INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 1

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Contents

I'm going to start by telling you about this course, and about the field of algebraic geometry.

Goals:

- geometric insight
- concrete examples (geometric and arithmetic)
- hands on calculations (no fear of commutative algebra)
- no cohomology, flatness, differentials

Modern algebraic geometry lies somewhere between differential geometry, number theory, and topology. In a loose sense, it is polynomial equations, and sets defined by polynomial equations. This seems to be extremely narrow and low-tech, but it surprisingly ends up being extremely broad, powerful, and abstract.

Some of the philosophy — get at geometry via algebra, algebra via "pictures". High school reference. Here's a high-powered example of the link between geometry and arithmetic. $x^n + y^n = z^n$. Finite number of solutions for each $n > 1$: the Mordell Conjecture, Faltings' Theorem. Vojta's conjecture. Weil conjectures.

Give out handout of motivating problems.

This will be a tools course: examples and pictures, but with generality.

- fast-moving, but grounded by intuition
- exercises are important
- concepts really generalize, but become more abstract.

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Objects:

smooth varieties over $\mathbb{C}$ (over k)

varieties over $\mathbb{C}$ (over k)

schemes

(stacks)

We won't be shy about schemes in this course.

Administrative stuff. Give out handout. Go through it. Headcount.

Theme: curves. Examples: $\mathbb{Z}$ and $\overline{k}[t]$.

1. Commutative algebra

Don't worry about it. See books on handout.

Ideas you should know: ring (commutative, has 1), field, integral domain, has quotient field, prime ideal, maximal ideal.

Sample problem (to appear on problem set):

Let A be a (commutative) ring. An element $a \in A$ is nilpotent (that is, $a^n = 0$ for some $n > 0$) if and only if a belongs to every prime ideal of A.

2. Algebraic sets

Throughout this course: k is a field. $\overline{k}$ is an algebraically closed field.

For now we work over $\overline{k}$. Feel free to think of this as $\mathbb{C}$ for now.

$\overline{k}^n$ will be rewritten $\mathbb{A}^n(\overline{k})$, affine n-space; we'll often just write $\mathbb{A}^n$ when there's no confusion about the field. Coordinates x_1 to x_n.

Algebraic geometry is about functions on the space, which form a ring. The only functions we will care about will be polynomials, i.e. $\overline{k}[x_1, \ldots, x_n]$. We'll eventually think of that ring as being the same thing as $\mathbb{A}^n$.

We'll next define subsets of $\mathbb{A}^n$ that we'll be interested in. Because we're being very restrictive, we won't take any subsets, or even analytic subsets; we'll only think of subsets that are in some sense defined in terms of polynomials.

Let S be a set of polynomials, and define $V(S)$ to be the locus where these polynomials are zero. ("Vanishing set".) Definition: Any subset of $\mathbb{A}^n(\overline{k})$ of the form $V(S)$ is an *algebraic set*.

Exercise (to appear on problem set): prove that the points of the form (t, t^2, t^3) in $\mathbb{A}^3$ form an algebraic set. In other words, find a set of functions that vanish on these points, and no others.

Example/definition: *hypersurface*, defined by 1 polynomial.

Facts.

- If $I = (S)$, then $V(I) = V(S)$. So we usually will care only about ideals. Hence: subsets of $\overline{k}[x_1, ..., x_n]$ give us subsets of $\mathbb{A}^n$; specifically, *ideals* give us *algebraic sets*
- $V(\cup I_a) = \cap V(I_a)$ (Say it in english.)
- $I \subset J$, then $V(I) \supset V(J)$
- $V(FG) = V(F) \cup V(G)$

Note: Points are algebraic. Finite unions of points are algebraic.

Definition. A radical of an ideal $I \subset R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n\}.$$

Exercise. Show that $\sqrt{I}$ is an ideal.

Definition. An ideal I is *radical* if $I = \sqrt{I}$.

Claim. $V(\sqrt{I}) = V(I)$. (Explain why.)

Conversely, subsets of $\mathbb{A}^n$ give us a subset of $\overline{k}[x_1, ..., x_n]$ For each subset X, let $I(X)$ be those polynomials vanishing on X.

Claim. $I(X)$ is a radical ideal. (Explain.)

Facts. If $X \subset Y$, then $I(X) \supset I(Y)$. $I(\emptyset) = \overline{k}[x_1, ..., x_n]$. $I(\mathbb{A}^n) = (0)$.

Question. What's $I((a_1, ..., a_n))$?

(Discuss.)

Notice: ideal is maximal. Quotient is field. Quotient map can be interpreted as "value of function at that point".

Exercise. (a) Let V be an algebraic set in $\mathbb{A}^n$, P a point not in V. Show that there is a polynomial F in $\overline{k}[x_1, ..., x_n]$ such that $F(Q) = 0$ for all Q in V, but $F(P) = 1$. Hint: $I(V) \neq I(V \cup P)$.

(b) Let $\{P_1, ..., P_2\}$ be a finite set of points in $\mathbb{A}^n(\overline{k})$. Show that there are polynomials $F_1, ..., F_r \in \overline{k}[x_1, ..., x_n]$ such that $F_i(P_j) = 0$ if $i \neq j$, and $F_i(P_i) = 1$.

Exercise. Show that for any ideal I in $\overline{k}[x_1, ..., x_n]$, $V(I) = V(\sqrt{I})$, and $\sqrt{I}$ is contained in $I(V(I))$.

3. Nullstellensatz (theorem of zeroes)

Earlier, we had: algebraic sets $\to$ radical ideals and ideals $\to$ algebraic sets.

This theorem makes an equivalence. In the literature, the word "nullstellensatz" is used to apply to a large number of results, not all of them equivalent.

Nullstellensatz Version 1. Suppose $F_1, \ldots, F_m \in \overline{k}[x_1, \ldots, x_n]$. If the ideal $(F_1, \ldots, F_m) \neq (1) = \overline{k}[x_1, \ldots, x_n]$ then the system of equations $F_1 = \cdots = F_m = 0$ has a solution in $\overline{k}$.

Proof next day. (There is a better version for fields that are not necessarily algebraically closed, but we're not worrying about that right now.)

Nullstellensatz Version 2. Supopse $\mathfrak{m}$ is a maximal ideal of $\overline{k}[x_1, \ldots, x_n]$. Then

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

for some $a_1, \ldots, a_n \in \overline{k}$.

Show that this is equivalent to version 1, modulo fact that ideals are finitely generated.

Nullstellensatz Version 3 (sometimes called the "Weak Nullstellensatz"). If I is a proper ideal in $\overline{k}[x_1, ..., x_n]$, then $V(I)$ is nonempty. (From Version 2.)

Nullstellensatz Version 4. Let I be an ideal in $\overline{k}[x_1, ..., x_n]$. Then $I(V(I)) = \sqrt{I}$. Equivalently: Radical ideals are in 1-1 correspondence with algebraic sets: If I is a radical ideal in $\overline{k}[x_1, ..., x_n]$ then $I(V(I)) = I$. So there is a 1-1 correspondence between radical ideals and algebraic sets.

Nullstellensatz Version 5. A radical ideal of $\overline{k}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it. This is the geometric rewording of 4. By version 4, a radical ideal is $I(X)$ for some algebraic set X. Functions vanishing on X are precisely those functions vanishing on all the points of X.

Nullstellensatz Version 6. If F_1, ..., F_r, G are in $\overline{k}[x_1, ..., x_n]$, and G vanishes wherever F_1, ..., F_r vanish, then there is an equation $G^N = A_1F_1 + ... + A_rF_r$ for some $N > 0$ and some A_i in $\overline{k}[x_1, ..., x_n]$.

This has a cute proof, with a useful trick in it.

Proof. The case $G = 0$ is obvious, so assume $G \neq 0$. Introduce a new variable U, and consider the polynomials

$$F_1, \ldots, F_m, \text{ and } UG - 1 \in \overline{k}[x_1, \ldots, x_n, U].$$

They have no common solutions in $\overline{k}$, so by Version 1 they generate the unit ideal, so there are polynomials $P_1, \ldots, P_m$, $Q \in \overline{k}[x_1, \ldots, x_n, U]$ such that

$$P_1 F_1 + \cdots + P_m F_m + Q(UG - 1) = 1.$$

Now set $U = 1/G$ in this formula, and multiply by some large power G^N of G to clear denominators. Then the right side is G^N, and the left side is in $(F_1, \ldots, F_m)$.