MODULI SPACES AND DEFORMATION THEORY, CLASS 16

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Contents

Again, I'll be out of town next Tuesday (Nov. 7), so Jason Starr will kindly be giving a guest lecture. He will likely tell you about applying Schlessinger's criteria to the Quot functor. I also encourage you to check out his talk at the Harvard-MIT algebraic geometry seminar (meeting that day at Harvard at 3 pm).

1. Where we are: Schlessinger's criterion for existence of universal deformations and hulls (miniversal deformations)

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

Schlessinger's Theorem. [Put on one board permanently!]

(1) F has a hull iff F has properties H1–H3:

- H1. (You can glue.) (1) is a surjection whenever $A'' \to A$ is a small extension. Equivalently whenever $A'' \to A$ is *any* surjection.
- H2. (Uniqueness of gluing on $k[\epsilon]/\epsilon^2$.) (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$. Equivalently, $A'' = k[V]$. Then by previous lemma, t_F is a k-vector space.
- H3. (finite-dimensional tangent space) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property

Date: Thursday, November 2, 2000.

H4. (bijection for gluing a small extension to itself)

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

is a *bijection* for any small extension $A' \to A$.

Recall from two times ago, repeated last time. Assume F satisfies H1–H3. Now given a fairly small extension $p : A' \to A$. Given any $a \in F(A)$, i.e. family over A, the set of lifts to $F(A')$ has a transitive action by the group $t_F \otimes I$. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)

2. Proof of Schlessinger, take 3

We've proved everything except that if H1–H3 are satisfied, then we have a hull, and we're much of the way through dealing with that.

We have used H1–H3 to build a candidate hull. Here were the steps so far. By H2 and H3, t_F is a finite-dimensional vector space. Let $T_1, \dots, T_r$ be a dual basis for t_F. Let $S = k[[T_1, \dots, T_r]]$, with maximal ideal $\mathfrak{n}$. We constructed $R = S/J$, with $r \in F(R)$, as follows.

We constructed $R_q = S/J_q$ and $r_q \in R_q$ for $q = 2, 3, \ldots$ $J_2 = \mathfrak{n}^2$; we had a canonical family over R_2, i.e. a canonical element $r_2 \in F(R_2)$.

Suppose we have (R_q, r_q), where $R_q = S/J_q$. We want J_{q+1} in S, minimal among $\mathcal{S} =$ (ideals J in S satisfying (a) $\mathfrak{n}J_q \subset J \subset J_q$, and (b) r_q lifts to S_J).

We then let $J = \cap_{q>1} J_q$, and $R = S/J$, and let r be the inverse limit of the r_q.

This will be our hull; we'll check that now.

We immediately have $h_R \to F$. Note that $t_F \cong t_R$ by construction. It remains to check that this is formally smooth, i.e. whether we always have lifts in the following situation:

$$\begin{array}{ccc} & & (A', a' \in F(A')) \\ & \overset{u'?}{\nearrow} & \downarrow p \, surjective \\ (R, r \in F(R)) & \overset{u}{\to} & (A, a \in F(A)) \end{array}$$

We can just deal with p small.

I claim it suffices to find that this is formally smooth.

$$\begin{array}{ccc} & & (A', b' \in F(A')) \\ & \overset{v'?}{\nearrow} & \downarrow p \\ (R, r \in F(R)) & \overset{u}{\to} & (A, a \in F(A)) \end{array}$$

for *any* b'. That's because we have a transitive action of $t_F \otimes I$ on $h_R(p)^{-1}(a)$ and $F(p)^{-1}(a)$ (by earlier comment) and a surjection $h_R(p)^{-1}(a) \to F(p)^{-1}(a)$ that respects this action, so by choosing $\sigma \in t_F \otimes I$ to change b' to a', and applying it to get v', we get the desired u'.

So now we have just to lift

$$\begin{array}{ccc} & & A' \\ & \overset{u'?}{\nearrow} & \downarrow \quad p \text{ small} \\ R & \overset{u}{\to} & A \end{array}$$

Now u factors through $R \to R_q \to A/I$ for some q, so it suffices to complete

$$\begin{array}{ccc} R_{q+1} & \overset{u'?}{\to} & A' \\ \downarrow & & \downarrow p \\ R_q & \overset{u}{\to} & A \end{array}$$

This is all that's left in the entire proof — and it's an explicit algebra question.

Choose *any* lift $S \to A'$ (possible as S is a power series). Rewrite as:

$$\begin{array}{ccccc} S & \overset{w}{\to} & R_q \times_A A' & \to & A' \\ \downarrow & \overset{v'}{\nearrow} & \downarrow \pi_1 \text{ small} & & \downarrow \quad p \text{ small} \\ R_{q+1} & \to & R_q & \overset{u}{\to} & A. \end{array}$$

If π_1 has a section, then we win. Otherwise, we use the following aside.

2.1. **Commutative algebra aside: Notation that will only be used once.**

Definition. A surjection $p : B \to A$ in $\mathcal{C}$ is *essential* if for any $q : C \to B$ such that pq is surjective, it follows that q is surjective.

Easy Lemma. Suppose $p \in \mathcal{C}$ is surjective. Then

(i) p is essential iff $p_* : t_B^* \to t_A^*$ is an isomorphism.
(ii) If p is small, then p is essential iff p has no section $s : A \to B$ (i.e. with $A \overset{s}{\to} B \overset{p}{\to} A$ the identity).

Now π_1 has no section, so it is essential. Hence $S \overset{w}{\to} R_q \times_A A'$ is a surjection. By H1, r_q lifts to something in $F(R_q \times_A A')$. Thus $\ker w \supset J_{q+1}$ by definition of J_{q+1}. So w factors thorugh R_{q+1} and v exists! We're done! □

3. Example: The Picard Functor

You'll see three examples: the Picard functor, the Functor of deformations of a scheme X, and the Quot functor (Jason). First, the Picard functor.

Here's the Picard functor I want to consider. Fix a scheme X. Recall the Picard group $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$ (Cech cohomology). In fact this is a group scheme.

Side comment to Jim and Sharon: We have infinitesimal automorphisms, so we don't expect prorepresentability if we hadn't done this "modulo isomorphism" thing.

For convenience, let $X_A = X \times_k \operatorname{Spec} A$. Fix $\mathcal{L}_0 \in \operatorname{Pic} X$. We will study deformations of this line bundle.

Let $P(A)$ be

$$\left\{ \begin{array}{rccc} & \mathcal{L} & \to & \mathcal{L}_0 \\ & | & & | \\ X \times \operatorname{Spec} A = & X_A & \to & X \end{array} \right\} /\text{isom.}$$

Those are the families; morphisms are pullback diagrams.

Theorem. Assume $H^0(X, \mathcal{O}_X) = k$, and $h^1(X, \mathcal{O}_X)$ is finite (e.g. if X is proper and connected). Then P is prorepresentable and $t_p \cong H^1(X, \mathcal{O}_X)$.

3.1. **Important remark: Relative version.** This result is more interesting in a relative situation, if X is deforming at the same time. Fix

$$\begin{array}{ccc} X_0 & \to & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spf} \Lambda \end{array} \quad \text{flat}$$

where Λ is a complete local Noetherian ring. (For example, $k[[x_1, \dots, x_n]]$; or else, $k = \mathbb{F}_p$, $\Lambda = \mathbb{Z}_p$.)

Schlessinger's theorem applies in this relative setting; the proof is actually identical to the one I've given to the more limited version. (I decided to just state the simpler version to keep notation to a minimum, as it's already pretty hairy.)

Theorem. If $H^0(X_0, \mathcal{O}_{X_0}) = k$, and $h^1(X_0, \mathcal{O}_{X_0})$ is finite, then P is prorepresentable (and $t_P = H^1(X_0, \mathcal{O}_{X_0})$).

What this means: $\operatorname{Spf} R \to \operatorname{Spf} \Lambda$.

The proof to this is also identical to the proof I'm going to give to the more limited case.

4. Flatness lemmas

In order to prove this theorem, I'll need a couple of flatness lemmas. The first is easy and fun. The second will need motivation, but is also easy once you parse the complicated diagram.

Fun lemma. Let A be a ring, J a nilpotenti ideal in A (e.g. $A \in \mathcal{C}$, $J \neq (1)$), and $u : M \to N$ a homomorphism of A modules, N flat over A. If $\overline{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism too.

Proof. (Let K be the cokernel, and show that it is zero. Then let K' be the kernel, and use flatness to show that it is zero too; see Schlessinger's paper.) □

Fun corollary. A module N over an Artinian ring A is flat if and only if it is free.

This is great, because flatness is sometimes scary, but freeness isn't!

Proof. Free modules are clearly flat, so that's easy. Suppose N is flat, and say $N/\mathfrak{m}N = k^n$ (where $\mathfrak{m}$ is the max ideal and k the res field of A; n needn't be finite). Let $M = A^n$, and pick $M \to N$ sending basis vectors of M to lifts of basis vectors of $N/\mathfrak{m}N$. By the lemma, this is an isomorphism. □

The next lemma looks scarier, but is fundamental, and in fact well-motivated. I'm going to try to motivate it, and then state it precisely. The proof will then be shorter than the statement!

In Schlessinger's criteria. we have a situation like this. We have a diagram of rings

$$\begin{array}{ccc} B := A' \times_A A'' & \to & A'' \\ \downarrow & & \downarrow \text{ surjection} \\ A' & \to & A \end{array}$$

and you want to check that if you have elements of $F(A')$ and $F(A'')$ restricting to the same thing in $F(A)$, then there is something in $F(B)$ mapping to it (H1), and in fact that it is uniquely defined (H2 and H4). (Say again in terms of gluing.) These functors in practice are often flat families of some sort. So you can imagine having a flat families over A' and A'' (restricting to the same thing over A), and that you can glue them together to get a family over B. Million dollar question: how do you know that the resulting family is flat?!! This lemma answers that question.

The lemma will be in terms of modules over a ring, as (a) that's the natural simplest way in which to discuss flatness, and (b) you really cook up families of all sorts by having modules or sheaves of modules with given additional structures.

Lemma.

The bottom square is

$$\begin{array}{ccc} B := A' \times_A A'' & \to & A'' \\ \downarrow & & \downarrow \text{ surjection} \\ A' & \to & A \end{array}$$

(in $\mathbb{C}$).

Over it, you have

$$\begin{array}{ccc} ? & \rightarrow & M'' \\ \downarrow & & \downarrow \quad u'' \\ M' & \stackrel{u'}{\rightarrow} & M \end{array}$$

Each of these three squares (incident to A in the cube) are tensor products, i.e. $M \cong M' \otimes_A A'$ (via u'), and $M \cong M'' \otimes_A A''$ (via u'').

Also, M' is a flat A'-module and M'' is a flat A''-module, and M is a flat A-module.

It's easy to fill in the upper corner with N: $N = M' \times_M M''$, which is indeed a B-algebra:

$$\begin{array}{ccc} N = M' \times_M M'' & \rightarrow & M'' \\ \downarrow & & \downarrow \\ M' & \rightarrow & M \end{array}$$

Then: N is a flat B-module! And the remaning two squares are pullback squares, i.e. $N \times_B A' \xrightarrow{\sim} M'$ and $N \times_B A'' \xrightarrow{\sim} M''$.

In terms of pseudo-geometry: we have this glued together family, and it has the properties we want: it is a flat family, and these two squares are pullback squares.

The proof of the lemma is actually shorter than the statement!

Proof. In our situation, flat is free by the fun corollary, so M' is free; choose a basis $(x'_i)_{i \in I}$ for it.

Then $M = M' \otimes_A A'$, so M is free too with basis $u'(x'_i)$.

Now as $A'' \to A$ is a surjection, M is M'' modulo J (the kernel of the surjection). Choose any lift $x''_i \in M''$ such that $u''(x''_i) = x_i$. Then there's a map of A''-modules $\sum A'' x''_i \to M''$ of A'' whose reduciton mod J is an isomorphism, so it is an isomorphism by the fun lemma. So M'' is free (which we already knew), with generators x''_i.

It follows quickly that $N = M' \times_M M''$ is free on generators $x'_i \times x''_i$ (hence flat), and that projections on the factors induce isomorphisms $N \times_B A' \xrightarrow{\sim} M'$ and $N \times_B A'' \xrightarrow{\sim} M''$. □

That lemma will be useful for testing H1, i.e. it shows existence of a gluing. We also want uniqueness for H2 and H4, and for this we will use

Corollary. With the same notation as above, let L be a B-module in a commutative diagram

$$\begin{array}{ccccc} & L & \stackrel{q''}{\rightarrow} & M'' & \\ q' & \downarrow & & \downarrow & u'' \\ & M' & \stackrel{u'}{\rightarrow} & M & \end{array}$$

such that q' induces an *iso*morphism $L \times_B A' \to M'$. Then the canonical morphism $q' \times q'' : L \to M' \times_M M'' = N$ is an isomorphism.

Proof. This follows by applying the fun lemma to $q' \times q''$. $B \to A'$ is a surjection (as $A'' \to A$ was), and $q' \times q''$ is an isomorhpism modulo that ideal, so that's it. $\square$

Next day (i.e. next Thurs.): Proof of existence of hull of the Picard functor