INTERSECTION THEORY CLASS 7

RAVI VAKIL

CONTENTS

1. INTERSECTING WITH A PSEUDODIVISOR

Here's where we are. We have defined divisors of 3 sorts: Weil divisors, Cartier divisors, and pseudo-divisors (L, Z, s).

I'd like to make something more explicit than I have. An *effective* Cartier divisor on a scheme is a closed subscheme locally cut out by *one* function, and that function is not a *zero-divisor*. (Translation: the zero-set does not contain any associated points.) $\operatorname{Pic} X =$ group of line bundles $=$ Cartier divisors modulo linear equivalence $=$ Cartier divisors modulo principal Cartier divisors. We get a map from Cartier divisors to Weil divisors that descends to $\operatorname{Pic} X \to A_{\dim X - 1} X$.

We defined intersection with pseudo-divisors by linearity starting with $D \cdot [V]$, where $j : V \hookrightarrow X$ is a variety, by $D \cdot [V] = [j^* D]$. We'll do three things with this. First (or more correctly, last), this will be leveraged to define more complicated intersections, and to show that they behave well. Second, we'll use this to define the first Chern class of a line bundle, denoted $c_1(L) \cap$. Third, we'll use it to define the Gysin pullback for a closed immersion of an effective Cartier divisor $j : D \hookrightarrow X$.

Proposition 2.3.

(a) (linearity in α) If D is a pseudo-divisor on X, and α and α' are k-cycles on X, then
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \quad \text{in } A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|)).$$
(b) (linearity in D) If D and D' are pseudo-divisors on X, and α is a k-cycle on X, then
$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha \quad \text{in } A_{k-1}((|D| \cup |D'|) \cap |\alpha|).$$

Date: Wednesday, October 13, 2004.

(c) (projection formula) Let D be a pseudo-divisor on X, $f : X' \to X$ a proper morphism, α a k-cycle on X', and g the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by f. Then

$$g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \quad \text{in } A_{k-1}(|D| \cap f(|\alpha|)).$$

(d) (commutes with flat base change) Let D be a pseudo-divisor on X, $f : X' \to X$ a flat morphism of relative dimension n, α a k-cycle on X, and g the induced morphism from $f^{-1}(|D| \cap |\alpha|)$ to $|D| \cap |\alpha|$. Then

$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \quad \text{in } A_{k+n-1}(f^{-1}(|D| \cap |\alpha|).$$

(e) If D is a pseudodivisor on X whose line bundle $\mathcal{O}_X(D)$ is trivial, and α is a k-cycle on X, then

$$D \cdot \alpha = 0 \quad \text{in } A_{k-1}(|\alpha|).$$

Proof. (a) This follows from the definition; $D \cdot \alpha$ is linear in the second argument because it was defined by linearity and $D \cdot [V]$ for V a subvariety! Hence for the rest of the proof we can assume $\alpha = [V]$.

(b) Recall the definition of $D \cdot [V]$: We pull the pseudo-Cartier divisor D back to V. We take any Cartier divisor giving that pseudo-divisor (let me sloppily call this D as well). We then take the Weil divisor corresponding to that Cartier divisor: $D \mapsto \sum_W \operatorname{ord}_W(D)$. This latter is a group homomorphism.

(c) It suffices to deal with the case $X' = V$ and $X = f(V)$:

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d, "\text{proper}"] & X' \arrow[r, dash] \arrow[d, "\text{proper}"] & V \\
f(V) \arrow[r, hook] & X \arrow[r, dash] & D
\end{tikzcd}$$

If we've proved the desired result for the left portion of the above diagram, then we've proved what we wanted.

D can be chosen to be some Cartier divisor on $X = f(V)$. Note that f^*D is *also* Cartier: the support of D doesn't contain the generic point of $F(V)$ hence f^*D doesn't contain the generic point of V. Then we want to prove: $f_*[f^*D] = \deg(X'/X)[D]$. This is a local question on X, so we can assume $D = \operatorname{div}(r)$ for some rational function on X. Then we have:

$$f_*[\operatorname{div}(f^*r)] = [\operatorname{div}(N(f^*r))]$$

(came up in discussion of why proper pushforwards exist)

$$= \operatorname{div}(r^{\deg(X'/X)})$$

(definition of norm)

$$= (\deg X'/X)[\operatorname{div} r].$$

(d) (Skipped for the sake of time) Again we can assume $V = X$, so D is represented by a Cartier divisor. We want to prove $[f^*D] = f^*[D]$ as cycles on X. Both sides are additive, so we need only prove it for the case where D is effective. But we've shown earlier (Lemma in Section 2 of Class 4 notes, Oct. 4) that fundamental classes of subschemes behave well with respective to flat pullbacks, so we're done.

(e) We may assume again that $\alpha = [V]$, $V = X$, and D is a Cartier divisor on X. We know that D is principal. Then we want to show that $[D] = 0$ in $A_{k-1}(X)$. This follows from the fact that there is a group homomorphism from the group of Cartier divisors modulo linear equivalence (i.e. modulo principal divisors) to the group of Weil divisors modulo linear equivalence (the latter is $A_{k-1}(X)$). □

2. The first Chern class of a line bundle

The key result of this chapter is:

Big Theorem 2.4. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-1}(|D| \cap |D'|)$.

Proof soon.

Given a line bundle L of a scheme X, for any sub*variety* V of X, $L|_V$ is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor C on V (determined up to linear equivalence). The Weil divisor $[C]$ determines a well-defined element in $A_{k-1}(X)$, denoted by

$$c_1(L) \cap [V] := [C].$$

We extend this by linearity to get a map $c_1(L)\cap : Z_k X \to A_{k-1}X$.

Proposition 2.5.

(a) If α is rationally equivalent to 0 on X, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism $c_1(L)\cap : A_k X \to A_{k-1}X$. (That's what we'll usually mean by $c_1(L) \cap \cdot$.)

(b) (commutativity) If L, L' are line bundles on X, α a k-cycle on X, then

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \quad \text{in } A_{k-2}X.$$

(c) (projection formula) If $f : X' \to X$ is a proper morphism, L a line bundle on X, α a k-cycle on X', then

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \quad \text{in } A_{k-1}X.$$

(d) (flat pullback) If $f : X' \to X$ is flat of relative dimension n, l a line bundle on X, α a k-cycle on X, then

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha) \quad \text{in } A_{k+n-1}X'.$$

(e) (additivity) If L and L' are line bundles on X, α a k-cycle on X, then

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L' \cap \alpha) \quad \text{and}$$

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \quad \text{in } A_{k-1}X.$$

Proof. (a) follows from a corollary to our big theorem that I stated last time: If D is a pseudo-divisor on X, α a k-cycle on X which is rationally equivalent to 0. Then $D \cdot \alpha = 0$ in $A_{k-1}(|D|)$.

(b) follows from another corollary to the big theorem that I stated last day: If D and D' are pseudo-divisors on a scheme X. Then for any k-cycle α on X, $D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$ in $A_{k-2}(|D| \cap |D'| \cap |\alpha|)$.

The remaining three follow immediately from Proposition 2.3 above.

3. Gysin pullback

We also defined the *Gysin pullback*: Suppose $i : D \to X$ is an inclusion of an effective Cartier divisor. Define $i^* : Z_k X \to A_{k-1} D$ by

$$i^* \alpha = D \cdot \alpha.$$

Proposition.

(a) If α is rationally equivalent to zero on X then $i^* \alpha = 0$. (Hence we get induced homomorphisms $i^* : A_k X \to A_{k-1} D$.)
(b) If α is a k-cycle on X, then $i_* i^* \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_{k-1} X$.
(c) If α is a k-cycle on D, then $i^* i_* \alpha = c_1(N) \cap \alpha$ in $A_{k-1} D$, where $N = i^* \mathcal{O}_X(D)$. N is the normal (line) bundle. (Caution to differential geometers: D could be singular, and then you'll be confused as to why this should be called the normal bundle.)
(d) If X is purely n-dimensional, then $i^*[X] = [D]$ in $A_{n-1} D$.
(e) (Gysin pullback commutes with $c_1(L) \cap$) If L is a line bundle on X, then

$$i^*(c_1(L) \cap \alpha) = c_1(i^* L) \cap i^* \alpha$$

in $A_{k-2} D$ for any k-cycle α on X.

Proof. (a) follows from the first corollary last time.
(b) follows from the definition: both are $D \cap \alpha$, as a class on X.
(c) too: both are $D \cap \alpha$, but as a class on D.
(d) says that $[D] = D \cdot [X]$, which we proved earlier, although you may not remember it.
(e) follows from the second corollary from last time. □

4. Towards the proof of the big theorem

Big theorem. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cap [D]$ in $A_{n-2}(|D| \cap |D'|)$.

The case where D and D' have no common components, so $|D| \cap |D'|$ is codimension 2, boils down to algebra, and the details are thus omitted here. Here's how it boils down to algebra: $A_{n-2}(|D| \cap |D'|) = Z_{n-w}(|D| \cap |D'|)$, so rational equivalence doesn't come into it. This is a local question, so we can consider a particular codimension 2 point, and then consider an affine neighborhood of that $\operatorname{Spec} A$. Upon localizing at that point, we have a question about a dimension 2 local ring A_P.

So the real problem is what to do if D and D' have a common component. We'll deal with this by induction on this:

$$\epsilon(D, D') := \max\{\operatorname{ord}_V(D)\operatorname{ord}_V(D') : \operatorname{codim}(V, X) = 1\}.$$

Note that we know the result when $\epsilon = 0$.

The proof involves an extremely clever use of blowing up.

4.1. **Crash course in blowing up.** I'm going to repeat this next time, in more detail. Let X be a scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals on X. (Technical requirement automatically satisfied in our situation: $\mathcal{I}$ should be a coherent sheaf, i.e. finitely generated.) Here is the "universal property" definition of blowing-up. Then the blow-up of $\mathcal{O}_X$ along $\mathcal{I}$ is a morphism $\pi : \tilde{X} \to X$ satisfying the following universal property. $f^{-1}\mathcal{I}\mathcal{O}_{\tilde{X}}$ (the "inverse ideal sheaf") is an invertible sheaf of ideals, i.e. an effective Cartier divisor, called the *exceptional divisor*. (Alternatively: the scheme-theoretic pullback of the subscheme $\mathcal{O}/\mathcal{I}$ is a closed subscheme of $\tilde{X}$ which is (effective) Cartier, and this is called the exceptional (Cartier) divisor E.) If $f : Z \to X$ is any morphism such that $(f^{-1}\mathcal{I})\mathcal{O}_Z$ is an invertible sheaf of ideals on Z (i.e. the pullback of $\mathcal{O}/\mathcal{I}$ is an effective Cartier divisor), then there exists a unique morphism $g : Z \to \tilde{X}$ factoring f.

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

In other words, if you have a morphism to X, which, when you pull back the ideal I, you get an effective Cartier divisor, then this factors through $\tilde{X} \to X$.

As with all universal property statements, any two things satisfying the universal property are canonically isomorphic.

Theorem: Blow-ups exist. The proof is by construction: show that $\mathbf{Proj} \oplus_{d \geq 0} \mathcal{I}^d$ satisfies the universal property. (See Hartshorne II.7, although his presentation is opposite.)

This construction shows that in fact π is projective (hence proper).

I'm going to start next day by discussing three examples: (i) blowing up a point in the plane, (ii) blowing up along an effective Cartier divisor, and (iii) blowing up X along itself.

E-mail address: vakil@math.stanford.edu