INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 6

Due Thursday October 28 in class (no lates). Hand in seven of the following questions. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument). Ask someone else about the non-scheme problems you skip.

The "miscellaneous" questions are more straightforward. The problems on the Hilbert polynomial and Veronese embeddings are harder, but there are fast short-cuts (which aren't necessarily easy to find).

Miscellaneous.

1. (a) Consider two varieties, $\mathbb{P}^1$ and $\mathbb{A}^1$. Let one of the standard affine opens of $\mathbb{P}^1$ have coordinate t, and let x be a coordinate on $\mathbb{A}^1$. Find a dominant rational map $f : \mathbb{P}^1 \dashrightarrow \mathbb{A}^1$ corresponding to the morphism of function fields $k(\mathbb{A}^1) = \overline{k}(x) \to k(\mathbb{P}^1) = \overline{k}(t)$ given by $x \to t^2$.
   (b) If X is a prevariety, give a natural identification between elements of $k(X)$ and the rational maps from X to $\mathbb{P}^1$.
2. (a) In Problem Set 4, you showed that the function field of the hypersurface X given by $wx = yz$ in $\mathbb{A}^4$ was isomorphic to $\overline{k}(t_1, t_2, t_3)$. By comments in class, X should be rational, i.e. birational to $\mathbb{A}^3$. Prove that it is.
   (b) If X is the line with the doubled origin, show that the diagonal $\Delta(X)$ is not closed in $X \times X$.
3. Consider the morphism $f : \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ given by $(x, y) \mapsto (x; y)$. Show that there is no morphism $\mathbb{A}^2 \to \mathbb{P}^1$ extending f.

Projective varieties: Hilbert functions, dimension, and degree. Suppose V is a projective variety in $\mathbb{P}^n$ (with projective coordinates $x_0, \dots, x_n$) with projective coordinate ring $R = \overline{k}[x_0, \dots, x_n]/I = \oplus_d R_d$, where R_d is the degree d part of R. It is a fact that there is a polynomial $f(d)$ and an integer d_0 such that $\dim R_d = f(d)$ for $d \geq d_0$ (see Lang's Algebra, the section on the Hilbert polynomial). $f(d)$ is called the *Hilbert polynomial* of $V \subset \mathbb{P}^n$. The *dimension* of V can be defined as the degree of $f(d)$ (although a proof is required to show that is the same as the definition I'll give in the course). The *degree* of V can be defined as $(\dim V)!$ times the leading coefficient of $f(d)$. In the next five problems, use these definitions of dimension and degree. (Another, more intuitive, definition of degree is: Intersect V with $\dim V$ randomly chosen hyperplanes, and count the points of intersection.)

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4. Calculate the Hilbert polynomial when $V = \mathbb{P}^n$. Verify that $\dim V = n$ and $\deg V = 1$. If V is a codimension m *linear subspace*, i.e. cut out by m linear independent linear forms in the x_i, verify that $\dim V = n - m$ and $\deg V = 1$.
5. Suppose V is cut out by a single equation $q(x_0, \ldots, x_n) = 0$, where q is irreducible and homogeneous of degree e, so $R = \overline{k}[x_0, \ldots, x_n]/(q)$. Calculate the Hilbert polynomial of $V \subset \mathbb{P}^n$, and verify that $\dim V = n - 1$ and $\deg V = e$.
6. *(The beginnings of intersection theory.)* Suppose $V' \subset \mathbb{P}^n$ is a projective variety of dimension D and degree d, with coordinate ring $R = \overline{k}[x_0, \ldots, x_n]/I(V')$, and suppose that q (as in problem 5) cuts out an irreducible subvariety W of V' (distinct from V'), which is equivalent to q being irreducible in the ring R. Prove that $\dim W = D - 1$ and $\deg W = de$.

The degree d Veronese embedding of $\mathbb{P}^n$. Consider the morphism

$$\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

given by $(x_0^d; x_0^{d-1}x_1; \ldots; x_n^d)$, where $(x_0^d; x_0^{d-1}x_1; \ldots; x_n^d)$ includes all degree d monomials in $x_0, \ldots, x_n$.

7. Verify that it is a morphism, and an isomorphism onto its image V.
8. If $R = \oplus_{e \geq 0} R_e$ is the projective coordinate ring of V, verify that R_e can be identified with the vector space of (homogeneous) degree de polynomials in the x_i. Hence verify that $\dim V = n$ and $\deg V = d^n$.

Schemes: nilpotents.

9. (a) If $f \in R$ is nilpotent, describe $D(f)$ in $\operatorname{Spec} R$.
   (b) A scheme $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every P in X, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements. (Translation: "reducedness" is a local property, in the sense that you can check it on points.)
10. Explain intuitively how the morphisms from $\operatorname{Spec} \overline{k}[t]/(t^2)$ (which has only one point) to $\mathbb{A}^2$ (over the field $\overline{k}$) correspond to the data of a point of $\mathbb{A}^2$, and a tangent vector of $\mathbb{A}^2$. Using this "definition", calculate the dimension of the tangent space to $\mathbb{A}^n$ at the origin; and the dimension of the tangent space to the curve $y^2 = x^2 + x^3$ in $\mathbb{A}^2$ at the origin.