INTERSECTION THEORY CLASS 9

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I have one update from last time, and this is aimed more at the experts. Rob pointed out that there was no reason that we know that a Cartier divisor can be expressed as a difference (or quotient) of effective Cartier divisors. More precisely, a Cartier divisor can be described cohomologically as follows. Let X be a scheme. We have a sheaf $\mathcal{O}^*$ of invertible functions. There is another sheaf $\mathcal{K}^*$ that are things that locally look like quotients of a function by a nonzerodivisor. (If X is a variety, then $\mathcal{K}^*$ is the constant sheaf with $\mathcal{K}^*(U) = R(X)$ for all U.) Then I informally described Cartier divisors of X as determined by certain data: there is an open cover of X by open sets U_i; we have an element of $\mathcal{K}^*$ for each U_i; and on $U_i \cap U_j$ the quotient of the two elements of $\mathcal{K}^*$ corresponding to i and j is an element of $\mathcal{O}_X^*$. We then mod out by an equivalence relation that I was careless about defining. This definition translates to the more compact notation: Cartier divisors are global sections of the (quotient) sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. (More generally, we get a sheaf of Cartier divisors $\mathcal{K}^*/\mathcal{O}_X^*$.) The description I gave was the Cech description of a quotient sheaf. This drives home the point that any Cartier divisor is *locally* the quotient of two effective Cartier divisors, but not necessarily globally. I don't know of any specific examples of a Cartier divisor that is not the quotient/difference of two effective Cartier divisors, and I would like to see one.

Fulton is then proving that even though we don't know for sure that any Cartier divisor D on X is a difference of two effective divisors, we can construct a proper surjective morphism $\pi : \tilde{X} \to X$ such that $\pi^* D$ is Cartier, and a difference of effective Cartier divisors. Moreover, he tells us what to do: define a closed subscheme by taking the "ideal sheaf of denominators" of the Cartier divisor, and blow it up. The example I said I'd like to see corresponds to the question: find a scheme and a Cartier divisor where this ideal sheaf of denominators is not Cartier. I'm still a bit perplexed; it seems to me that it should always be Cartier, as "Cartier-ness" is a local condition, and locally every Cartier divisor is principal. (I'm assuming, as we are throughout this course, that all schemes are essentially of finite type, so the Cech description certainly holds.)

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As an aside: when you see that Cartier divisors are global sections of a quotient sheaf, you should immediately be curious about the corresponding long exact sequence of cohomology.

$$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$$

gives us

$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{K}^*) \to H^0(X, \mathcal{K}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{K}^*) = 0.$$

The right term is 0 because $\mathcal{K}$ is a flasque (=flabby) sheaf. All the other terms have obvious meanings too. The image of $H^0(X, \mathcal{K}^*)$ is the set of principal Cartier divisors. (An element of $H^0(X, \mathcal{O}^*)$ gives a trivial principal divisor.) $H^1(X, \mathcal{O}_X^*) = \operatorname{Pic} X$. So this shows that $\operatorname{Pic} X \cong$ Cartier divisors modulo principal divisors.

1. Vector bundles, and Segre and Chern classes

1.1. **Segre classes of vector bundles.** Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. Note that it is both flat and proper (explain).

Define homomorphisms

$$s_i(E) \cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha)$. Note that this indeed maps from $A_k X \to A_{k-i} X$.

Segre class Theorem. (a) for all $\alpha \in A_k X$, (i) $s_i(E) \cap \alpha = 0$ for $i < 0$, and (ii) $s_0(E) \cap \alpha = \alpha$.

(b) (commutativity) If E and F are vector bundles on X, and $\alpha \in A_k X$, then for all i, j,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(c) (Segre classes behave well with respect to proper pushforward) If $f : X' \to X$ is proper, E a vector bundle on X, $\alpha \in A_* X'$, then for all i,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(d) (Segre classes behave well with respect to flat pullback) If $f : X' \to X$ is flat, E a vector bundle on X, $\alpha \in A_* X$

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha).$$

(a) and (c) proved last time. (d) **Exercise.** Before proving (b), let me mention some useful consequences.

Corollary. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap \beta)$.

(We're going to use this soon in the proof of the splitting principle, so don't forget this. We'll also soon see that $A_m \mathbb{P}E \cong \oplus_{i=0}^e A_{m-i} X$. The inclusion $A_{m-i} X \hookrightarrow A_m \mathbb{P}E$ will be given by $c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap \beta$. The projection will be given by: "cap with $c_1(\mathcal{O}_{\mathbb{P}E}^{e-i}$ and push forward.)

Corollary. It makes sense to multiply by various polynomials in Segre classes of various bundles, by the commutativity part (b).

Proof of (b). It won't be surprising how we get commutativity. Consider the fibered square:

$$\begin{array}{ccccc} & & Q & & \\ & \swarrow^{q'} & & \searrow^{p'} & \\ \mathbb{P}E & & & & \mathbb{P}F \\ & \searrow_{p} & & \swarrow_{q} & \\ & & X & & \end{array}$$

where p and q are the projections; all morphisms are projective bundles. Let $f+1$ be the rank of F (and as usual $e+1$ is the rank of E). Then:

$$\begin{aligned} s_i(E) \cap (s_j(F) \cap \alpha) &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p^*(q_*(c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap q^*\alpha))) \\ &\qquad \text{(left side of desired equality)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap {q'}_*({p'}^*(c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap q^*\alpha))) \\ &\qquad \text{(pr. pushforwards and fl. pullbacks ``commute'')} \\ &= p_* q'_*(c_1({q'}^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap ({p'}^* c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap {p'}^* q^*\alpha)) \\ &\qquad \text{(proj. form. and flat pull. behaves well w.r.t. } c_1) \\ &= q_* p'_*({p'}^* c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap (c_1({q'}^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap {p'}^* q^*\alpha)) \\ &\qquad \text{(prop. pushforwards commute, and } c_1\text{'s commute)} \\ &= \text{(then go backwards to get desired result)} \end{aligned}$$

□

Exercise. Let E be a vector bundle of rank $e+1$, L a line bundle. Show that

$$s_p(E \otimes L) = \sum_{i=0}^{p} (-1)^{p-1} \binom{e+p}{e+i} s_i(E) c_1(L)^{p-i}$$

(Hint: Identify $\mathbb{P}E$ with $\mathbb{P}(E \otimes L)$, with universal subbundle $\mathcal{O}_{\mathbb{P}E}(-1) \otimes p^*L$. Then $s_p(E \otimes L) \cap \alpha = p_*((c_1(\mathcal{O}_{\mathbb{P}E}(1)) - c_1(p^*L))^{e+p} \cap p^*\alpha)$.)

1.2. **Chern classes.** We now define Chern classes. Define the Segre power series $s_t(E)$ to be the generating function of the s_i:

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i = 1 + s_1(E)t + s_2(E)t^2 + \cdots .$$

Define the *Chern power series* (soon to be Chern polynomial!) as the inverse of $s_t(E)$:

$$c_t(E) = \sum_{i=0}^{\infty} c_i(E) t^i = 1 + c_1(E)t + c_2(E)t^2 + \cdots .$$

$$c_t(E) s_t(E) = 1.$$

Hence $c_0(E) = 1$, $c_1(E) = -s_1(E)$, $c_2(E) = s_1(E)^2 - s_2(E), \dots,$
$$c_n(E) = -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - \cdots - s_n(E).$$

Note: The old-fashioned definition of $c_1(L)$ agrees with the new definition of $c_1(L)$, by the last part of the previous Theorem.

Chern class Theorem. The Chern classes satisfy the following properties.

(a) (vanishing) For all bundles E on X, and all $i > \operatorname{rank} E$, $c_i(E) = 0$.

(b) (commutativity) For all bundles E, F on X, integers i and j, and cycles α on X,
$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha).$$

(c) (projection formula, i.e. Chern classes behave well with respect to proper pushforward) Let E be a vector bundle on X, $f : X' \to X$ a proper morphism. Then
$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$
for all cycles α on X' and all i.

(d) (Chern classes behave well with respect to flat pullback) Let E be a vector bundle on X, $f : X' \to X$ a flat morphism. Then
$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$
for all cycles α on X, and all i.

(e) (Whitney sum) For any exact sequence
$$0 \to E' \to E \to E'' \to 0$$
of vector bundles on X, then $c_t(E) = c_t(E') \cdot c_t(E'')$, i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

(f) (Normalization) If E is a line bundle on a variety X, D a Cartier divisor on X with $\mathcal{O}(D) \cong E$, then $c_1(E) \cap [X] = [D]$.

(b), (c), and (d) follow from the Segre class theorem above. I explained (f) last time. Thus we have to show (a) and (e). I'll set up the right way of thinking about (a) and (e), and then prove them next day.

Splitting principle. This uses a very nice (and very important) construction, the splitting principle. It is *not* true that the every vector bundle splits into a direct sum of line bundles. However, the splitting principle in essence tells that we can pretend that every vector bundle splits, not into a direct sum, but into a nice filtration.

Given a vector bundle E on a scheme X, there is a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective, and
(2) f^*E has a filtration by subbundles
$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0.$$

Injectivity shows that if we can show some equality involving Chern classes on the pull-back to X', then it will imply the same equality downstairs on X.

The construction is pretty simple: it will be a tower of projective bundles. Recall that we showed earlier today that if F is any vector bundle on Y, and $g : \mathbb{P}F \to Y$, then $g^* : A_k X \mapsto A_{k+e}\mathbb{P}E$ is an injection, so we'll get (1) immediately. We'll constructive the tower of projective bundles inductively on the rank of E. If $r = 1$, we're already done. Otherwise, let $g : \mathbb{P}E \to X$. On $\mathbb{P}E$, we can split off the tautological subline bundle.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}E}(-1) & \longrightarrow & g^*E & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \Big| & & & & \\ & & & & \mathbb{P}E & & & & \\ & & & & \Big\downarrow{\scriptstyle g} & & & & \\ & & & & X & & & & \end{array}$$

Here Q is the quotient bundle of rank one less than that of E.

Thus we've shown how to split a single vector bundle. Clearly we can split any finite number of vector bundles in this way as well.

I stated the following result, and will prove it next time.

Lemma. Assume that E is filtered with line bundle quotients $L_1, \dots, L_r$. Let s be a section of E, and let Z be the closed subset of X where s vanishes. Then for any k-cycle α on X, there is a $(k-r)$-cycle β on Z with

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta$$

in $A_{k-r}X$. (Even better, we will see that we will get equality in $A_{k-r}(Z)$: we have pinned down (or "localized") this class even further.) In particular, if s is nowhere zero, then $\prod_{i=1}^{r} c_1(L_i) = 0$. (Recall $r = \operatorname{rank} E$.)

I suggested that people browse through the many examples in this chapter, including the Chern character and the Todd class.

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