INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 18

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Contents

No class Thursday. Problem sets back at end. New problem set handed out.

1. Extending rational maps of nonsingular curves

1.1. **More on integral closure in a field extension.** I neglected to say a few small facts about integral closure in a field extension. Recall:

Let A be an integral domain, which is a finitely generated algebra over $\overline{k}$. Let K be the quotient field of A, and let L be a finite algebraic extensions of K. Then the *integral closure of* A *in* L consists of those elements of L satisfying $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in A$.

For example, the *integral closure of* A *in its quotient field* is *the integral closure of* A.

It isn't hard to show that integral closure A' of A in L must be integrally closed, i.e. the only solutions to such equations in L, where now the a_i are supposed to lie in A', lie in A'.

Remark 1. $R \subset S$. Indeed, if $r \in R$, then it is a solution to the equation $x - r = 0$.

Remark 2. The quotient field of A' is L. This follows from:

Remark 3. If $l \in L$, then some multiple of it al ($a \in A$) is in A'. *Proof.* l satisfies some $l^n + a_{n-1}l^{n-1} + \cdots + a_0 = 0$ for some $a_i \in K$. Clear denominators,

Date: Tuesday, November 9, 1999.

to write $b_n l^n + b_{n-1} l^{n-1} + \cdots + b_0 = 0$ for some $b_i \in A$. Then rewrite as $(b_n l)^n + b_{n-1} b_n (b_n l)^{n-1} + \cdots + b_0 b_n^n = 0$. Thus $b_n l \in A'$. $\square$

Remark 4. A reminder: A theorem I proved (using a couple of theorems I *didn't* prove), that we'll use today:

Theorem. Take any Dedekind domain R that is a finitely generated algebra over $\overline{k}$, and let K be its field of fractions. Let L be a finite extension of K, and let S be the integral closure of R in L. Then S is also a Dedekind domain, and is also a finitely generated algebra over $\overline{k}$.

1.2. **Last time.** Goal: Rational maps of nonsingular curves to projective varieties can be extended to morphisms.

We discussed reasons why you can't extend $\mathbb{P}^1 \dashrightarrow \mathbb{A}^1$, $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$.

Lemma. Let Y be a prevariety, and suppose P and Q are two points contained in a single affine open U, and $\mathcal{O}_{Y,Q} \subset \mathcal{O}_{Y,P}$ (as subrings of $k(Y)$). Then $P = Q$.

Key Technical Theorem (tricky). Let K be a finitely generated function field of dimension one over $\overline{k}$, and let $x \in K$. Then the set of discrete valuations of $K/\overline{k}$ where $v(x) < 0$ is *finite.*

Warning: We're going to get a lot of mileage out of the proof this, so pay attention to it, even though it will get heavy!

Here's the geometric idea. Suppose $K = \overline{k}(t)$, the function field of $\mathbb{P}^1$. $x \in K$, so x is a rational function of $\mathbb{P}^1$. Then the discrete valuations of $K/\overline{k}$ correspond to points of $\mathbb{P}^1$ (exercise). The points/valuations v_p where $v_p(x) < 0$ are precisely the points where x has poles; there are only a finite number of such points/valuations.

1.3. **New material starts here.** The proof will use the following:

Commutative algebra lemma. Suppose $(S, \mathfrak{n})$, $(R, \mathfrak{m})$ are two discrete valuation rings of $K/\overline{k}$, with $S \subset R$ and $\mathfrak{n} = S \cap \mathfrak{m}$. Then $S = R$ and $\mathfrak{n} = \mathfrak{m}$: they are the *same* valuation ring.

(Compare to lemma just above.)

Proof. The two valuation rings give two discrete valuations $v_\mathfrak{n}$ and $v_\mathfrak{m}$ on K (so S are the elements s of K with $v_\mathfrak{n}(s) \geq 0$, and R are the elements r of K with $v_\mathfrak{m}(r) \geq 0$; we can recover the DVRs from the valuations.) Let u be a uniformizaer of S (i.e. a generator of $\mathfrak{n}$), so $v_\mathfrak{n}(u) = 1$ (and $v_\mathfrak{m}(u) > 0$). Suppose $r \in K$, $r \notin S$; we'll see that $r \notin R$. As $v \notin S$, $v_\mathfrak{n}(r) < 0$, so $r = u^{-n} w$ with $n > 0$, $v_\mathfrak{n}(w) = 0$, so $w \in S \backslash \mathfrak{n}$. Then $w \in R \backslash \mathfrak{m}$, hence $v_\mathfrak{m}(w) = 0$. But then $v_\mathfrak{m}(r) = -n v_\mathfrak{m}(u) + v_\mathfrak{m}(w) < 0$, so $r \notin R$ as desired. $\square$

Proof of "Key Technical Theorem". i) The set-up. Note that if $(R, \mathfrak{m}_R)$ is a valuation ring of $K/\overline{k}$, then if x is not in R then it has negative valuation, so $y := 1/x$, having positive valuation, is in $\mathfrak{m}_R$. So we have to show that if $y \in K$, $y \neq 0$, then the set of discrete valuations where $y \in \mathfrak{m}_R$ is a finite set. (Geometric picture: show that a non-zero rational function on $\mathbb{P}^1$ has only a finite number of zeroes.) If $y \in \overline{k}$, then $v(y) = 0$ for all discrete valuations of $K/\overline{k}$, so we can assume y is not in $\overline{k}$. (Geometric picture: can ignore constant functions.)

ii) Part of proof involving y. Consider $\overline{k}[y]$, the subring of K generated by y. Since $\overline{k}$ is algebraically closed, y is transcendental over $\overline{k}$, so $\overline{k}[y]$ is a polynomial ring, i.e. y doesn't satisfy any relations. Hence K is a finite field extension of $\overline{k}(y)$. Let B be the integral closure of $\overline{k}[y]$ in K. Then by an earlier theorem proved last day (and repeated as Remark 4 above), B is a Dedekind domain, and is also a finitely generated algebra over $\overline{k}$.

Hence B corresponds to an affine variety Y, which maps to $\mathbb{A}^1$ with coordinate y; draw it! It is dimension 1 and nonsingular. Also, y is a function on Y.

iii) Now bring in the valuation. If y is contained in a discrete valuation ring $(R, \mathfrak{m}_R)$ of $K/\overline{k}$, then $\overline{k}[y] \subset R$, and since R is integrally closed in K, we have $B \subset R$ as well. Let $\mathfrak{n} = \mathfrak{m}_R \cap B$. Then $\mathfrak{n}$ is a maximal ideal of B. (Hence it corresponds to a point of Y mapping to zero.) Also, $B_\mathfrak{n} \subset R$. And also $\mathfrak{n}B_\mathfrak{n} \subset \mathfrak{m}_R$. Now $B_\mathfrak{n}$ is also a DVR of $K/\overline{k}$, so $B_\mathfrak{n} = R$ by the Commutative Algebra Lemma.

Hence if y is in $\mathfrak{m}_R$, then y is in $\mathfrak{n}$. To say that y is in $\mathfrak{n}$, means that y, as a regular function on Y, vanishes at the corresponding point. As non-zero functions vanish at only finitely many points (closed zero-dimensional subsets are only finite numbers of points), we're done. □

From the proof, we can also extract another result, which we can use later.

Corollary. Let v be a discrete valuation of the field $K/\overline{k}$, a finitely-genenerated function field of transcendence degree 1. Then there is a nonsingular curve C, with function field K, and a point $p \in C$, such that v is the valuation induced by p.

Proof. Take any $y \in K$ of positive valuation. Then construct Y as in the previous proof. □

Corollary. Given $y \in K \setminus \overline{k}$, all discrete valuations of $K/\overline{k}$ such that $v(y) \geq 0$ (i.e. that y is in the corresponding DVR) are accounted for by points on the curve Y above.

(Remark: just proof above.)

The only "missing" valuations are those for which $v(y) < 0$, so by the Key Technical Theorem, we're only missing a finite number.

Proof. Let $(R, \mathfrak{m})$ be the corresponding DVR. As $v_{\mathfrak{m}}(y) \geq 0$, $y \in R$. Then let $a \in \overline{k}$ be the residue of y modulo $\mathfrak{m}$, i.e. $y \equiv a \pmod{\mathfrak{m}}$. If $a = 0$, then the proof above shows that $(R, \mathfrak{m})$ is the local ring of one of the points of Y mapping to 0, so we're done.

If $a \neq 0$, then replace y with $y - a$, and we're done again. □

This confused them.

1.4. **Extension of morphisms to projective varieties, over nonsingular points of curves.** Now we're ready to prove the main result of this section.

Theorem. Let X be a nonsingular curve, p a point of X, Y a projective variety, and $\phi : X - p \to Y$ a morphism. Then there exists a unique morphism $\overline{\phi} : X \to Y$ extending ϕ.

By separatedness, as Y is a variety, we have uniqueness; all that is necessary is existence.

Last day, we discussed an example of a map from $\mathbb{P}^1$ to $\mathbb{P}^n$: $[x; y] \mapsto [x^3; x^2y; xy^2]$ is defined away from $[0; 1]$; it's clear how to extend. You divide by x. We'll carry this example along as we do the proof.

Hence this is l'Hopital's rule in a vague sense.

Proof. It is sufficient to show that f extends to a morphism of X into $\mathbb{P}^n$ (with coordinates $x_0, \ldots, x_n$. Let U be the open set where all x_i are non-zero. By changing coordinates if necessary, assume $f(X - p)$ meets U.

For each i, j, x_i/x_j is a regular function on U; pulling back by f, we have regular function f_{ij} on an open subset of X, which we view as a rational function on X, so f_{ij} is in $k(X)$.

Let v be the valuation associated with p. Let $r_i = v(f_{i0})$ for $i = 0, 1, \ldots, n$. Then $v(f_{ij}) = r_i - r_j$. Choose r_k minimal, so $v(f_{ik}) \geq 0$.

Consider the map of sets $\overline{\phi} : X \to \mathbb{P}^n$, given by $\overline{\phi}$ is the same as ϕ on $X - p$, and $\overline{\phi}(p) = (f_{0k}(p); \ldots; f_{nk}(p))$ (observe that not all coordinates are 0). I claim this is a morphism. To show this, I need only show an affine neighbourhood X_{aff} of X that maps pointwise to an affine U of $\mathbb{P}^n$, such that the pullback of every regular function on U is a regular function on X_{aff}.

Let $U = U_k$ be the open set where $x_k \neq 0$ (so $\overline{\phi}(p) \in U_k$, since $f_{kk}(p) = 1$). The coordinate ring of the affine U_k is $\overline{k}[x_0/x_k, \ldots, \widehat{x_k/x_k}, \ldots, x_n/x_k]$. These functions pull back to $f_{0k}, \ldots, f_{nk}$ which are regular at p by construction. Let X_{aff} be any affine neighbourhood of p where the rational functions f_{ik} are defined.

So we're done. □

As consequences, we will get (next day):

Theorem. Every nonsingular separated curve C is quasiprojective.

Corollary. Hence every nonsingular separated curve is birational to a projective curve.

Proposition. Let C_1 and C_2 be two separated nonsingular curves, and let $\alpha : C_1 \dashrightarrow C_2$ be a *birational map* between them. Then they can be glued together via α, i.e. there is another nonsingular curvve $C = U_1 \cup U_2$, with C_i isomorphic to U_i, and the birational map $U_1 \dashrightarrow U_2$ induced by α.

Proposition. If two nonsingular projective curves C_1 and C_2 are birational, then they are isomorphic.

Coming up: We'll discuss these, and begin talking about why finitely-generated fields of transcendence dimension 1 correspond to nonsingular projective curves (over $\overline{k}$). started just before the introduction