t H(t)(\zeta_F(2(1+2it))^{-1}L_F(\chi,1+2it)\zeta_F(1+4it)X^{2it}+$$
$$\zeta_F(2(1-2it))^{-1}L_F(\chi,1-2it)\zeta_F(1-4it)X^{-2it})dt$$

The function $t\mapsto\zeta_F(1+4it)$ is singular as $t\to 0$. One studies (73) by means of:

Lemma 10. *Let f be a differentiable function belonging to $L^1(\mathbb{R})$. Then*

$$PV\int_x x^{-1}f(x)e^{ikx}dx \equiv \lim_{\epsilon\to 0^+}\int_{|x|\geq\epsilon} x^{-1}f(x)e^{ikx}dx \to \pm\pi i f(0)$$

as $k\to\pm\infty$.

Proof. The Riemann-Lebesgue Lemma shows that the limit vanishes if $f(x)=0$; this reduces for checking it for a *single* f such that $f(0)\neq 0$, and then one can (e.g.) take $f(x)=1/(1+x^2)$ and use contour integration. □

Applying Lemma 10 to (73), with $k=2\log(X)$, we obtain:

$$(74) \qquad \lim_{X\to\infty}\frac{\mathrm{CSC}_\mu(X)}{X}=\zeta_F(2)^{-1}\mathrm{Res}_{s=1}(\zeta_F(s))L_F(1,\chi)\frac{\pi}{2}H(0)$$

This proves, in view of the definition of H, that $\lim_{X\to\infty}\mathrm{CSC}_\mu(X)X^{-1}$ is indeed a multiple of $h(0)$. One may carry out this procedure for all μ, eventually recovering (63).

5. Higher Symmetric Powers, and Counting Special Forms

We will sketch in Section 5.1 how the procedure of the previous section, when applied to Sym^r for $r>2$, fails.

However, this procedure is not without value: philosophically speaking it still yields a *density estimate* for those forms so that $L(s,\mathrm{Sym}^r\pi)$ has a pole; the estimate is sharp for $r=2$, and decreases in sharpness from the (presumably) "correct" answer is r increases. Rather than answer precisely this question, we answer an intimately related – but more arithmetically interesting – question: bounding the number of *forms associated to complex Galois representations*; it is believed, in any case, that all non-dihedral π for which $L(s,\mathrm{Sym}^r\pi)$ have a pole, for some $r\geq 3$, are associated to complex Galois representations.

We give some discussion of this estimate and its proof in Section 5.3; the ideas are essentially the same as [10], but since it is short, shows some new features over a number field, and fits well into the framework of this paper, we have presented it here. The first estimate of this nature was given by Duke, [3].

5.1. **Higher Symmetric Powers.** The central difficulty with higher symmetric powers was observed by Peter Sarnak. We sketch the main difficulty here.

We return to the language of Section 1.1 and Section 3. We have seen how one can spectrally average the residue of $L(s, \mathrm{Sym}^2\pi)$, and how this leads to the classification of dihedral forms. We now explain the difficulties in the case $r > 2$, in a similar context to that of Section 3.

Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. For simplicity we shall assume that π is unramified at all finite places and satisfies the Ramanujan conjecture; we will also also assume the analytic continuation of $L(s, \mathrm{Sym}^r\pi)$. These assumptions do not affect the underlying difficulty. Let $\lambda_\pi(n)$ be the nth Hecke eigenvalue of π, and $a_n(f)$ the nth Fourier coefficient.

A simple computation, using these assumptions, shows that $L(s, \mathrm{Sym}^r\pi)$ has a pole if and only the limit

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n<X, (n,N)=1} \lambda_\pi(n^r)$$

is nonvanishing; here $N = N(r)$ is chosen sufficiently divisible to eliminate difficulties from small primes.

The analogue of the earlier analysis, then, is to carry out the spectral sum (c.f. (18)):

$$\Sigma(X) = \sum_f h(t_f) \frac{1}{X} \left(\sum_{n=1}^{\infty} a_{n^r}(f) g(n/X) \right) \overline{a_m(f)}$$

where the sum is over, say, an orthonormal basis of Maass forms for $\mathrm{SL}_2(\mathbb{Z})$. This is done, again, via the Kuznetsov formula. The geometric side of the formula will resemble (c.f. (19))

$$\sum_{c=1}^{\infty} \sum_{n=1}^{\infty} g(n/X) KS(n^r, m, c) \varphi(\frac{4\pi n^{r/2}\sqrt{m}}{c}) \frac{1}{c}$$

The Kloosterman sum $KS(n^r, m, c)$ is dependent only on the residue class of n mod c. The technique used in Section 3 was to break the n-sum into residue classes mod c. Since $\varphi(x)$ decays as $x \to \infty$, the n-sum has effective length around $c^{2/r}$.

If $r = 2$, the n-sum has length c, and therefore one expects the n sum to "touch" all residue classes mod c; it is therefore reasonable to expect that the exponential sums that occur are *complete* – summed over complete residue classes mod c. These exponential sums are the $S(\nu, m, c)$ that were introduced after (20).

If $r > 2$, however, the n sum becomes much shorter: even if $r = 3$, the sum over n is only of length about $c^{2/3}$. The exponential sums that occur are therefore fundamentally *incomplete* and their analysis becomes progressively harder as r increases. Even the gap between $r = 2$ and $r = 3$ is enormous. However, with advances with exponential sum "technology" the resulting exponential sums may be someday amenable to analysis; for small values of r this does not seem totally beyond reach.

Therefore, an exact treatment of higher symmetric powers in this fashion is expected to be a rather difficult endeavor. The goal of the rest of Section 5 is to show how one obtains interesting, though approximate, results, by relaxing our requirements: roughly speaking, rather than trying to exactly characterize those f such that $m(f, \mathrm{Sym}^{12}) = 1$, we merely try to bound the *number* of such f; see remarks at beginning of Section 5.

This is interesting for other reasons: such forms are associated to Galois representations, and one can deduce interesting arithmetic information from analytic bounds; see [3] and [15].

Remark 4. The idea of the bound is to spectrally average a sum that represents the pole of $L(s, \mathrm{Sym}^r f)$. However, there is a slight problem with doing this.

Suppose, for example, that f is attached to an icosahedral Galois representation σ. Then $\mathrm{Sym}^{12}\sigma$ contains the identity representation; in particular, $L(s, \mathrm{Sym}^{12} f)$ – which can be defined as a meromorphic function– has a pole at $s = 1$. However, to bound the number of such fs, one requires that this pole may be "detected" by a sum

$$\sum_{n<Y} a_{n^{12}}(f) \tag{75}$$

where Y is small compared to the conductor. This is difficult to establish unconditionally, as it is related to issues of Siegel zeros; if one assumes GRH one can proceed along these lines. However, one can sacrifice a tiny amount of precision and construct a "replacement" for (75) which is unconditionally large; this was the idea of Duke in [3]. Representation-theoretically, it rests on the fact that one can write the identity representation, *as a virtual representation*, as a certain combination of $\mathrm{Sym}^r\sigma$, for various $r \leq 12$.

5.2. **Automorphic Forms over a Number Field: Notations and Normalizations.** See Remark 4 above and the preamble to Section 5 for motivation of the problem and technique used. We also suggest reading [3] or [10] for more background. In this section, we do not require F to be totally real.

An automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is said to be of *Galois type* if there is a Galois representation $\rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\mathbb{C})$ that corresponds (under the local Langlands correspondence) to π at all places.

We will prove bounds for the number of Galois type automorphic forms on GL_2 over F of specified conductor, and with specified central character. One of the bounds will be absolute for F of fixed degree over $\mathbb{Q}$; there is no "totally real" restriction. (Thus the notation $\ll$ should be understood as $\ll_{[F:\mathbb{Q}]}$; the constant depends on $[F:\mathbb{Q}]$ but not F.)

Here, in accordance with the formula we derive in the Appendix, we will restrict ourselves to proving the result for "weight 0" Galois type automorphic forms: that is, the $\mathrm{GL}_2(F_\infty)$-representation associated to π should be spherical.

5.3. **Main Theorem and its proof.** If χ is a Grossencharacter of F, with $\chi_\infty \equiv 1$, let $N_\chi(\mathfrak{q})$ be the space of dimensions of automorphic forms on GL_2 over F of Galois type, spherical at ∞, with central character χ and conductor $\mathfrak{q}$. The assumption "spherical at ∞" could be dropped, c.f. [10] over $\mathbb{Q}$, but that would require a generalization of the Bruggeman-Miatello formula, so we will not attempt it here.

Volume considerations suggest that the "trivial bound" for $N_\chi(\mathfrak{q})$ would be $D_F^{3/2}\mathrm{Norm}(\mathfrak{q})$.

Theorem 2. *Let $N_\chi(\mathfrak{q})$ be as above. Set $a = \frac{1}{12[F:\mathbb{Q}]}$. Then one has both:*

$$N_\chi(\mathfrak{q}) \ll_{F,\epsilon} \mathrm{Norm}(\mathfrak{q})^{6/7+\epsilon} \tag{76}$$

$$N_\chi(\mathfrak{q}) \ll_{[F:\mathbb{Q}],\epsilon} (D_F^{3/2}\mathrm{Norm}(\mathfrak{q}))(D_F\mathrm{Norm}(\mathfrak{q}))^{-a} \tag{77}$$

Remark 5. The first result is a direct generalization of [10]. The second result is a weak hybrid result in a slightly different direction, offering a bound uniform in F (for fixed degree). There is a significant cost to the exponent due to the need to avoid Siegel zeros. The constant a could probably be significantly improved.

The proof will follow [10], but there are some subtleties in dealing with a number field. In part, these are dealt with by the following three Lemmas. The first clarifies the relation between Hecke eigenvalues and Fourier coefficients; the second helps us deal with sums over the units, and is a variant of a Lemma due to Bruggeman-Miatello. The final asserts that there are "not too many" ideals in a fixed ideal class.

Let Res_F be the residue of $\zeta_F(s)$ at $s=1$. By the Brauer-Siegel theorem, $D_F^{-\epsilon} \ll_\epsilon \mathrm{Res}_F \ll_\epsilon D_F^{\epsilon}$. Secondly, it is easy to see that the regulator $R_F \gg_{[F:\mathbb{Q}]} 1$ (we even have $R_F \gg 1$ uniformly, but we do not need this). In particular, $h_F \ll_\epsilon D_F^{1/2+\epsilon}$. We will make use of these bounds without explicit comment.

Lemma 11. *Let $\mathbf{f} \in \mathrm{FS}_\chi$ be a Hecke eigenform of L^2 norm 1. Let $C_{\mathbf{f}}$ be the constant such that $a_{\mathbf{f}}(\mathfrak{a},\alpha) = C_{\mathbf{f}}\lambda_{\mathbf{f}}(\mathfrak{a}\alpha\mathfrak{d})$ (see Lemma 1). Then $|C_{\mathbf{f}}|^2 \gg (h_F D_F \mathrm{Norm}(\mathfrak{q}))^{-1}\mathrm{Res}_F$; in particular, by Brauer-Siegel $|C_{\mathbf{f}}|^2 \gg (h_F D_F\mathrm{Norm}(\mathfrak{q}))^{-1-\epsilon}$.*

Proof. This follows from the Rankin-Selberg method – explicitly writing the L-function as an integral against an Eisenstein series – together with Iwaniec's technique for bounding Rankin-Selberg L-functions at 1. An implementation of the former, in the holomorphic Hilbert modular case, may be found in Shimura, [12]; for the latter (over $\mathbb{Q}$), see Iwaniec's book [6]. The result in question is obtained by straightforward modification of the techniques in these references. □

The next result is a variant of [2], Lemma 8.1, and is also a discrete version of Lemma 3.

Lemma 12. *Fix $f : F_\infty^\times \to \mathbb{C}$ satisfying the following condition, for some $k > 0$:*

$$|f(x)| \ll \prod_v \min(|x|_v^k, |x|_v^{-k}) \tag{78}$$

Then, for all x in $F_\infty^\times$

$$\sum_{\epsilon \in \mathfrak{o}_F^*} |f(x\epsilon)| \ll_\epsilon \max(\mathrm{Norm}(x), \mathrm{Norm}(x)^{-1})^\epsilon \min(\mathrm{Norm}(x)^k, \mathrm{Norm}(x)^{-k})$$

the implicit constant depending only on the implicit constant in (78), k and $[F : \mathbb{Q}]$.

Proof. For $x \in F_\infty$, define $\log_F(x)$ as in Section 2.2. By Dirichlet's unit theorem, $\log_F$ maps $\mathfrak{o}_F^*$ homomorphically to a lattice of dimension $d_\mathbb{R} + d_\mathbb{C} - 1$ in $\mathbb{R}^{d_\mathbb{R}+d_\mathbb{C}}$; the kernel has size ω_F, which is bounded in terms of $[F : \mathbb{Q}]$. Let $|| \cdot ||_1$ be the L^1 norm on $\mathbb{R}^{d_\mathbb{R}+d_\mathbb{C}}$. Thus:

$$\sum_{\epsilon \in \mathfrak{o}_F^*} |f(x\epsilon)| \ll \sum_\epsilon e^{-k||\log_F(x)+\log_F(\epsilon)||_1}$$

There is a constant A, depending only on $[F : \mathbb{Q}]$, such that $||\log_F(\epsilon)||_1 > A$ for all ϵ which are not roots of unity. (This follows from the evident fact that there are only finitely many algebraic integers of *fixed degree*, all of whose Galois conjugates lie in a fixed compact subset of $\mathbb{C}$.)

It follows that we can cover $\mathbb{R}^{d_\mathbb{R}+d_\mathbb{C}}$ with cubes of fixed side length in such a way that each cube contains at most one of the $\log_F(x) + \log_F(\epsilon)$ (for any F).

Let $R > 0$. The number of cubes intersecting an $|| \cdot ||_1$-ball of radius R is $\ll R^{d_\mathbb{R}+d_\mathbb{C}}$. We sum over cubes, applying the trivial bound $|f(x\epsilon)| \ll \min(\mathrm{Norm}(x)^k, \mathrm{Norm}(x)^{-k})$ inside an $||\cdot||_1$-ball of radius R, and the sharper bound $|f(x\epsilon)| \ll e^{-k||\log_F(x\epsilon)||_1}$ outside it. Optimizing R, we obtain the result. □

Lemma 13. *The number of integral ideals of F having norm $\leq X$ is, for any $\epsilon > 0$, $\ll_{[F:\mathbb{Q}],\epsilon} X^{1+\epsilon}$.*

Proof. For $n \in \mathbb{Z}$, let $d(n)$ be the number of divisors of n. One checks that the number of ideals of F with norm n is $\leq d(n)^{[F:\mathbb{Q}]-1} \ll_\epsilon n^\epsilon$. □

Proof. (of Theorem 2) We give the proof in the case of icosahedral Galois representations, the other cases being similar; the reader may consult [10] for the proof over $\mathbb{Q}$ and [15] for a very detailed exposition over $\mathbb{Q}$ of the techniques of amplification needed. We shall use (15), and refer to Remark 4 for motivation of the method. We continue with the notation of Section 2.

First note that it is possible to choose a test function $h \in \mathcal{H}(M, N)$ so that $h(\nu) > 0$ for all $\nu \in Y$ (notation of Section 2.2), and so that $\varphi(x)$ on F_∞ satisfies, for $m < M$:

$$|\varphi(x)| \ll \prod_v \min(|x_v|^m, |x_v|^{-1/4}) \tag{79}$$

In other words, φ is "mildly" decreasing near ∞ and 0. We may also take M arbitrarily large. It is easy to choose h, and one verifies the conditions on φ using standard properties of Bessel functions (c.f. (104) in the Appendix; this statement holds even in presence of complex places). Set $h_0 = \int_{\nu \in \mathfrak{a}} h(\nu) d\mu_\nu$.

As in [3], [15] and [10], we define an amplifier: a function $\mathfrak{I} \mapsto c_\mathfrak{I}$ on integral ideals of F, defined as:

$$c_\mathfrak{I} = \begin{cases} \chi(\mathfrak{p})^6, & \text{for } \mathfrak{I} = \mathfrak{p}^{12}, \mathrm{Norm}(\mathfrak{p}) \le N^{1/12}, \mathfrak{p} \text{ prime}, \ (\mathfrak{p}, \mathfrak{q}) = 1 \\ -\chi(\mathfrak{p})^4, & \text{for } \mathfrak{I} = \mathfrak{p}^{8}, \mathrm{Norm}(\mathfrak{p}) \le N^{1/12}, \mathfrak{p} \text{ prime}, \ (\mathfrak{p}, \mathfrak{q}) = 1 \\ -\chi(\mathfrak{p}), & \text{for } \mathfrak{I} = \mathfrak{p}^{2}, \mathrm{Norm}(\mathfrak{p}) \le N^{1/12}, \mathfrak{p} \text{ prime}, \ (\mathfrak{p}, \mathfrak{q}) = 1 \\ 0 & \text{else.} \end{cases}$$

Set T to be the number of $\mathfrak{I}$ for which $c_\mathfrak{I}$ is nonvanishing. One cannot make very strong assertions about T, owing to the possibility of Siegel zeros, but one can at least say $T \gg N^{\frac{1}{12[F:\mathbb{Q}]} - \epsilon}$, by counting primes above primes of $\mathbb{Q}$. The sequence $(c_\mathfrak{I})$ has the following properties: $|c_\mathfrak{I}| \le 1$ always; the number of nonvanishing $c_\mathfrak{I}$ for $\mathrm{Norm}(\mathfrak{I}) \le N$ is T; and, if $\mathbf{f}$ is associated to an icosahedral Galois representation, one has:

$$\sum_{\mathfrak{I} : \mathrm{Norm}(\mathfrak{I}) \le N} c_\mathfrak{I} \lambda_\mathbf{f}(\mathfrak{I}) \gg T \tag{80}$$

For the proof of the last, see [15]. We will square the left-hand side of (80) and sum over $\mathbf{f} \in \mathbf{B}(\mathrm{FS}_\chi)$; as in Section 4.4 we replace a sum over ideals with a sum over pairs $(\mathfrak{a}, \alpha)$ with $\mathfrak{a} \in \tilde{C}_F, \alpha \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}$. Using also Lemma 11, we obtain:

$$S = \sum_{\mathbf{f} \in \mathbf{B}(\mathrm{FS}_\chi)} h(\nu_\mathbf{f}) \left| \sum_{\mathfrak{a} \in \tilde{C}_F} \sum_{\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^*} c_{\mathfrak{d}\mathfrak{a}\alpha} a_\mathbf{f}(\mathfrak{a}, \alpha) \right|^2 \gg N_\chi(\mathfrak{q}) (h_F D_F \mathrm{Norm}(\mathfrak{q}))^{-1-\epsilon} T^2 \tag{81}$$

We now apply the GL(2)-Bruggeman-Miatello formula (15) to bound S from above; this, combined with the above equality, will bound $N_\chi(\mathfrak{q})$. We separate the parts of the constants c_1, c_2 in (15) that involve h_F or D_F, denoting by c_1', c_2' the factors of $2, \pi$ that remain. The positivity of h allows us to neglect the continuous-spectral contribution altogether; we obtain:

$$S \leq c_1' h_F^{-1} D_F^{1/2} h_0 \sum_{\mathfrak{a},\alpha} |c_{\mathfrak{d}\mathfrak{a}\alpha}|^2 + \tag{82}$$

$$c_2' h_F^{-1} \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{c}:\mathfrak{c}^2\sim\mathfrak{a}_1\mathfrak{a}_2 \\ \epsilon\in\mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2}} \sum_{\substack{\alpha_1\in\mathfrak{a}_1^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^* \\ \alpha_2\in\mathfrak{a}_2^{-1}\mathfrak{d}^{-1}/\mathfrak{o}_F^*}} c_{\alpha_1\mathfrak{a}_1\mathfrak{d}} \overline{c_{\alpha_2\mathfrak{a}_2\mathfrak{d}}} \sum_{c\in\mathfrak{c}^{-1}\mathfrak{q}} \varphi\left(\frac{\epsilon\alpha_1\alpha_2}{c^2[\frac{\mathfrak{c}^2}{\mathfrak{a}_1\mathfrak{a}_2}]}\right) \frac{KS(\dots)}{\mathrm{Norm}(c\mathfrak{c})}$$

The argument of KS is as specified in (15).

To prove (76) from here, one sets $N = \mathrm{Norm}(\mathfrak{q})^{12/7}$ and applies the trivial bound to the right hand-side; the only "ingredient" is (13) for Kloosterman sums, c.f. [10]. (To deal with the c-sum, one may separate the c-sum into all c that generate a given principal ideal (c), and then sum over principal ideals (c); one can control the φ-sum in the first step by Lemma 12, and the second step by Lemma 13. One could also easily proceed directly, since uniformity in F is not important.)

To prove (77), fix $\delta > 0$ and set $N = (D_F \mathrm{Norm}(\mathfrak{q}))^{1-\delta}$. If $\alpha_1, \mathfrak{a}_1$ is such that $c_{\alpha_1\mathfrak{a}_1\mathfrak{d}} \neq 0$, we have (by definition of the assigment $\mathfrak{J} \mapsto c_{\mathfrak{J}}$) that $\mathrm{Norm}(\alpha_1\mathfrak{a}_1\mathfrak{d}) \leq N$; similarly for $\mathfrak{a}_2, \alpha_2$. If $x = \frac{\alpha_1\alpha_2}{c^2[\mathfrak{c}^2\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}]}$ is the argument of φ in (82), one must have (for any $\mathfrak{a}_1, \alpha_1, \mathfrak{a}_2, \alpha_2$ so that $c_{\alpha_1\mathfrak{a}_1\mathfrak{d}} \neq 0, c_{\alpha_2\mathfrak{a}_2\mathfrak{d}} \neq 0$) $\mathrm{Norm}(x) \ll N^2 D_F^{-2} \mathrm{Norm}(\mathfrak{q})^{-2}$; this follows as $c \in \mathfrak{c}^{-1}\mathfrak{q}$. Thus, for our choice of N, $\mathrm{Norm}(x) \ll D_F^{-\delta} \mathrm{Norm}(\mathfrak{q})^{-\delta}$. In view of (79) this shows that $\varphi(x)$ is $\ll_m D_F^{-m\delta} \mathrm{Norm}(\mathfrak{q})^{-m\delta}$. From this one easily sees that the contribution of the latter term in (82) is negligible (one can again formalize this argument easily with the help of Lemmas 12 and 13.)

It follows that the dominant contribution to S comes from the h_0 term, whence $S \ll h_F^{-1} D_F^{1/2} T$. Thus by (81):

$$\begin{aligned} N_\chi(\mathfrak{q}) &\ll_\epsilon (D_F \mathrm{Norm}(\mathfrak{q}))^\epsilon D_F^{3/2} \mathrm{Norm}(\mathfrak{q}) T^{-1} \\ &\ll_\epsilon D_F^{3/2+\epsilon} \mathrm{Norm}(\mathfrak{q}) (D_F \mathrm{Norm}(\mathfrak{q}))^{\frac{1-\delta}{12[F:\mathbb{Q}]}} \end{aligned}$$

where we have used $T \gg_\epsilon N^{\frac{1}{12[F:\mathbb{Q}]}-\epsilon}$ (see prior discussion). Allowing $\delta \to 0$ gives (77). $\square$

6. Appendix: Derivation of PGL(2) sum formula

We shall derive a version of the Bruggeman-Miatello formula for $\mathrm{PGL}_2(F_\infty)$ with Nebentypus; we follow Bruggeman-Miatello in [2]. We use freely the notations stated in Sections 2.2 and 2.3. In particular, we will fix a conductor $\mathfrak{J}$ and a central character χ, whose conductor divides $\mathfrak{J}$; we will assume always that $\chi_\infty = 1$.

Roughly speaking, in the notation of Section 2.2, we produce a sum formula for the space FS, i.e. a sum formula including forms whose central character is any twist of χ by a class group character; from this we deduce by an averaging process the sum formula for FS_χ.

6.1. **Action of the class group.** Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals, and $\pi_\mathfrak{a}, \pi_\mathfrak{b}$ as in Section 2.1. Our immediate aim will be to explicate the action of the element $\mathrm{diag}(\pi_\mathfrak{b}, \pi_\mathfrak{b}) \in Z(\mathbb{A}_F)$ on FS. The connected components of the symmetric space $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/K_\infty K_0(\mathfrak{I})Z(F_\infty)$ are parameterized by ideal classes of F (c.f. Section 2.3) and the action of $Z(\mathbb{A}_F)$ permutes them. The resulting computations are reminiscent of those associated to the multiple cusp setting in the upper half-plane.

Given $g_\infty \in GL_2(F_\infty)$, it is a consequence of strong approximation (for the group SL_2) that we may write the following equality in $\mathrm{GL}_2(\mathbb{A}_F)$:

$$\mathrm{diag}(\pi_\mathfrak{b}, \pi_\mathfrak{b})\mathrm{diag}(\pi_\mathfrak{a}, 1)g_\infty = \gamma^{-1}\mathrm{diag}(\pi_\mathfrak{b}^2\pi_\mathfrak{a}, 1)g'_\infty\kappa_\gamma \tag{83}$$

for some $\kappa_\gamma \in K_0(\mathfrak{I}), \gamma \in \mathrm{GL}_2(F)$ and $g'_\infty = \gamma g_\infty \in \mathrm{GL}_2(F_\infty)$. It follows that:

$$\gamma \in GL_2(F) \cap GL_2(F_\infty)\mathrm{diag}(\pi_\mathfrak{b}^2\pi_\mathfrak{a}, 1)\kappa_\gamma\mathrm{diag}(\pi_\mathfrak{a}\pi_\mathfrak{b}, \pi_\mathfrak{b})^{-1} \tag{84}$$

We will denote by $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ the set $\{\gamma : \gamma \text{ satisfies (84)}\}$. For $\gamma \in \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ we denote by κ_γ the element of $K_0(\mathfrak{I})$ defined by (84). Now fix an element $\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2} \in \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$. One easily verifies that $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2) = \gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2}\Gamma_0(\mathfrak{I}, \mathfrak{a}) = \Gamma_0(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2}$, and that any element of $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ conjugates $\Gamma_0(\mathfrak{I}, \mathfrak{a})$ to $\Gamma_0(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)$.

Let $\mathbf{f} = (\mathbf{f}_\mathfrak{a}) \in \mathrm{FS}$. The action of $z = \mathrm{diag}(\pi_\mathfrak{b}, \pi_\mathfrak{b})$ on $\mathbf{f}$ may then be written as $\pi(z) \cdot \mathbf{f} = \mathbf{f}'$, where

$$\mathbf{f}'_\mathfrak{a}(g_\infty) = \chi(\pi_\mathfrak{b})^2\chi_f(\kappa_{\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2}})\mathbf{f}_{\mathfrak{a}\mathfrak{b}^2}(\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2}g_\infty) \tag{85}$$

This follows from (83) and (9). We now record some simple facts about $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$.

Lemma 14. *One has explicitly:*

$$\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in \mathfrak{b}, b \in \mathfrak{a}\mathfrak{b}, c \in \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{I}, d \in \mathfrak{b}^{-1}, ad-bc \in \mathfrak{o}_F^\times\} \tag{86}$$

Set $\Gamma_N(\mathfrak{I}, \mathfrak{a}) = N(F) \cap \Gamma_0(\mathfrak{I}, \mathfrak{a})$*; similarly define* $\Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)$*. Then the map* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c, a, ad-bc)$ *sets up a bijection from* $\Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I}, \mathfrak{a})$ *to*

$$\{(c, x, \epsilon) : c \in \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{I}, \epsilon \in \mathfrak{o}_F^\times, x \in \mathfrak{b}, x \text{ generates } \mathfrak{b}/\mathfrak{a}\mathfrak{b}^2(c)\}$$

Here "generates" means "generates as an $\mathfrak{o}_F$*-module.*

The double coset corresponding (under the bijection above) to (c, x, ϵ) *has as representative* $\begin{pmatrix} x & (xy-\epsilon)c^{-1} \\ c & y \end{pmatrix}$, *where the image of* y *in* $\mathfrak{b}^{-1}/\mathfrak{a}c$ *is such that* $xy \equiv \epsilon$ *mod* $c\mathfrak{a}\mathfrak{b}$.

Remark 6. $\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$ is contained entirely in the large Bruhat cell, unless $\mathfrak{b}$ is principal. Indeed, if $c = 0$ we see that $ad \in \mathfrak{o}_F^\times$; since $a \in \mathfrak{b}, d \in \mathfrak{b}^{-1}$ the claim follows.

Explicit Bruhat Decomposition: Set $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On the large Bruhat cell we note:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} c & 0 \\ 0 & c^{-1}(ad-bc) \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix} \tag{87}$$

We denote this by $\gamma = n_{1,\gamma} w a_\gamma n_{2,\gamma}$.

6.2. **Derivation of Bruggeman-Miatello Formula.** We will derive the formula by taking the inner product of two Poincaré series. What follows is only a sketch, to indicate how the formula emerges; the real work is really due to Bruggeman-Miatello, in establishing details of convergence as well as the integral formulas we use. In those respects, the case of GL_2 is identical to SL_2, and that is contained in [2]. Indeed, if K_{SL} is the maximal compact of $\mathrm{SL}_2(F_\infty)$ and K_{PGL} that of $\mathrm{PGL}_2(F_\infty)$, we have an isomorphism

$$\mathrm{SL}_2(F_\infty)/K_{\mathrm{SL}} \approx \mathrm{PGL}_2(F_\infty)/K_{\mathrm{PGL}}$$

The details we must attend to arise from units and the class group. We will therefore proceed formally.

Some notations, and normalization of measures: Set $G = \mathrm{PGL}_2(F_\infty)$. We will always work with $Z(F_\infty)$-invariant functions and only need measures on $\mathrm{PGL}_2(F_\infty)$. We assign Haar measure 1 to the maximal compact K_{PGL} in $\mathrm{PGL}_2(F_\infty)$. To define the Haar measure in general, one uses the Iwasawa decomposition. Set A to be the image in $\mathrm{PGL}_2(F_\infty)$ of diagonal matrices with *totally real, positive entries and determinant* 1. (We say an element $y \in F_\infty = \prod_{v|\infty} F_v$ is totally real and positive if y_v is real and positive for each place v.) Then $\mathrm{PGL}_2(F_\infty) = N(F_\infty) \cdot A \cdot K$; now $N(F_\infty), A$ lie in $\mathrm{SL}_2(F_\infty)$, and we accordingly use the measure from [2] on those. It should be noted that the corresponding measure on $N(F_\infty)$ does not correspond to Lebesgue measure on F_∞ under the identification $x \to n(x)$. These choice of measures induce, in the usual way, inner products on FS and FS_χ (notation of 2.3).

Let $\rho : A \to \mathbb{R}_+$ be the character $\mathrm{diag}(x, x^{-1}) \to \prod_v |x|_v^{\deg(v)}$, for x a totally positive, real element of F_∞; here $\deg(v) = 1$ or 2 according to whether v is real or complex, respectively.

We set $Z(F_\infty)$ to be the center of $\mathrm{GL}_2(F_\infty)$, and we set $Z_\Gamma = Z(F_\infty) \cap \Gamma$.

Characters: Let $\psi_1^{(\mathfrak{a})}$ be a character of $N(F_\infty)$ trivial on $\Gamma_N(\mathfrak{J}, \mathfrak{a}) = \Gamma_0(\mathfrak{J}, \mathfrak{a}) \cap N(F_\infty)$, and let $\psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$ be a character of $N(F_\infty)$ trivial on $\Gamma_N(\mathfrak{J}, \mathfrak{a}\mathfrak{b}^2)$.

Here $N(F_\infty)$ is considered as a locally compact topological group. The superscripts are only present as a mnemonic aid, and it is hoped they do not clutter the notation too much. Via the identification $x \mapsto n(x)$ of F_∞ with $N(F_\infty)$ (see Section 2) we identify $\psi_1^{(\mathfrak{a})}$ and $\psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$ with characters, also denoted $\psi_1^{(\mathfrak{a})}$ and $\psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$, of F_∞ which are trivial on $\mathfrak{a}$ and $\mathfrak{a}\mathfrak{b}^2$, respectively. Let $\alpha \in \mathfrak{a}^{-1}\mathfrak{d}^{-1}, \alpha' \in \mathfrak{a}^{-1}\mathfrak{b}^{-2}\mathfrak{d}^{-1}$ be so that

$$\psi_1^{(\mathfrak{a})}(z) = \psi_\infty(\alpha z),\ \psi_2^{(\mathfrak{a}\mathfrak{b}^2)}(z) = \psi_\infty(\alpha' z) \tag{88}$$

Construction of Poincare Series: Recall the notation of Section 2.3, which we shall follow. Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals. We will shortly define two Poincaré series, $P_1, P_2 \in \mathrm{FS}$. The element $\mathrm{diag}(\pi_\mathfrak{b}, \pi_\mathfrak{b}) \in Z(\mathbb{A}_F)$ acts on FS, and we may form: $\langle \pi_\mathfrak{b} P_2, P_1\rangle$, where $\pi_\mathfrak{b} P_2$ is a shorthand for $\mathrm{diag}(\pi_\mathfrak{b}, \pi_\mathfrak{b}) \cdot P_2$. This will be computed in two ways – geometrically and spectrally. This is the approach of [2], and we will follow it in the $\mathrm{PGL}_2(F_\infty)$-case.

Let $\psi_1^{(\mathfrak{a})}, \psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$ be as above. Let $f^{(\mathfrak{a})}$ be a function on $\mathrm{GL}_2(F_\infty)$ that satisfies, for $x \in F_\infty, z \in F_\infty^\times, k \in K_\infty$:

$$f^{(\mathfrak{a})}(n(x)\mathrm{diag}(z,z)gk) = \psi_1^{(\mathfrak{a})}(x) f^{(\mathfrak{a})}(g) \tag{89}$$

(In general one would have a factor $\chi_\infty(z)$, but we are assuming that $\chi_\infty \equiv 1$.) Similarly define $f^{(\mathfrak{a}\mathfrak{b}^2)}$ using $\psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$.

Now define P_1 to be an element of FS so that for $g \in \mathrm{GL}_2(F_\infty)$:

$$(P_1)_\mathfrak{a}(g) = \sum_{Z_\Gamma \Gamma_N(\mathfrak{I},\mathfrak{a})\backslash \Gamma_0(\mathfrak{I},\mathfrak{a})} \chi_f(\gamma) f^{(\mathfrak{a})}(\gamma g) \tag{90}$$

and all other components of P_1 are zero (that is, for $\mathfrak{c} \in \tilde{C}_F$, $(P_1)_\mathfrak{c} = 0$ if $\mathfrak{c}$ not in the same ideal class as $\mathfrak{a}$). Similarly define P_2 using $f^{(\mathfrak{a}\mathfrak{b}^2)}$, so that $(P_2)_\mathfrak{c}$ vanishes if $\mathfrak{c}$ is not in the ideal class of $\mathfrak{a}\mathfrak{b}^2$, and is defined using the analogue of (90) for $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^2$.

We refer to [2] for details of convergence, at least for the specific $f^{(\mathfrak{a})}, f^{(\mathfrak{a}\mathfrak{b}^2)}$ we will end up using.

6.3. **Geometric Evaluation.** One may compute $\pi_\mathfrak{b} \cdot P_2$ via (85):

$$\begin{aligned}(\pi_\mathfrak{b} \cdot P_2)_\mathfrak{a}(g) &= \chi(\pi_\mathfrak{b}^2) \sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash \Gamma_0(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)} \chi_f(\gamma)\chi_f(\kappa_{\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2}}) f^{(\mathfrak{a}\mathfrak{b}^2)}(\gamma\gamma_{\mathfrak{a}\mapsto\mathfrak{a}\mathfrak{b}^2} g) \\ &= \chi(\pi_\mathfrak{b}^2) \sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash \Gamma_0(\mathfrak{I},\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)} \chi_f(\kappa_\gamma) f(\gamma g)\end{aligned} \tag{91}$$

where for $\gamma \in \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)$, $\kappa_\gamma \in K_0(\mathfrak{I})$ is defined by means of (83). Set $I = \chi(\pi_\mathfrak{b})^{-2}\langle \pi_\mathfrak{b} P_2, P_1\rangle$. We evaluate it by unfolding the defining integral.

One obtains:

$$I = \int_{\Gamma_N(\mathfrak{I},\mathfrak{a})Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)}$$
$$\cdot \left(\sum_{\gamma \in Z_\Gamma \Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash \Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)} \chi_f(\kappa_\gamma) f(\gamma g)\right) dg \tag{92}$$

Let B be the Borel subgroup of GL_2 consisting of the upper triangular matrices; we treat separately the terms in the inner sum of (92) that correspond to $\gamma \in B(F_\infty)$ (we denote this part by $I^{(1)}$) and $\gamma \notin B(F_\infty)$ (we denote this part of the sum by $I^{(2)}$).

As seen in Remark 6, $I^{(1)} \neq 0$ only if $\mathfrak{b}$ is principal. In that case, a set of representatives for the quotient $\Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)Z_\Gamma\backslash(\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)\cap B(F_\infty))$ is given by (see (86) $\{\mathrm{diag}([\mathfrak{b}]\epsilon, [\mathfrak{b}]^{-1})\}$, as ϵ varies over $\mathfrak{o}_F^*$. For such γ one computes $\chi_f(\kappa_\gamma) = \chi(\pi_\mathfrak{b})^{-1}$, and so:

$$I^{(1)} = \sum_\epsilon \int_{\Gamma_N(\mathfrak{I},\mathfrak{a})Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \chi(\pi_\mathfrak{b})^{-1}\overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}(\mathrm{diag}([\mathfrak{b}]\epsilon, [\mathfrak{b}^{-1}]g)dg$$

To simplify, define $\Delta(\alpha, \alpha'\mathfrak{b}^2)$ to equal 1 exactly if, for some $\epsilon_0 \in \mathfrak{o}_F^*$, $\alpha = \alpha'[\mathfrak{b}^2]\epsilon_0$, i.e. $(\alpha) = (\alpha')\mathfrak{b}^2$; otherwise set $\Delta = 0$. Then, considering the $N(F_\infty)$-action on the left, we see:

$$I^{(1)} = \Delta(\alpha, \alpha'\mathfrak{b}^2)\mathrm{vol}(\Gamma_N(\mathfrak{I},\mathfrak{a})\backslash N(F_\infty))\chi(\pi_\mathfrak{b})^{-1}$$
$$\cdot \int_{N(F_\infty).Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}(\mathrm{diag}([\mathfrak{b}]\epsilon_0, [\mathfrak{b}]^{-1})g)dg$$

Here $\epsilon_0 \in \mathfrak{o}_F^*$ is so that $\alpha = \alpha'[\mathfrak{b}^2]\epsilon_0$, if such an element exists.

To evaluate $I^{(2)}$, we use Lemma 14. For $\omega \in \Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I},\mathfrak{a})$ we denote by $[\omega]$ any element of $\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)$ representing ω. Then:

$$I^{(2)} = \int_{\Gamma_N(\mathfrak{I},\mathfrak{a})Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \sum_{\substack{\omega \in Z_\Gamma\Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I},\mathfrak{a}) \\ [\omega]\notin B(F_\infty)}}$$
$$\sum_{\nu \in \Gamma_N(\mathfrak{I},\mathfrak{a})} \overline{f^{(\mathfrak{a})}(g)}\chi_f(\kappa_{[\omega]}) f^{(\mathfrak{a}\mathfrak{b}^2)}([\omega]\nu g)dg \tag{93}$$

We can fold together the integral and the ν sum, giving:

$$I^{(2)} = \sum_{\substack{\omega \in Z_\Gamma\Gamma_N(\mathfrak{I},\mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a}\to\mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I},\mathfrak{a}) \\ [\omega]\notin B(F_\infty)}} \int_{Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}([\omega]g)\chi_f(\kappa_{[\omega]})dg$$

For each $[\omega]$, we can decompose according to the Bruhat decomposition $[\omega] = n_{1,[\omega]} w a_{[\omega]} n_{2,[\omega]}$, as per (87). We obtain:

$$I^{(2)} = \sum_{\substack{\omega \in Z_\Gamma \Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\backslash \Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I}, \mathfrak{a}) \\ [\omega] \notin B(F_\infty)}} \psi_1^{(\mathfrak{a})}(n_{2,[\omega]}) \psi_2^{(\mathfrak{a}\mathfrak{b}^2)}(n_{1,[\omega]}) \chi_f(\kappa_{[\omega]})$$

$$\int_{Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}(w a_{[\omega]} g) dg \tag{94}$$

The latter integral depends only on the class of $a_{[\omega]}$ modulo the center $Z(F_\infty)$. We now use the explicit description of $\Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I}, \mathfrak{a})$ from Lemma 14, together with the following simple computation involving the definition of KS (Definition 2):

$$\begin{aligned} \sum_{\omega \to (c,?,\epsilon)} \chi_f(\kappa_{[\omega]}) \psi_1^{(\mathfrak{a})}(n_{2,[\omega]}) \psi_2^{(\mathfrak{a}\mathfrak{b}^2)}(n_{1,[\omega]}) &= \sum_{x \in (\mathfrak{b}/\mathfrak{a}\mathfrak{b}^2(c))^\times} \psi_\infty\left(\frac{\alpha' x + \alpha \epsilon x^{-1}}{c}\right) \chi(\pi_{\mathfrak{b}}^{-1} x) \\ &= \chi(\pi_{\mathfrak{b}})^{-1} KS(\epsilon\alpha, \mathfrak{a}; \alpha', \mathfrak{a}\mathfrak{b}^2; c, \mathfrak{a}\mathfrak{b}) \end{aligned} \tag{95}$$

The first sum is over ω that map to (c, x, ϵ), for *some* x, under the map of Lemma 14; and KS is as in Definition 2. Combining this with $I = I^{(1)} + I^{(2)}$, we get:

$$\begin{aligned} I = {} & \Delta(\alpha, \alpha'\mathfrak{b}^2) 2^{-d_{\mathbb{C}}} \pi^{-d_{\mathbb{R}} - d_{\mathbb{C}}} \chi(\pi_{\mathfrak{b}})^{-1} \mathrm{Norm}(\mathfrak{a}) D_F^{1/2} \\ & \cdot \left(\int_{NZ(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}(\mathrm{diag}([\mathfrak{b}]\epsilon_0, [\mathfrak{b}]^{-1}) g) dg \right) \\ & + \frac{1}{2} \sum_{\substack{c \in \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{I} \\ \epsilon \in \mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2}} \chi(\pi_{\mathfrak{b}})^{-1} KS(\epsilon\alpha, \mathfrak{a}; \alpha', \mathfrak{a}\mathfrak{b}^2; c, \mathfrak{a}\mathfrak{b}) \\ & \cdot \left(\int_{Z(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \overline{f^{(\mathfrak{a})}(g)} f^{(\mathfrak{a}\mathfrak{b}^2)}(w \mathrm{diag}(1, \epsilon c^{-2}) g) dg \right) \end{aligned} \tag{96}$$

The factor $2^{-d_{\mathbb{C}}} \pi^{-d_{\mathbb{R}} - d_{\mathbb{C}}}$ arises from the normalization of measure in [2]. Also, since we are dealing with $Z_\Gamma \Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I}, \mathfrak{a})$, rather than $\Gamma_N(\mathfrak{I}, \mathfrak{a}\mathfrak{b}^2)\backslash\Gamma(\mathfrak{a} \to \mathfrak{a}\mathfrak{b}^2)/\Gamma_N(\mathfrak{I}, \mathfrak{a})$, we replace the range $\epsilon \in \mathfrak{o}_F^*$ (from the parameterization of Lemma 14) by $\epsilon \in \mathfrak{o}_F^*/(\mathfrak{o}_F^*)^2$; this process introduces a $\frac{1}{2}$ due to the "multiplication by -1" symmetry. Note that the Kloosterman sum is invariant under $c \to -c$, since by assumption $\chi_\infty = 1$.

6.4. **Spectral Evaluation.** We can expand the desired inner product spectrally:

$$\langle \pi_{\mathfrak{b}} \cdot P_2, P_1 \rangle = \sum_{\mathbf{f} \in \mathbf{B}(\mathrm{FS})} \overline{\langle P_1, \mathbf{f} \rangle} \langle \pi_{\mathfrak{b}} P_2, \mathbf{f} \rangle + (\text{Eisenstein term})$$

Let $f \in \mathbf{B}(\mathrm{FS})$. Unfolding and using the Iwasawa decomposition:

$$\langle P_1, \mathbf{f}\rangle = \int_{Z(F_\infty)\Gamma_N(\mathfrak{I},\mathfrak{a})\backslash \mathrm{GL}_2(F_\infty)} f^{\mathfrak{a}}(g)\overline{\mathbf{f}_{\mathfrak{a}}(g)}$$

$$= \int_A f^{\mathfrak{a}}(a)a^{-2\rho}da \int_{\Gamma_N(\mathfrak{I},\mathfrak{a})\backslash N(F_\infty)} \psi_1^{(\mathfrak{a})}(n)\overline{\mathbf{f}_{\mathfrak{a}}(ng)}dn \tag{97}$$

Noting that $a_{\mathbf{f}}(\mathfrak{a},\alpha) = a_{\mathbf{f}}(\mathfrak{a},-\alpha)$, the definitions (Section 2.5) give:

$$\int_{\Gamma_N(\mathfrak{I},\mathfrak{a})\backslash N(F_\infty)} \mathbf{f}_{\mathfrak{a}}(na)\overline{\psi_1^{(\mathfrak{a})}(n)}dn = 2^{-d_{\mathbb{C}}}\pi^{-d_{\mathbb{R}}-d_{\mathbb{C}}}a_{\mathbf{f}}(\mathfrak{a},\alpha)\frac{\sqrt{\mathrm{Norm}(\mathfrak{a})}}{\sqrt{\mathrm{Norm}(\alpha)}}W_{\nu_{\mathbf{f}}}(\mathrm{diag}(\alpha,1)a)$$

where $2^{-d_{\mathbb{C}}}\pi^{-d_{\mathbb{R}}-d_{\mathbb{C}}}$ again arises from the measure normalization in [2]. Similarly one evaluates $\langle P_2, \mathbf{f}\rangle$. Note that $\langle \pi_{\mathfrak{b}}P_2, \mathbf{f}\rangle = \omega_{\mathbf{f}}(\pi_{\mathfrak{b}})\langle P_2, \mathbf{f}\rangle$, if $\mathbf{f}$ transforms under the character $\omega_{\mathbf{f}}$ of $Z(\mathbb{A}_F)$. We see:

$$\langle \pi_{\mathfrak{b}}P_2, P_1\rangle = \sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS})} \omega_{\mathbf{f}}(\pi_{\mathfrak{b}})\langle P_2, \mathbf{f}\rangle\overline{\langle P_1, \mathbf{f}\rangle} + \mathrm{CSC} = \tag{98}$$

$$\sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS})} \omega_{\mathbf{f}}(\pi_{\mathfrak{b}})a_{\mathbf{f}}(\mathfrak{a},\alpha)\overline{a_{\mathbf{f}}(\mathfrak{a}\mathfrak{b}^2,\alpha')}$$

$$(2^{-d_{\mathbb{C}}}\pi^{-d_{\mathbb{R}}-d_{\mathbb{C}}})^2\frac{\mathrm{Norm}(\mathfrak{a}\mathfrak{b})}{\sqrt{\mathrm{Norm}(\alpha)\mathrm{Norm}(\alpha')}}\overline{\left(\int_{a\in A} a^{-2\rho}f^{\mathfrak{a}}(a)W_{\nu_{\mathbf{f}}}(\mathrm{diag}(\alpha,1)a)da\right)}$$

$$\left(\int_{a\in A} a^{-2\rho}f^{\mathfrak{a}\mathfrak{b}^2}(a)W_{\nu_{\mathbf{f}}}(\mathrm{diag}(\alpha,1)a)da\right)$$

6.5. **Further analysis.** We will cite the results from [2] that we need. Recall the definition of $\mathcal{B}(x,\nu)$ from Definition 4, and continue to assume that $\psi_1^{(\mathfrak{a})}, \psi_2^{(\mathfrak{a}\mathfrak{b}^2)}$ are given as in (88).

As in [2], given a function $h(\nu) \in \mathcal{H}(M,N)$, we set

$$\varphi(x) = \int_{\nu\in\mathfrak{a}} h(\nu)\mathcal{B}(x,i\nu)d\mu_\nu \tag{99}$$

It is shown in [2] that, given $h(\nu) \in \mathcal{H}(M,N)$, we may find functions $f^{(\mathfrak{a})}, f^{(\mathfrak{a}\mathfrak{b}^2)}$ satisfying (89) so that, for any $\nu \in Y$ one has:

(1)

$$\left(\frac{\pi^2}{2}\right)^{2d_{\mathbb{R}}+2d_{\mathbb{C}}}\mathrm{Norm}(\alpha\alpha')h(\nu) = \overline{\int_A f^{(\mathfrak{a})}(a)W_\nu(\mathrm{diag}(\alpha,1)a)a^{-2\rho}da} \cdot$$

$$\int_A f^{(\mathfrak{a}\mathfrak{b}^2)}(a)W_\nu(\mathrm{diag}(\alpha',1)a)a^{-2\rho}da$$

(2)

$$(\frac{\pi^2}{2})^{d_{\mathbb{R}}+d_{\mathbb{C}}}\pi^{d_{\mathbb{C}}}\varphi(\alpha\alpha' x)\sqrt{|\mathrm{Norm}(x\alpha\alpha')|} = \int_G \overline{f^{(\mathfrak{a})}(g)}f^{(\mathfrak{a}\mathfrak{b}^2)}(w\mathrm{diag}(1,x)g)dg$$

(3) Let $\mathfrak{b}, \epsilon_0$ be so that $\alpha = \alpha'[\mathfrak{b}^2]\epsilon_0$; then

$$\int_A \overline{f^{(\mathfrak{a})}(a)} f^{(\mathfrak{a}\mathfrak{b}^2)}((\mathrm{diag}([\mathfrak{b}]^2\epsilon_0, 1)a)a^{-2\rho}da = \left(\frac{\pi^2}{2}\right)^{d_\mathbb{R}+d_\mathbb{C}} \mathrm{Norm}(\alpha) \int_{\nu\in\mathfrak{a}} h(\nu)d\mu_\nu$$

Indeed, this follows from (25), (26) and Proposition 9.4 of [2], transferring the functions to PGL_2 via the isomorphism of $\mathrm{SL}_2/K_{SL} \equiv \mathrm{PGL}_2/K_{PGL}$; see also [2], Section 10 on "Test Functions." In the notation of [2], we take both $f^{(\mathfrak{a})}, f^{(\mathfrak{a}\mathfrak{b}^2)}$ to be of the form $K_r h$, for appropriate r and $h = h_1, h_2$; in proving the second claim, one needs to treat x positive and negative at real places separately.

6.6. **Final formula.** We put together the geometric evaluation (96), the spectral evaluation (98), and the formulas stated in the previous section. Note that the diagonal term is nonvanishing if and only if $\mathfrak{a}(\alpha) = \mathfrak{a}\mathfrak{b}^2\alpha'$; this allows us to slightly simplify this term. The result is, after some manipulation:

$$(100) \qquad (2^{-d_\mathbb{C}}\pi^{-d_\mathbb{R}-d_\mathbb{C}})^2 \sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS})} \omega_\mathbf{f}(\pi_\mathfrak{b}) a_\mathbf{f}(\mathfrak{a},\alpha)\overline{a_\mathbf{f}(\mathfrak{a}\mathfrak{b}^2,\alpha')} h(\nu) + \mathrm{CSC} =$$

$$\chi(\pi_\mathfrak{b})\left(\frac{1}{2}\cdot\frac{2^{d_\mathbb{R}+d_\mathbb{C}}}{\pi^{2d_\mathbb{R}+d_\mathbb{C}}} \sum_{\substack{c\in\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{I}\\ \epsilon\in(\mathfrak{o}_F^*)/(\mathfrak{o}_F^*)^2}} KS(\alpha,\mathfrak{a};\alpha'\epsilon,\mathfrak{a}\mathfrak{b}^2;c,\mathfrak{a}\mathfrak{b})\varphi(\frac{\epsilon\alpha\alpha'}{c^2})\mathrm{Norm}(c\mathfrak{a}\mathfrak{b})^{-1}\right)$$

$$+2^{-d_\mathbb{C}}\pi^{-d_\mathbb{R}-d_\mathbb{C}}\frac{2^{d_\mathbb{R}+d_\mathbb{C}}}{\pi^{2d_\mathbb{R}+2d_\mathbb{C}}}\chi(\pi_\mathfrak{b})\Delta(\alpha,\alpha'\mathfrak{b}^2)D_F^{1/2}\int_{\nu\in\mathfrak{a}} h(\nu)d\mu_\nu$$

where h and φ are related by (99).

On the other hand, if $\mathfrak{a}, \mathfrak{a}'$ are such that the ideal class of $\mathfrak{a}^{-1}\mathfrak{a}'$ is not the same as that of $\mathfrak{b}^2$, we see by twisting by class group characters (c.f. Remark 3) that:

$$(101) \qquad \sum_{\mathbf{f}\in\mathbf{B}(\mathrm{FS})} \omega_\mathbf{f}(\pi_\mathfrak{b}) a_\mathbf{f}(\mathfrak{a},\alpha)\overline{a_\mathbf{f}(\mathfrak{a}',\alpha')} h(\nu_\mathbf{f}) = 0$$

Finally, we note that, although this is a sum over all of $\mathbf{B}(\mathrm{FS})$, we can restrict to $\mathbf{B}(\mathrm{FS}_\chi)$ by introducing a sum over $\mathfrak{b}$ to "pick out" a given central character:

$$\sum_{\mathbf{f}:\omega_\mathbf{f}=\chi} a_\mathbf{f}(\mathfrak{a}_1,\alpha_1)\overline{a_\mathbf{f}(\mathfrak{a}_2,\alpha_2)} = \frac{1}{h_F}\sum_\mathbf{f} a_\mathbf{f}(\mathfrak{a}_1,\alpha_1)\overline{a_\mathbf{f}(\mathfrak{a}_2,\alpha_2)}\left(\sum_{\mathfrak{b}\in\tilde{C}_F}\omega_\mathbf{f}(\pi_\mathfrak{b})\chi(\pi_\mathfrak{b})^{-1}\right)$$

Every $\mathbf{f} \in \mathbf{B}(\mathrm{FS})$ transforms under $Z(\mathbb{A}_F)$ by a twist of χ by a class group character. Thus, combining the above equation with (100), we obtain after some manipulation (15).

6.7. **Archimedean computation.** We use the notations of Definitions 3 and 4.

Proposition 7. *Let F be a totally real field. Suppose $h \in \mathcal{H}(M,N)$ and $\varphi : F_\infty^\times \to \mathbb{C}$ is the integral transform given by:*

$$\varphi(x) = \int_{\nu \in \mathfrak{a}} B(x, i\nu) h(\nu) d\mu_\nu$$

Define $\underline{\cosh}$ as in Section 2.1. Define $\widehat{h \cdot \underline{\cosh}} : \mathfrak{a} \to \mathbb{C}$ to be the Fourier transform of $h(\nu)\underline{\cosh}(\nu)$, so for $t \in \mathfrak{a}$:

$$\widehat{h \cdot \underline{\cosh}}(t) = \int_{\nu \in \mathfrak{a}} e^{2\pi i \langle \nu, t \rangle} h(\nu) \underline{\cosh}(\nu) d\nu$$

where $\langle t, \nu \rangle = \sum_{v | \infty} t_v \nu_v$. Defining $\log_F : F_\infty^\times \to \mathbb{C}$ as in Section 2.2, one has:

$$\widehat{h \cdot \underline{\cosh}}(\log_F |\kappa|) = \int_{F_\infty} \varphi(\frac{1}{2 + \kappa + \kappa^{-1}} x^2) \psi_\infty(x) \frac{dx}{\mathrm{Norm}(x)}$$

(One is guaranteed convergence as long as M, N are sufficiently large.)

Proof. Since the situation in the presence of multiple real places is a "product" of situations involving just one real place, it suffices to check it in the case that $F_\infty = \mathbb{R}$. Over $\mathbb{R}$, this follows, after some manipulations, from (106) and (112), proven in the next section. Note that, in the notation of that section, $\phi(x) = \varphi(\pm x^2)$. □

Over $\mathbb{C}$, the Proposition (or a slight variant, perhaps using a different version of $\underline{\cosh}$) should still be true. This amounts to a finite integral equality and should be verifiable (and has been in some special cases), but the author has not yet managed to accomplish the general case. We now give the proof over $\mathbb{R}$.

6.7.1. *Proof of Proposition 7 over $\mathbb{R}$.* The question reduces to an explicit computation with Bessel transforms, that we shall do as directly as possible. Throughout this section, suppose $\nu, s \in \mathbb{C}$.

Set $B_\nu(x) = \frac{1}{\sin(\pi\nu/2)}(J_{-\nu}(x) - J_\nu(x))$ and as usual $K_\nu(x) = \frac{\pi}{2} \frac{(I_{-\nu}(x) - I_\nu(x))}{\sin(\nu\pi/2)}$. We treat the cases of the Proposition corresponding to $\kappa > 0$ and $\kappa < 0$ separately; they correspond to transforms involving B and K, respectively.

Let h be a holomorphic even function in the strip $|\Im(z)| \le M$, satisfying decay estimates

$$|h(i\sigma + t)| \ll e^{-\pi|t|}(1 + |t|)^{-N} \quad (|\sigma| \le M) \tag{102}$$

6.7.2. *The B-transform.* Set for $x > 0$

$$\phi(x) = \frac{1}{2} \int_{\nu=-\infty}^{\infty} B_{2i\nu}(4\pi x) h(\nu) \nu \sinh(\pi\nu) d\nu = i \int_{-\infty}^{\infty} t J_{2it}(4\pi x) h(t) dt \tag{103}$$

Then one has decay estimates

$$|\phi(x)| \ll \min(x^{2M}, x^{-1/2}) \tag{104}$$

Indeed, suppose $s = \sigma + it \in \mathbb{C}$ is so that σ is constrained to a fixed compact subset of $(-1/2, \infty)$. One then has an estimate, uniform in x and $s = \sigma + it$:

$$\left| \frac{J_{\sigma+it}(x)}{\cosh(\pi t)} \right| \ll (1 + |t|)^2 \min(x^\sigma, x^{-1/2}) \tag{105}$$

Indeed, this follows for $x \gg 1 + |t|^2$ directly from the asymptotics and remainders given in Section 7.21 of [14], whereas in the region $x \ll 1 + |t|^2$ one may use any of several integral representations for the J-Bessel function (for instance use [5], Section 8.41, formula (13) in the range $1 \leq x \ll 1 + |t|^2$, and formula (4) in the range $x \leq 1$.) From this, together with holomorphic extension of h (shift the line of integration) one verifies the claimed decay estimates on ϕ.

We now wish to evaluate the Fourier cosine transform of $\phi(x)/x$; this is given by an absolutely convergent integral in view of (104). Applying (103):

$$\int_0^\infty \cos(2\pi kx)\phi(x)x^{-1}dx = i\int_0^\infty x^{-1}dx\cos(2\pi kx)\left(\int_{-\infty}^\infty tJ_{2it}(4\pi x)h(t)dt\right)$$

Let $\sigma > 0$ be "small" – say $\sigma < 1/2$. We may shift the line of integration in the inner integral to $\Im(t) = -\sigma$, obtaining:

$$\int_0^\infty \cos(2\pi kx)\phi(x)x^{-1}dx = i\int_0^\infty x^{-1}\cos(2\pi kx)dx\int_{-i\sigma-\infty}^{-i\sigma+\infty} sJ_{2is}(4\pi x)h(s)ds$$

Now, making crucial use of (105) and (102), one verifies that the double integral is absolutely convergent; in particular, one may switch orders of integration to obtain:

$$\int_0^\infty \cos(2\pi kx)\phi(x)x^{-1}dx = i\int_{-i\sigma-\infty}^{-i\sigma+\infty} sh(s)ds\int_{-\infty}^\infty x^{-1}J_{2is}(4\pi x)\cos(2\pi kx)dx$$

The inner integral is computed (see [5]); the answer is distinct in the regions $k \geq 2$ and $k < 2$, but we only need the former.

$$\int_0^\infty \phi(x)\cos(2\pi kx)x^{-1} = \frac{1}{2}\int_{-i\sigma-\infty}^{-i\sigma+\infty} ds(h(s)\cosh(\pi s))(k/2+\sqrt{\frac{k^2}{4}-1})^{-2is} \quad (k > 2)$$

Utilizing again (102), we are now free to move the line of integration back to $\Im(s) = 0$, obtaining:

$$\int_0^\infty \phi(x)\cos(2\pi kx)x^{-1}dx = \frac{1}{2}\int_{-\infty}^\infty ds(h(s)\cosh(\pi s))(\frac{k+\sqrt{k^2-4}}{2})^{-2is} \quad (k > 2) \tag{106}$$

6.7.3. *The K-transform.* In this case we must reason slightly differently to the reasoning above, mainly for the reason of availability of necessary references.

Now set

$$\phi(x) = \frac{1}{\pi}\int_\nu h(\nu)K_{2i\nu}(4\pi x)\nu\sin(\pi\nu)d\nu = i\int_{-\infty}^\infty tI_{2it}(4\pi x)h(t)dt \tag{107}$$

In a similar manner to (105), one may establish the following estimate on the K-Bessel function: For every $k > 0$ one has, uniformly for $\Re(s)$ in a compact set and for $x \geq 1$:

$$K_s(x) \ll (1+x)^{-k} \quad (x \geq 1) \tag{108}$$

The implicit constant depends on the compact set.

This implies that $\phi(x) \ll (1+|x|)^{-k}$ for any $k > 0$ and for $x \geq 1$. On the other hand, one may shift lines of integration in the second equality of (107); this (and similar estimates on the I-Bessel function) shows that $|\phi(x)| \ll x^{2M}$. Thus:

$$|\phi(x)| \ll \min(x^{2M}, (1+x)^{-k}) \tag{109}$$

Note in particular that now $\phi(x)$ is rapidly decaying at ∞. In particular, (109) guarantees enough regularity that we may apply Lebedev-Kontorovitch inversion to (107), obtaining:

$$h(\nu) = \frac{2}{\pi} \int_0^\infty K_{2i\nu}(x)\phi(\frac{x}{4\pi})x^{-1}dx = \frac{2}{\pi}\int_0^\infty K_{2i\nu}(4\pi x)\phi(x)x^{-1}dx$$

Applying the integral representation $K_\nu(x) = \frac{\int_0^\infty \cos(x\sinh(t))\cosh(\nu t)dt}{\cos(\nu\pi/2)}$, we obtain:

$$\cosh(\pi\nu)h(\nu) = \frac{2}{\pi}\int_0^\infty dx\, \phi(x)x^{-1}\int_0^\infty \cos(4\pi x\sinh(t))\cos(2\nu t)dt \tag{110}$$

This integral is not absolutely convergent; nevertheless we may switch orders of integration. Indeed, the integral is absolutely convergent if one replaces the inner integral by an integral from 0 to T, for $T < \infty$, and one obtains:

$$\begin{aligned} &\int_0^\infty dx\phi(x)x^{-1}\int_0^T \cos(4\pi x\sinh(t))\cos(2\nu t)dt \\ &= \int_0^T \cos(2\nu t)dt\int_0^\infty \phi(x)x^{-1}\cos(4\pi x\sinh(t))dx \end{aligned} \tag{111}$$

and it remains to verify that one may take the limit as $T \to \infty$ on both sides. However some substitution and integration by parts demonstrates: $|\int_T^\infty \cos(4\pi x\sinh(t))\cos(2\nu t)dt| \ll_\nu (x\log(T))^{-1}$ and $|\int_0^\infty \phi(x)x^{-1}\cos(4\pi x\sinh(t))dx| \ll (\sinh(t))^{-1}$. Together with (109), this verifies that it is admissible to switch orders in (110), giving:

$$\cosh(\pi\nu)h(\nu) = \frac{2}{\pi}\int_{-\infty}^\infty \cos(2\nu t)d\nu\left(\int_0^\infty x^{-1}\phi(x)\cos(4\pi x\sinh(t))dx\right)$$

It is easy to check this cosine transform may be inverted, yielding:

$$\frac{1}{2}\int_{-\infty}^\infty \cosh(\pi\nu)h(\nu)\cos(2\nu t)d\nu = \int_0^\infty x^{-1}\phi(x)\cos(4\pi x\sinh(t))dx$$

and thus after some manipulation:
(112)

$$\int_0^\infty x^{-1}\phi(x)\cos(2\pi kx)dx = \frac{1}{2}\int_{-\infty}^{\infty}\cosh(\pi\nu)h(\nu)(\frac{-k+\sqrt{k^2+4}}{2})^{2i\nu}d\nu$$

We note that it is presumably true that one can give a similar proof in the case of the B-transform, but published proofs of the inversion formula in that case require somewhat better estimates than (104).

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