MODULI SPACES AND DEFORMATION THEORY, CLASS 15

RAVI VAKIL

Contents

1. Where we are 1
2. Schlessinger's criterion for existence of universal deformations and hulls (miniversal deformations) 2
3. Proof of Schlessinger, take 2 3

In case I forget to tell you on Thursday: I'll be out of town next Tuesday (Nov. 7), so Jason Starr will kindly be giving a guest lecture. He will likely tell you about applying Schlessinger's criteria to the Quot functor.

1. Where we are

$\mathcal{C}$, $\hat{\mathcal{C}}$. $k[V]$. $A \times_C B$. $k[V \oplus W] = k[V] \times_k k[W]$.

Lemma. Suppose F is a functor (covariant, on $\mathcal{C}$) such that

$$F(k[V] \times_k k[W]) \xrightarrow{\sim} F(k[W]) \times F(k[W])$$

for *finite-dimensional* vector spaces V and W over k. Then $F(k[V])$ and in particular $t_F = F(k[\epsilon])$, has a canonical vector space structure, such that $F(k[V]) \cong t_F \otimes V$.

Patched proof. $k[V]$ is a "vector space object" in $\hat{\mathcal{C}}$, via

$$\mathrm{Hom}(A, k[V]) \cong \mathrm{Der}_k(A, V).$$

For the last statement: Note that $\mathrm{Hom}(k[\epsilon]/\epsilon^2, k[V])$ is naturally identified with V. For any element of $\mathrm{Hom}(k[\epsilon]/\epsilon^2, k[V])$ we get a map $t_F = F(k[\epsilon]/\epsilon^2) \to F(k[V])$, hence $t_F \times V \to F(k[V])$. In fact this is $\otimes$. The desired result is true if V is one-dimensional; then use induction, as V is finite-dimensional, and $k[V] = \times_k^{\dim V} k[\epsilon]/(\epsilon^2)$. (I said something wrong in class.) □

Date: Tuesday, October 31, 2000.

2. Schlessinger's criterion for existence of universal deformations and hulls (miniversal deformations)

Recall: In $\mathcal{C}$, define a *small extension* to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''} I = 0$, and I is one-dimensional.

For the purposes of this course *only*, define a *fairly small extension* to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''} I = 0$, without requiring that I is one-dimensional. Then for any A in $\mathcal{C}$, you can filter A into a series of fairly small extensions (by powers of the maximal ideal). Then you can filter A into a series of small extensions.

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

I explained last time that if F is a prorepresentable functor, by $R \in \mathcal{C}$ say, then this map is always a bijection — this is because $\times$ is a categorical product!

Schlessinger's Theorem. [Put on one board permanently!]

It has two parts, and I'll say it slowly, with translations and remarks.

(1) F has a hull iff F has properties H1–H3:

- H1. (You can glue.) (1) is a surjection whenever $A'' \to A$ is a small extension. Equivalently whenever $A'' \to A$ is *any* surjection.
- H2. (Uniqueness of gluing on $k[\epsilon]/\epsilon^2$.) (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$. Equivalently, $A'' = k[V]$. Then by previous lemma, t_F is a k-vector space.
- H3. (finite-dimensional tangent space) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property

- H4. (bijection for gluing a small extension to itself)

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

is a *bijection* for any small extension $A' \to A$.

Recall from last time. Suppose F satisfies H1–H3.

Consider any fairly small extension $p : A' \to A$, i.e. $0 \to I \to A' \to A \to 0$, so $m_{A'} I = 0$. We have an isomorphism

$$A' \times_{A'/I} A' \xrightarrow{\sim} A' \times_k k[I]$$

induced by the map $(x, y) \mapsto (x, x_0 + y - x)$ (e.g. $A' = k[u]/u^3 = k[v]/v^3$, $A = k[u]/u^2$).

Now given a small extension $p : A' \to I$, By H2, we get

$$F(A' \times_A A') = F(A' \times_k k[I]) \xrightarrow{\sim} F(A') \times_{F(k)} F(k[I]) = F(A') \times (t_F \otimes I).$$

Hence we get

$$F(A') \times (t_F \otimes I) \to F(A') \times_{F(A)} F(A').$$

For each $\eta \in F(A)$, this determines a group action of $t_f \otimes I$ on $F(p)^{-1}(\eta)$, i.e. those $F(A')$'s lifting $F(A)$, assuming the set is nonempty. The fact that this is a surjection (H1) means that the action is transitive. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)

So explicitly, what this is telling us is explicitly is that if F already has a hull, then its obstruction to be representable is the existence of an automorphism of an object y in some $F(A)$, that cannot be extended to an automorphism of some object $y' \in F(A')$ for some A'.

3. Proof of Schlessinger, take 2

• Immediate: If F is prorepresentable, then H1–H4 are all satisfied, as (1) is *always* a bijection, and t_F is finite-dimensional.

• Hull and H4 imply prorepresentable (shown last time).

• Next: Then I'll show that hull implies H1–H3.

• Finally: H1–H3 imply hull.

Hull implies H1–H3.

Recall the definition of a hull or miniversal deformation space (e.g. def space of a node). $(R, r \in F(R)) \to F$, formally smooth.

Translation: for each surjection $B \to A$ in $\mathcal{C}$, $\mathrm{Hom}(R, B) \to \mathrm{Hom}(R, A) \times_{F(A)} F(B)$ is surjective.

Translation:

$$\begin{array}{ccc} & & (B, b \in F(B)) \\ & ? \nearrow & \downarrow \\ (R, r \in F(R)) & \to & (A, a \in F(A)) \end{array}$$

(Note phantom left corner.)

H3: $t_R \cong t_F$ proves H3.

H1: Want:

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

is a surjection whenever $A'' \to A$ is a small extension (or indeed any surjection).

Suppose we have $p' : (A', \eta' \in F(A')) \to (A, \eta)$ and $p'' : (A'', \eta'') \to (A, \eta)$ are morphisms of couples or deformations, and p'' is a surjection. (Put in diagram, A' to left, A'' above).

$$\begin{array}{ccc} (A' \times_A A'', ?) & \to & (A'', a'') \\ \downarrow & & p'' \downarrow \\ (A', a') & \stackrel{p'}{\to} & (A, a) \end{array}$$

Since $h_R \to F$ is surjective, there is $u' : (R, r) \to (A', \eta')$. By smoothness of h_R over F, we get $u'' : (R, r) \to (A'', \eta'')$. We get R maps to $A' \times_A A''$; use covariant F to get η there. So H1 is satisfied!

H2 (restate). Suppose now that $(A, \eta) = (k, pt)$ and $A'' = k[\epsilon]/\epsilon^2$. We want there to be only one such $(A' \times_A A'', \eta)$. Suppose there are two, η_1 and η_2. Then make that diagram, add in (R, r). Choose u' as before. Then there is only one u'' possible, thanks to $t_R = t_F$. Then there is only one lift $R \to A' \times_A A''$, and that's it.

Diagram (mostly for my own benefit), of maps from (R, r) to various parts of the diagram.

$$\begin{array}{ccc} \text{second, lift by smoothness of } h_R \to F & \to & \text{finally, unique by } t_R = t_F \\ \downarrow & & \downarrow \\ \text{pick } u \text{ first} & \to & \end{array}$$

Finally, H1–H3 implies hull. This is the trickiest part of the proof. Most of it has the same level of complexity, but there's a trick at the end.

Assume H1–H3. We will build a hull. By H2, t_F is a vector space, and by H3, it is finite-dimensional. Let $T_1, \dots, T_r$ be a dual basis for t_F. Let $S = k[[T_1, \dots, T_r]]$, with maximal ideal $\mathfrak{n}$. Our goal is to get $R = S/J$, with $r \in F(R)$, that is a hull.

$R_2 = S/\mathfrak{n}^2$. (Think: $S/(J, \mathfrak{n}^2)$.) $= k[\epsilon]/\epsilon^2 \times_k \cdots \times_k k[\epsilon]/\epsilon^2$.

By H2, there is some $r_2 \in F(R_2)$ inducing a bijection between t_{R_2} (which is $\mathrm{Hom}(R_2, k[\epsilon])$) and t_F. (Say this differently! Use the lemma; get canonical family over this ring.)

Now we build up R inductively.

Suppose we have (R_q, r_q), where $R_q = S/J_q$. We want J_{q+1} in S, minimal among $\mathcal{S} =$ (ideals J in S satisfying (a) $\mathfrak{n}J_q \subset J \subset J_q$, and (b) r_q lifts to S_J). Those satisfying J correspond to subspaces of $J_q/\mathfrak{n}J_q$. (Picture?)

So we need to show that there is a minimal element. So we need to see that $\mathcal{S}$ is stable under pairwise intersection. Say $J, K \in \mathcal{S}$. Enlarge J so that $J + K = J_q$, without changing their intersection. Then

$$S/J \times_{S/J_q} S/K \cong S/(J \cap K)$$

If you want an example: $J_q = (x, y)^2$. $J = (x^2, xy, y^3)$, $K = (x^3, xy, y^2)$.

By H1, $F(S/J \cap K) \to F(S/J) \times_{F(S/J_q)} F(S/K)$ surjects, so that's okay.

So now let $J = \cap_{q>1} J_q$, and $R = S/J$. Let r be the inverse limit of the r_q. (As J_q form a base for the topology, as $\mathfrak{n}^q \subset J_q$, can set $r = \lim_{\leftarrow} r_q \in F(R)$.) This will be our hull; we'll check that next day.