INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 13

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Contents

Hand in problem sets. Problem set policy: How does this sound? 11 or 12 sets due, depending on Thanksgiving. Count best 8. If last 4 are great, then you'll do fine, but I'd strongly suggest you don't rely on that.

(Note added later: Where I say "finite", I might want to often say "generically finite" instead.)

1. Rational maps

We can reinterpret the definition of separatedness as follows. Suppose I'm thinking of a morphism $f : Y \to X$, where X is a *variety*. And suppose I tell you what the morphism is on a non-empty open set $U \subset Y$, i.e. I tell you $f|_U : U \to X$. Then there is only one way for you to recover the "full" morphism f. Because if you have two different morphisms f_1 and f_2 extending f, then you have two morphisms $f_1, f_2 : Y \to X$ which agree on a dense open set (the set U; recall that dense means that the closure of U is Y), and agree on a closed set (as X is separated). Hence they have to agree everywhere.

As an example, in an earlier problem set, I defined a map from the curve $x^2+y^2 = z^2$ in $\mathbb{P}^2$, minus a point, to $\mathbb{P}^1$, and asked you to extend it. As $\mathbb{P}^1$ is a *variety*, I didn't have to worry about different people extending it in different ways.

Morphisms from open sets come up a lot, as do questions about extending them, so there is common terminology that is used.

This leads to a natural question: when can a morphism on an open set be extended? The answer, as we shall see, is not always (we'll see soon), but always when the source Y is a *smooth curve* (which we'll eventually define).

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Definition. A *rational map* of prevarieties $Y \to X$ is the data of a map $f : U \to X$ (where U is a non-empty hence dense open set of Y) modulo the equivalence relation $f_1 : U_1 \to X$ is equivalent to $f_2 : U_2 \to X$ if there is a smaller non-empty open set $W \subset U_1 \cap U_2$ where f_1 and f_2 agree on W. A rational map is written $f : Y \dashrightarrow X$.

Remark. When the target X is a *variety*, we can take $W = U_1 \cap U_2$; because if f_1 and f_2 (considered as morphisms from $U_1 \cap U_2$ to X) agree on a dense open set W of $U_1 \cap U_2$, and they both agree on the *closure* of W in $U_1 \cap U_2$, i.e. all of $U_1 \cap U_2$.

Remark. Note that the elements of the *function field* correspond to rational maps to $\mathbb{A}^1$.

Example of sloppy notation. Consider the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{A}^1$, where $\mathbb{A}^2$ has coordinates x, y and $\mathbb{A}^1$ has coordinates t, given by $t = x/y$, or if you prefer, $(x, y) \mapsto x/y = t$. You'll have some natural questions: what is the open set $U \subset Y$? The obvious answer is $D(y)$, where y isn't zero. You clearly can't extend it further. And also, because the target $\mathbb{A}^1$ is separated, there is no question what the morphism $D(y) \to \mathbb{A}^1$ *is.*

I might have been silly, and chosen $D(yx)$, and you might have been silly and chosen $D(y(x-1))$, but we would have gotten together, and realized that we could "glue together" our open sets. Also, by the separatedness property, which can be thought of as a "uniqueness of extension" property, we know that our functions agree on the overlap.

Observation from the example / Definition. Given a rational map $Y \dashrightarrow X$ to a *variety*, there is a largest open set of definition $U \subset Y$. (This requires proof, but isn't hard.)

Example. Notice what goes wrong if X isn't separated. Consider the rational map $\mathbb{A}^1 \to X$, where X is the line with the doubled origin, and the rational map is given by $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ (by which I mean the obvious identity). Then it isn't clear how to extend this over 0.

Exercise. In all of the examples so far, if we have $U \subset Y$, and X a variety, morphisms $U \to X$ have always been "extendable" to $Y \to X$. Show that isn't always the case, by considering the case $Y = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus \{(0, 0)\}$, $X = \mathbb{P}^1$, and $U \to X$ given by $(x, y) \mapsto (x; y)$. Show that there is no morphism $Y \to X$ extending this.

Some more definitions.

Definition. A rational map of prevarieties $f : Y \dashrightarrow X$ is *dominant* if a representative $U \to X$ has dense image. Examples: projection $\mathbb{A}^2 \to \mathbb{A}^1$;

Proposition. A dominant rational map $f : Y \dashrightarrow X$ induces a morphism of function fields $k(X) \to k(Y)$.

Proof 1. Elements of $k(X)$ are of the form $(U, g \in \mathcal{O}_X(U))$, i.e. functions defined on an open set of X, subject to an equivalence relation $(U, g \in \mathcal{O}_X(U)) \sim (V, h \in \mathcal{O}_X(V))$ if there is a smaller open in both U and V where g and h restrict to the same function. Because the image of Y is dense, the preimage $f^{-1}(U)$ is non-empty in Y, so we get a section $(f^{-1}(U), f^*g \in \mathcal{O}_Y(f^{-1}(U)))$. Then you just need to check that if you pull back different representatives of an element of $k(X)$, you get representatives of the same element of $k(Y)$, which I'll leave to you. $\square$

Here's a different proof.

Proof 2. Note that you can compose dominant rational maps! Slightly more generally, if you have $Y \dashrightarrow X \dashrightarrow Z$ where $Y \dashrightarrow X$ is dominant, then there is a composed map $Y \dashrightarrow Z$; and if $X \dashrightarrow Z$ is also dominant, then the composition is dominant too. *Exercise:* You can identify elements of $k(Y)$ with rational maps from Y to $\mathbb{A}^1$. Hence given $f : Y \dashrightarrow X$, given any element of $k(X)$, i.e. a rational map $X \dashrightarrow \mathbb{A}^1$, you can compose them to get a rational map $Y \dashrightarrow \mathbb{P}^1$, i.e. an element of $k(Y)$. $\square$

It is also true that a morphism of function fields induces a dominant rational map; the following example will help convince you of this, although a general proof is of course required. See Hartshorne p. 26.

Exercise. Consider two varieties, $\mathbb{P}^1$ and $\mathbb{A}^1$. Let one of the affine covers of $\mathbb{P}^1$ have coordinate x. For $\mathbb{A}^1$, let the coordinate be t. Find a dominant rational map $f : \mathbb{P}^1 \dashrightarrow \mathbb{A}^1$ corresponding to the morphism of function fields $k(\mathbb{A}^1) = \overline{k}(x) \to k(\mathbb{P}^1) = \overline{k}(t)$ given by $x \to t^2$.

Hence there is a correspondence between dominant rational maps $Y \dashrightarrow X$ and maps of function fields in the opposite direction.

Definition. A rational map $f : Y \dashrightarrow X$ is a *birational map* if it has a representative $U \to X$ that is an open immersion. If such a map exists, we say that Y and X are *birational*. If X is birational to $\mathbb{A}^n$, X is said to be *rational*.

Through the correspondence between dominant rational maps $Y \dashrightarrow X$ and maps of function fields, one can show that a variety is rational if and only if its function field is a pure transcendental extension of $\overline{k}$, i.e. isomorphic to $\overline{k}(t_1, \ldots, t_n)$.

Exercise. In an earlier problem set (4, question 3), you showed that the function field of the hypersurface $wx = yz$ in $\mathbb{A}^4$ was isomorphic to $\overline{k}(t_1, t_2, t_3)$. By my previous comments, you'd expect the hypersurface to be birational to $\mathbb{A}^3$. Prove that it is.

Definition. If $f : Y \to X$ is dominant and induces a morphism $k(X) \to k(Y)$ that is an inclusion and a finite degree d field extension, then we say that f is a degree d rational map. *(Possibly mention "generically finite" here; "degree d rational map" might be bad notation.)*

Exercise (on next problem set). Let C be the projective variety in $\mathbb{P}^2$ defined by $x^2 + y^2 + z^2 = 0$. Find the degree of $f : C \to \mathbb{P}^1$ given by $[x; y; z] \to [x; y]$.

2. Dimension: a beginning

I'll repeat the statement of most of these results on Tuesday.

A long time ago, I defined dimension. Here, again, is the definition. As always, we'll fix an algebraically closed field $\overline{k}$.

Definition. If X is a prevariety defined over $\overline{k}$, define $\dim X = tr.d._{\overline{k}} k(X)$. If Z is a closed subset of X, then Z has *pure dimension* r if each of its components has dimension r. A variety of dimension 1 is a *curve*, a variety of dimension 2 is a *surface*, a variety of dimension n is an n-fold.

If R is a domain over $\overline{k}$, we might as well *define tr.d.R* to be the transcendence degree of the quotient field.

Note immediately that if U is open in X, then $\dim U = \dim X$, and $\dim X = 0 \Leftrightarrow k(X) = \overline{k} \Leftrightarrow X$ is a point.

We'll need some facts from basic commutative algebra, which are covered for example in Steve Kleiman's Commutative Algebra course that some of you are in. The rest of you can take these as black boxes.

First, a relatively straightforward algebraic fact (although I didn't explain the proof well in class).

Lemma. Let R be an integral domain over $\overline{k}$, $\mathfrak{p} \subset R$ a prime ideal. Then $tr.d._{\overline{k}} R \geq tr.d._{\overline{k}} R/\mathfrak{p}$, with equality iff $P = \{0\}$ or both sides are infinite.

Proof. Say $\mathfrak{p} \neq 0$, and the the right side is $n < \infty$. If the statemetn is false, then there are n elements $x_1, \ldots, x_n$ in R such that their imagse in R/P are algebraically independent over $\overline{k}$. Choose any non-zero $q \in \mathfrak{p}$. Then q and the x_i can't be algebraically independent over $\overline{k}$, so there is a polynomial $f(z, y_1, \ldots, y_n)$ (*with coefficients in* $\overline{k}$; this is what I flubbed in class) such that $f(q, x_1, \ldots, x_n) = 0$; R is a domain, so assume f is irreducible. Now f can't be a multiple of z, as q is non-zero; hence the reduction of $f(0, x_1, x_2, \ldots, x_n) = 0 \bmod \mathfrak{p}$ (*remember that the co-efficients are in* $\overline{k}$*, and aren't affected by this*) imposes a nontrivial algebraic relation on the images of $x_1, \ldots, x_n$ in $R/\mathfrak{p}$. But we said they were algebraically independent, yielding a contradiction. □

This gives us something geometrically very quickly:

Proposition. If Y is a closed subprevariety of X, then $\dim Y < \dim X$.

(I'll start dropping the "pre"s, but you can check that most things I say still apply in the general prevariety case.)

Definition. Call the difference the *codimension* of Y in X.

So for example, if the codimension is 1, there are no other subvarieties in between Y and X.

Proof. Choose an affine subset U of X that meets Y. Let R be its coordinate ring, and $\mathfrak{p}$ the prime ideal corresopnding to $U \cap Y$. Now $\mathfrak{p} \neq 0$, as $U \cap Y \neq U$. Then $k(X)$ is the quotient field of R, and $k(Y)$ is the quotient field of $R/\mathfrak{p}$, so we're done by the Lemma. $\square$

For our next result, we'll need a result from commutative algebra:

Theorem (Krull's Hauptidealsatz, or Principal Ideal Theorem). Suppose R is a finitely generated integral domain over $\overline{k}$, $f \in R$, $\mathfrak{p}$ a minimal prime of (f) (i.e. minimal among the prime ideals containing it). Then if $f \neq 0$, $tr.d.R/\mathfrak{p} = tr.d.R - 1$. (Proof omitted.)

In geometrical situations, this is very reasonable. For example, let $R = \overline{k}[x, y, z]$, and let f be some polynomial, say $xy(x - y^3 - z^4)$. Notice that the vanishing set $V(f) = \{f = 0\}$ consists of two planes and this weird surface. The minimal primes containing (f) correspond to maximum subvarieties of $\mathbb{A}^3$ contained in the vanishing set of f, so there are 3 of them. Intuitively, it is reasonable that all 3 have dimension 2.

Coming soon: More on dimension. Non-singularity. Curves.