# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 6

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Contents



*At the end, I collected problem sets, and handed out new ones. If anyone is not on my e-mail list let me know!*

**Where we are.**

Over the last couple of days, we talked about sheaves. We saw axioms (the presheaf axioms, plus gluability and identity). I mentioned the fact that you can "recover" a sheaf knowing the sections over a base of the topology, and the restriction maps between them. We defined stalks at a point.

Then we constructed the structure sheaf $\mathcal{O}_X$ of an irreducible algebraic set.

I concluded by defining affine varieties and prevarieties, which I'll do again today.

**Today:** Loose ends. Then, I'll go through an explicit example in detail, of a structure sheaf, to illustrate the theoretical statements I made last time, and to show that despite the theory, this is once again a very hands-on topic. Finally, prevarieties.

(A brief comment: we've been covering theory quite heavily, and following it up with examples. We've been moving quite fast. Lots of people have been dropping by asking good questions, so if you haven't, please feel free to do so.)

## 1. Loose ends

I made some comments on problems 6, 10, and 7 of Problem Set 2.

*Date*: Tuesday, September 28, 1999.

**Sheaves.** I should clear up some confusion of the two-point space examples I gave of presheaves without identity or gluability. Despite the fact that my primordial example of a sheaf, and all of the other examples I've given, including the structure sheaf, involve thinking of the sheaf as in some sense a *sheaf of functions* on a space, not all sheaves need be of this form.

I'm also going to put some sheaf-related problems on the set, to give you some practice and confidence. In particular, you'll define the *pushforward sheaf*.

**Small point: the obnoxious empty set.** If $F$ is a sheaf on some topological space, there are various ways you can go when dealing with the empty set. This is all abstract nonsense, so feel free to decide what you want $F(\emptyset)$ to be. I wanted to partially clear this up because two of the references say different things. If you're curious, ask me later, and I'll clear it up completely; it involves abstract nonsense, and I don't want to waste too much time on that.

There are 4 ways you can go, depending on your definition of ring and/or sheaf, and depending on how anal you are.

(Ask me if you're curious on this point.)

(0) You could not worry about it, or think about it only as entertainment. I strongly endorse this option.

(i) Suppose you think the 0-ring is a ring. Then you can show that $F(\emptyset) = 0$. In the end, the abstract nonsense approach shows that $F(\emptyset)$ must be the final object in the category of rings, i.e. the ring $R$ that all other rings have a unique map to. Indeed the 0-ring has this property. (Hartshorne uses this definition.)

In general, if you have sheaf with values in a category (e.g. sets if you're thinking about set), then $F(\emptyset)$ must be the final object in that category.

(ii) If you don't think the 0-ring is a ring, then you have to redefine sheaf so that the empty set doesn't turn up.

(iii) Mumford doesn't follow this, and you don't argue with Mumford. I followed his definition of the sections of a structure sheaf on an affine variety $V$ with coordinate ring $R$: if $K(R)$ is the quotient ring of $R$, then sections form a subring of $K(R)$: $\mathcal{O}_V(U)$ are those rational functions that, for all $x \in U$, can be written as $a/b$ where $b(x) \neq 0$. Then by this definition, $F(\emptyset) = K(R)$. To compare this to (i), Mumford is really dealing with a sheaf with values in the category of subrings of $K(R)$ (where objects are subrings, and morphisms are inclusions), and the final object in this category is $K(R)$ itself.

My conclusion is: don't worry about it.

## 2. Playing around with the structure sheaf of the plane

Now let's get our hands dirty with the structure sheaf $\mathcal{O}_{\mathbb{A}^2}$ of $\mathbb{A}^2$ (with coordinates $x$, $y$). Let's find sections over various open sets. First, recall how you should informally think of sections of the structure sheaf: functions are are locally quotients of polynomials. (In the case of $\mathbb{A}^2$, it will turn out that you can cross out the word "locally".)

$\mathbb{A}^2$: $\overline{k}[x, y]$.

$\mathbb{A}^2 \setminus \{x = 0\}$: $\overline{k}[x, y, 1/x]$. Restriction map is the inclusion.

$\mathbb{A}^2 \setminus \{y = 3x^2\}$: $\overline{k}[x, y, 1/(y - 3x^2)]$.

$\mathbb{A}^2 \setminus \{xy = 0\}$: $\overline{k}[x, y, 1/xy]$. Restriction map is the inclusion.

$\mathbb{A}^2 \setminus \{(0, 0)\}$. Hartogs' theorem.

Stalk at $(0, 0)$.

JP pointed out that if $X$ is an affine variety, then the ring of sections for any open lies in $K(R)$ (where $R$ is the ring of regular functions, and $K(R)$ is its fraction field), i.e. if $U$ is open, then $\mathcal{O}_X(U) \subset K(R)$. Then via this inclusion, $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$. This makes calculations much simpler. Also, $\mathcal{O}_X(U \cap V)$ is the subring of $K(R)$ generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$. On Thursday, we'll define a generalization of $K(R)$ for prevarieties (the *function field* of a prevariety $X$); one of the advantages of this definition will be that the properties described earlier in this paragraph still hold.

## 3. Defining affine varieties and prevarieties

Next on the agenda: we'll define affine varieties and prevarieties, give some examples, and define morphisms between them. This will take some time. I'd like to start with definitions, including that of rational functions, and only then will I give lots of examples. Morphisms will come next day.

Here's some intuition first. Let me describe to you the complex manifold $\mathbb{CP}^1$; it's a sphere. Points are given by the set $\mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ modulo the equivalence $(x, y) \sim (ax, ay)$ where $a \in \mathbb{C}^*$. I'll write points as $(x; y)$ where $(x; y) = (ax; ay)$ ($a \in \mathbb{C}$). That's just a set. To describe it as a topological space, or even as a complex manifold: glue together $\mathbb{C}$ (with coordinate $x$) with $\mathbb{C}$ (with coordinate $y$) along $\mathbb{C} \setminus 0$ using $x = 1/y$.

**Definition.** An *affine variety over* $\overline{k}$ $(X, \mathcal{O}_X)$ is a topological space $X$ plus the *structure sheaf* $\mathcal{O}_X$, a sheaf of $\overline{k}$-valued functions $\mathcal{O}_X$ on $X$ which is isomorphic to an irreducible algebraic subset of some $\mathbb{A}^n$ plus the structure sheaf. (I said a few words of explanation here.)

Often we will just say $X$ when the structure sheaf is clear.

*Remark.* We can now define an affine variety structure on $\mathbb{A}^n$.

**Definition.** A *prevariety over* $\overline{k}$ is a topological space $X$ plus a sheaf $\mathcal{O}_X$ of $\overline{k}$-valued functions on $X$ such that

1. $X$ is connected,
2. There is a finite open covering $\{U_i\}$ of $X$ such that for all $i$, $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

In particular, irreducible affine algebraic sets can be naturally given the structure of a prevariety, and we call these *affine varieties.*

**Definition.** An open subset $U$ of a prevariety $X$ is called an *open affine set* if $(U, \mathcal{O}_X|_U)$ is an affine variety.

*Exercise.* Prove that every prevariety $X$ is an irreducible topological space. (In particular, by a result on problem set 2, every open set is dense.) Prove that every prevariety is a Noetherian space.

Last day I mentioned 2 reasons we like distinguished open sets on irreducible algebraic sets: i) they form a base, and ii) we know the sections of the structure sheaf over them ($\mathcal{O}_X(D(f)) = R_f$, where $R$ is the ring of regular functions on $X$). Now we're ready for the long-awaited third reason we like distinguished open sets:

**Theorem.** Let $X$ be an affine variety, and $f \in A(X)$ a regular function. Then the distinguished open set $D(f)$ is an affine variety. (More precisely, the subset $D(f) \subset X$, along with the induced topology, and the restriction of the structure sheaf $\mathcal{O}_X|_U$, is affine.)

Before I prove it, let's see the fundamental idea in a straightforward example. Take the affine line $\mathbb{A}^1$ with co-ordinate $x$ (draw it), and the regular function $f = x$. Then $D(f) = \mathbb{A}^1 \setminus \{0\}$. Let's see how this will be an affine variety.

Consider the hyperbola $xy = 1$, with corresponding ring $\overline{k}[x, y]/(xy - 1)$. Its projection to the $x$-axis affine line is precisely $D(f)$. It's an affine variety, so it has the data of points, a topology, and a structure sheaf. So we need to show

(i) that the projection gives a bijection of points
(ii) that we have a homeomorphism of topological spaces
(iii) that we can produce an isomorphism between the structure sheaf on the hyperbola with the restriction of the structure sheaf of $\mathbb{A}^1$ to the distinguished open $D(f)$.

*Proof of Theorem.* This will be the most intricate argument we'll see today.

(Draw pictures throughout.) $X$ is an affine variety, so consider $X$ as lying in $\mathbb{A}^n$ (with co-ordinates $x_1, \ldots, x_n$), cut out by radical ideal $I = I(X)$. Now $f \in \overline{k}[x_1, \ldots, x_n]/I$, so let $f_1$ be an element of $\overline{k}[x_1, \ldots, x_n]$ restricting to $f$ (i.e. choose a polynomial "representing" $f$).

Let $J$ be the ideal in $\overline{k}[x_1, \ldots, x_{n+1}]$ generated by $I$ and by $1 - f_1 x_{n+1}$.

We claim that $J$ is prime, and $V(J)$ is isomorphic (as a prevariety) to $D(f) \subset X$.

Primeness: By the definition of $J$,

$$\overline{k}[x_1, \ldots, x_{n+1}]/J = (\overline{k}[x_1, \ldots, x_n]/A)[1/f_1] = (\overline{k}[x_1, \ldots, x_n]/A)_{f_1} = \mathcal{O}_X(D(f))$$

which is an integral domain, so $J$ is prime.

Define a morphism (of affine algebraic sets) from $V(J)$ to $X$ by

$$\pi : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$

It's an injection with image $D(f)$.

We can describe this morphism as a morphism of rings of regular functions: $A(X) \hookrightarrow A(X)_f$.

*Exercise.* $\pi$ gives a homeomorphism from $V(J)$ to $D(f)$. (Hints: see rest of proof. Use the base of the topology.)

Finally, we need to know that the structure sheaves are the same. It suffices to know that the sections over the elements of a base are the same, and the restriction morphisms are the same.

Consider $D(g) \subset D(f)$ in $X$. Then $D(g) = D(fg)$. Let $g_1$ be a polynomial restricting to $g$ in $\overline{k}[x_1, \ldots, x_n]/I$.

Sections over $D(g)$ are:

$$\mathcal{O}_X(D(fg)) = \frac{\overline{k}[x_1, \ldots, x_n]}{I}\left[\frac{1}{f_1 g_1}\right].$$

The corresponding open set in $V(J)$ is given by $f_1 g_1 \neq 0$ as well; it's also a distinguished open set, so we know the sections there too.

The ring of sections is:

$$\begin{aligned} \overline{k}[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}f_1 - 1)\left[\frac{1}{f_1 g_1}\right] &= \overline{k}[x_1, \ldots, x_n]/I\left[\frac{1}{f_1}\right]\left[\frac{1}{f_1 g_1}\right] \\ &= \overline{k}[x_1, \ldots, x_n]/I\left[\frac{1}{f_1 g_1}\right]. \end{aligned}$$

so the sections over $D(g_1)$ *are* the same. The restriction maps are the same too; omitted. $\square$

*Corollary.* (1) Every affine prevariety has a base of distinguished affine open sets.

(2) Every prevariety has a base of affine open sets. (Give proof.)

(3) If $(X, \mathcal{O}_X)$ is a prevariety, and $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is a prevariety too. (Just check the conditions. In lecture, I showed that $U$ is covered by affine opens. Leo pointed out that one of the conditions is that $U$ is covered by a *finite* number of affine opens. This is a consequence of the fact that $X$ is a Noetherian space, which is a problem on Problem Set 3.)

Next (on Thursday), I'll give a bunch of examples of prevarieties.

**Coming in the next few lectures:**

1. Examples of prevarieties.
2. Function fields of prevarieties. (We should get at least this far on Thursday.)
3. Morphisms of prevarieties. (Then we'll have fully defined the *category of prevarieties.*
4. Open subprevarieties. Closed subprevarieties. Rational maps. Birationality.
5. Projective varieties.