INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 2

Due Tuesday September 28 in class. No lates will be accepted, so Ana-Maria will be able to return it in time for Thursday's class. Throughout this set, we will only deal with morphisms between affine algebraic sets, so use the "naive" definition which works in this case.

1. Prove that any non-empty open subset of an irreducible space is irreducible and dense.
2. Suppose X is an algebraic subset of $\mathbb{A}^n$. Show that $I(X)$ is a prime ideal if and only if X is irreducible.
3. Suppose Y is a Noetherian topological space. Prove that it can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i in a unique way (up to relabelling) assuming no redundancies (i.e. $Y_i \subseteq Y_j$ implies $i = j$).
4. (a) Let Y be the twisted cubic that is the subset (t, t^2, t^3) in $\mathbb{A}^3(\overline{k})$. Show that $A(Y)$ (the ring of regular functions on Y, which you essentially calculated in the last set) is isomorphic to $\overline{k}[x]$. As this is an integral domain, Y is irreducible.
   (b) Let Z be the plane curve $xy = 1$ (in $\mathbb{A}^2(\overline{k})$). Show that $A(Z)$ is *not* isomorphic to $\overline{k}[t]$. Prove that Z is irreducible.
   (c) Let Z be the plane curve $y^2 = x(x-1)(x-\lambda)$ (where $\lambda \in \overline{k}$). (Some of you will recognize this as an elliptic curve minus the origin.) Prove that Z is irreducible.
5. Prove that the morphism f from $\mathbb{A}^1$ (with co-ordinate t) to the curve C given by $y^2 = x^3$ in the xy-plane $\mathbb{A}^2$, given by $t \mapsto (t^2, t^3)$ is not an isomorphism, by showing directly that the induced map on rings is not an isomorphism.
6. (Practice with morphisms and algebra.) The plane curve $C \subset \mathbb{A}^2(\mathbb{C})$ given by $y^2 = x^3 + x$ is actually isomorphic to $\mathbb{C}$ modulo a lattice (the Gaussian integers, i.e. $\{a + bi | a, b \in \mathbb{Z}\}$), minus one point. The symmetry group of $\mathbb{C}$ preserving the lattice has order 4 *(that's too vague, and technically incorrect)*, and as a result C has an automorphism group of order 4. Find the group.
7. Let R be the ring whose elements are finite sums of terms of the form ax^iy^j where $a \in \overline{k}$, and i and j are non-negative integers such that $i + j \neq 1$. This is a finitely generated $\overline{k}$-algebra with no nilpotents (generated by x^2, xy, and y^2 — *that's not right*), so we know that it is the ring of regular functions for some affine algebraic set. Find such a set. (Give equations cutting out the set. Don't bother proving that your answer is correct.)
8. Let F be the sheaf of differentiable real-valued functions on the unit disc $\{x^2 + y^2 < 1\}$ (in the classical topology). Show that the stalk $F_{(0,0)}$ of F at the origin is a local ring (i.e. that it has one maximal ideal). (Hint: first find the maximal ideal $\mathfrak{m}$, by describing a surjection from the stalk $F_{(0,0)}$ onto some field. Then show that anything not in this maximal ideal is invertible.

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Finally, show that if I is any other proper ideal of $F_{(0,0)}$, then it must be contained in $\mathfrak{m}$.)

9. *Images of morphisms.* If $f : X \to Y$ is a morphism of algebraic sets, then the image of $f(X)$ in Y need not be an algebraic set (although it can still be "described in terms of polynomials"). To see some pathologies that can come up, find the image of $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (xy, y)$ (as a set).
10. *Fibers of morphisms.* Each fiber of a morphism $f : X \to Y$ (i.e. the set $f^{-1}(y)$ for some $y \in Y$) can be easily understood algebraically. For example, suppose let C be the curve $y^2 = x^3$ in the plane, and $f : C \to \mathbb{A}^2$ the morphism $(x, y) \to (x^2, y^2)$. Then the fiber $f^{-1}(1, 1)$ can be found as follows. (i) Informally: Solve $y^2 = x^3$, $x^2 = 1$, $y^2 = 1$ to find $(x, y) = (1, 1)$ or $(1, -1)$. (ii) More correctly: Consider the ideal $I = (y^2 - x^3, x^2 - 1, y^2 - 1)$, and show that its radical is $\mathfrak{m}_{(1,1)} \cap \mathfrak{m}_{(1,-1)}$.

    (a) Consider the morphism $f : \mathbb{A}^1 \to \mathbb{A}^1$, given by $t \mapsto t^2$. Calculate the fiber $f^{-1}(y)$ for all y in $\overline{k}$ (using (ii) above).

    (b) Consider the morphism described in the previous question. Calculate the fibers $f^{-1}(0, 0)$, $f^{-1}(0, 1)$, and $f^{-1}(1, 0)$ (using (ii) above).

    In the future, feel free to use (i) above; by the Nullstellensatz, they are the same.