

INTERSECTION THEORY CLASS 4

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CONTENTS

1. Proper pushforwards	1
2. Flat pullback	3
3. Parsimonious definition of Chow groups	7
4. Things Rob will tell you about on Wednesday	7
4.1. Excision exact sequence	7

Homework due on Monday:

1. Find the order of y/x at origin in $y^2 = x^3$ using the length definition.
2. In no more than half a page, explain why Bezout's Theorem for plane curves is true (i.e. explicate Fulton's Example 1.4.1). Feel free to assume that F is irreducible.

You can also get a "bye" for two weeks of homework by (at some point in the future) explaining to me the "rational equivalence pushes forward under proper morphisms" result (Prop. 1.4).

We're in the process of seeing that cycles (proper) pushforward and (flat) pullback, and that rational equivalences do to.

We need a lot of algebra to set ourselves up. This will decrease in later chapters.

Also, Rob will give Wednesday's class; he'll end Chapter 1 and start Chapter 2.

1. PROPER PUSHFORWARDS

1.1. For any subvariety V of X , let $W = f(V)$ be the image; it is closed (image of closed is closed for proper morphisms). I want to define f_*V . If $\dim W < \dim V$, define $f_*V = 0$. Otherwise, $R(V)$ is a finite field extension of $R(W)$ (both are field extensions of K of transcendence degree $\dim V$). Set

$$\deg(V/W) = [R(V) : R(W)].$$

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In the complex case, this degree is what you think it is: it's the number of preimages of a general point. In positive characteristic, this needn't be true; $K[t^p] \rightarrow K[t]$ gives a map of schemes that is degree p but is one-to-one on points.

Define $f_*: Z_k X \rightarrow Z_k Y$ by

$$f_*[V] = \deg(V/W)[W].$$

Note: If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_* f_*$.

Example: the parabola example (what happens to points, and to the entire parabola). Normalization.

Big Theorem. If $f : X \rightarrow Y$ is a *proper* morphism, and α is a k -cycle on X which is rationally equivalent to zero, then $f_* \alpha$ is rationally equivalent to zero on Y .

Hence there is a pushforward for Chow groups: $f_* : A_k X \rightarrow A_k Y$.

I'm not going to prove this; I'll only point out that we can reduce this statement to something simpler:

Little theorem. Let $f : X \rightarrow Y$ be a proper surjective morphism of varieties, and let $r \in R(X)^*$. Then

- (a) $f_*[\text{div}(r)] = 0$ if $\dim Y < \dim X$
- (b) $f_*[\text{div}(r)] = [\text{div } N(r)]$ if $\dim Y = \dim X$

In (b), $R(X)$ is a finite extension of $R(Y)$, and $N(r)$ is the *norm* of r .

This is a really natural reduction. We need only to prove it for a generator of rational equivalence, which involves $X' \hookrightarrow X$ of dimension $k + 1$, and $\alpha = \text{div}(r)$ for $r \in K(X')$. We can just work on X' instead. We can also replace Y by $f(X)$, because this construction doesn't care about anything else.

So we can now deal with two varieties. ...

Here are some consequences.

Bonus 1. We can now define the *degree* of a dimension 0 cycle class (= cycle mod rational equivalence) on something proper over K . (Definition: *complete* = *proper over K*. This is a common word, but I may try to avoid it.)

Definition. If $\alpha = \sum n_P P$ is a zero-cycle on X , define the *degree* of α to be $\sum n_P \deg[P/K]$ (the sum of the degree extensions). Example: $\text{Spec } \mathbb{Q}[x]/(x^2 + 2)$ over $\text{Spec } \mathbb{Q}$, there is one point, that *counts for 2*.

Definition/theorem. If X is a *complete scheme* then define the *degree* of an element of $A_0 X$ to be the degree of the pushforward to a point $\text{Spec } K$. This makes sense by the big theorem.

Homework: As a corollary, “prove” Bezout’s theorem for plane curves. Fulton essentially does this in Example 1.4.1, so read what he has to say, and write it up in your own words. (Less than a page is fine.)

Bonus 2. Recall that we were annoyed at having to working out $\text{ord}_{\mathcal{O}_{V,X}}(r)$ for $r \in R(X)$, and needing to use lengths, and not what we know about DVR’s. This theorem tells us we don’t have to. We could pull r back to the normalization \tilde{X} of X , which is regular in codimension 1. We work out how it vanishes on all the divisors mapping to V . (There are a finite number, by finiteness of normalization, which I said earlier.)

Exercise. Check your answer to $\text{ord}(y/x)$ at $(0,0)$ on $y^2 = x^3$, i.e. $k[x,y]/(y^2 - x^3)$ by pulling it back to the normalization, which is $k[t]$, given by $t \mapsto (t^2, t^3)$.

2. FLAT PULLBACK

2.1. Crash course in flat morphisms. A morphism $f : X \rightarrow Y$ is *flat* if locally it can be written as $f : \text{Spec } A \rightarrow \text{Spec } B$ (so $B \rightarrow A$) where A is *flat* B -module. A B -module M is *flat* if for every exact sequence

$$0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0,$$

the sequence

$$0 \rightarrow M \otimes P \rightarrow M \otimes Q \rightarrow M \otimes R \rightarrow 0$$

is also exact. (The only issue is the inclusion $M \otimes P \hookrightarrow M \otimes Q$.)

Idea “flat morphisms are nice”. They are more general than fibrations, but have all the same properties.

Easy facts to know:

- flatness is preserved by base change
- anything is flat over a point (as all modules over a field are flat!)
- the composition of flat morphisms is again flat
- open immersions are flat. projections from a vector bundle or \mathbb{A}^n -bundles are flat ($R[x_1, \dots, x_n]$ is a flat R -module). The projection $Y \times Z \rightarrow Z$ is flat (using base change and the 2nd bullet point).

Harder facts to know:

- A dominant morphism $X \rightarrow Y$ from a variety to a smooth curve is flat.
- More generally, a morphism from a scheme to a smooth curve is flat iff all associated points of X map to the generic point of Y .
- Hence: If $X \rightarrow Y$ is a morphism of varieties, there is no “dimension-jumping”. (Otherwise, if $X \rightarrow Y$ has dimension-jumping, basechange to a smooth curve that “sees” the dimension-jumping, and then use this fact.)
- More general fact still: If $X \rightarrow Y$ is a morphism of schemes, then associated points of X map to associated points of Y .

Joe asked about another useful fact: in the case of morphisms of finite type, *flat morphisms are open*, i.e. the image of an open set is an open set.

Examples of flat morphisms: the map of the parabola to the line is one. Reason: $k[x]$ is a flat $k[x^2]$ -module, as it is a free $k[x^2]$ -module.

(Draw also a family of nodal curves.)

Goal: flat pullbacks exist. In other words, we'll define out how cycles pullback, and then we'll check that rational equivalences pull back to rational equivalences.

You can see why we don't like dimensional jumping. But it's interesting that we don't mind degenerations as in the family of nodal curves, or in the parabola example.

Definition. Let Y be a pure k -dimensional scheme, with irreducible components Y_1, \dots, Y_q . Then define the *fundamental cycle* $[Y]$ to be $\sum_1^q m_i [Y_i]$ in $Z_k(Y)$. where m_i is the length of $\mathcal{O}_{Y_i, Y}$. (The local rings $\mathcal{O}_{Y_i, Y}$ are "local Artin rings", corresponding to zero-dimensional local schemes.)

Example: $k[x, y]/(y^2(x + y)^3)$. The length of the local rings at the two generic points are 2 and 3 respectively.

(Note: if Y is a subscheme of X , then $[Y]$ is naturally in $Z_k(X)$ of course; $Z_k[Y] \hookrightarrow Z_k[X]$ of course.)

Definition (pulling back cycles). Suppose $f : X \rightarrow Y$ is flat of relative dimension n . If V is an irreducible subvariety of Y , let $f^*[V] := [f^{-1}(V)]$. Then by linearity, I know how to pull back any linear combination of subvarieties. Hence I've defined $f^* : Z_k Y \rightarrow Z_{k+n} X$.

What's wrong with that? Well, we don't know that $(gf)^* = f^*g^*$. Example (picture omitted in notes): branched double cover of a branched double cover. Fortunately we get 4 both ways. But does this work in general?

Lemma (pulling back fundamental classes). If $f : X \rightarrow Y$ is flat, then for any equidimensional subscheme Z of Y , $f^*[Z] = [f^{-1}(Z)]$. In other words, the pullback of a fundamental class of a scheme is the fundamental class of the pullback of a scheme.

(Direct algebra from the appendix; omitted.)

This makes us happy, because schemes pullback nicely; $f^{-1}g^{-1}(Z) = (gf)^{-1}(Z)$. Thus pullbacks are functorial.

Proposition (Flat pullback commutes with proper pushforward). Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibered square, with g flat and f proper (so g' flat and f' proper). Then $f'_*g'^*\alpha = g_*f_*\alpha$.

This is on the level of *cycles*. We don't yet know that we can flat-pullback cycle classes.

Proof also by direct algebra. Reduce first to the case where X and Y are varieties, and f is surjective. Here are the reductions: it suffices to do this for a generator of $Z_k X$, so $\alpha = [V]$ where V is a variety. $f(V)$ is also a variety (remember f is proper, hence $f(\text{closed})$ is closed). Base change the entire square by $f(V) \rightarrow Y$. Then we can assume $f(V) = Y$. Next base change the upper arrow by $V \rightarrow X$:

$$\begin{array}{ccc}
 X' \times_X V & \longrightarrow & V \\
 \downarrow & & \downarrow \\
 X' & \xrightarrow{g'} & X \\
 \downarrow f' & & \downarrow f \\
 Y' & \xrightarrow{g} & Y
 \end{array}$$

Then turn this into a calculation involving local rings (omitted). □

We'll next show that rational equivalences flat-pullback to rational equivalences. Hence we'll have shown that we have flat pullback of Chow groups.

Preliminary Algebraic Lemma. Let X be a purely n -dimensional scheme, with irreducible components X_1, \dots, X_r , and geometric multiplicities m_1, \dots, m_r . Let D be an effective Cartier divisor on X . Let $D_i = D \cap X_i$ be the restriction of D to X_i . Then $[D] = \sum m_i [D_i]$ in $Z_{n-1}(X)$.

(An *effective Cartier divisor* is a subscheme locally cut out by a single function that is not a zero-divisor.)

This is certainly reasonable! (Draw a picture, when D doesn't have a component along the intersection of two of the X_i 's.) I omitted this explanation in class due to time.

Proof. One checks this along each Weil divisor V of X . Immediately reduces to algebra. Let me get us to the algebra. We'll check that each codimension one subvariety V of X appears with the same multiplicity on both sides of the equation. We reduce to the local situation: let A be the local ring of X along V , and $a \in A$ a local equation for D . The minimal prime ideals p_i in A correspond to the irreducible components X_i of X which contain V .

$m_i = l_{A/p_i}(A/p_i)$. The multiplicity of $[V]$ in $[D]$ is $l_A(A/aA)$. The multiplicity of $[V]$ in $[D_i]$ is $l_{A/p_i}(A/(p_i + aA))$. So we want to show:

$$l_A(A/aA) = \sum m_i l_{A/p_i}(A/p_i + aA).$$

This is shown in the appendix. □ □

Preliminary Geometric Lemma. A cycle α in $Z_k X$ is rationally equivalent to zero if and only if there are $(k + 1)$ -dimensional subvarieties V_1, \dots, V_t of $X \times \mathbb{P}^1$, such that the projections from V_i to \mathbb{P}^1 are dominant, with

$$\alpha = \sum_{i=1}^t ([V_i(0)] - [V_i(\infty)])$$

(Draw picture.)

Before I get into it, notice that flatness is already in the picture here: each $V_i \rightarrow \mathbb{P}^1$ is flat. We'll see that $[V_i(0)]$ is the flat pullback of 0, and ditto for ∞ .

Proof. (I only roughly sketched this proof in class.) If there are such subvarieties, then $\alpha \sim 0$: Certainly the classes on $X \times \mathbb{P}^1$ are each rationally equivalent to 0 by the definition of rational equivalence. The projection $X \times \mathbb{P}^1 \rightarrow X$ is proper (because $\mathbb{P}^1 \rightarrow \text{pt}$ is proper, and properness is preserved by base change).

Now for the other direction. We need to show this for a generator of rational equivalence on X , so there is a subvariety W of dimension $k + 1$ in X , and a rational function on W $r \in R(W)^*$. This gives a rational map $W \dashrightarrow \mathbb{P}^1$. Let V be the closure of the graph of this rational map, so $V \subset W \times \mathbb{P}^1 \hookrightarrow X \times \mathbb{P}^1$. (The generic point of W maps to the generic point of \mathbb{P}^1 , so the same is true of V .) V maps birationally and properly onto W . That morphism is degree 1. If f is the induced morphism to \mathbb{P}^1 , then $\text{div}(r) = p_*[\text{div}(f)]$ by our big theorem on proper pushforwards, which in turn equals $[V(0)] - [V(\infty)]$. \square

Theorem. Let $f : X \rightarrow Y$ be flat of relative dimension n , and $\alpha \in Z_k(Y)$ which is rationally equivalent to 0. Then $f^*\alpha$ is rationally equivalent to 0 in $Z_{n+k}X$.

Thus we get flat pullbacks $f^* : A_k Y \rightarrow A_{k+n} X$.

Proof. (I did not give this proof in class.) We may deal with a generator of rational equivalence. Thanks to the geometric lemma, we can take our generator to be of the form $\alpha = [V(0)] - [V(\infty)]$.

$$\begin{array}{ccccc}
 W = & & (f \times 1)^{-1}(V) \xrightarrow{\text{cl. imm.}} & X \times \mathbb{P}^1 & \xrightarrow[p_{\text{flat prop.}}]{p} & X \\
 & \searrow h & \downarrow \text{flat} & \downarrow \text{flat} & \text{flat} & \downarrow f \\
 & & V & \xrightarrow{\text{cl. imm.}} & Y \times \mathbb{P}^1 & \xrightarrow[q_{\text{flat prop.}}]{q} & Y \\
 & & \downarrow g & & & & \\
 & & \mathbb{P}^1 & & & &
 \end{array}$$

We have a cycle α that is rationally equivalent to 0 on Y . It is the proper pushforward of $[g^{-1}(0)] - [g^{-1}(\infty)]$ from V . When we pull back this class from Y to X , we want to see that it is rationally equivalent to 0. But by our lemma showing that proper pushforwards and flat pullbacks commute, that's the same as pulling back to W , and pushing forward to X . The pullback to W is $[h^{-1}(0)] - [h^{-1}(\infty)]$.

We feel like we're done: we just push this forward to X , and that should be it. But: W may not be a variety (it may be reducible and nonreduced), so we don't (yet) know that this class is rationally equivalent to 0. This is why we need our algebraic lemma. Let $[W] = \sum m_i [W_i]$. Since

$$[h_i^{-1}(0)] - [h_i^{-1}(\infty)] = \text{div}(h_i)$$

is rationally equivalent to 0, it suffices to verify that $[h^{-1}(P)] = \sum m_i [h_i^{-1}(P)]$ (and then plug in $P = 0$ and ∞). And that's precisely what the algebraic lemma tells us. \square

3. PARSIMONIOUS DEFINITION OF CHOW GROUPS

(I discussed this aside rather quickly.)

I'd promised earlier that Chow groups would satisfy 3 conditions: (a) 0 would be rationally equivalence to ∞ in \mathbb{P}^1 . (b) They would satisfy flat pullbacks. (c) They would satisfy proper pushforward.

We've shown this. Now note that these three things define Chow groups. Translation: anything satisfying these three things is a quotient of the Chow group, so the Chow group is the "minimal" thing satisfying these three conditions. To prove this, all we have to do is show that if W is a $(k + 1)$ -dimensional subvariety of X , and r is a rational function on W , then $\text{div}(r)$ is forced to be 0. $W \xrightarrow{r} \mathbb{P}^1$. Pullback $(0) - (\infty)$. Pushforward by closed immersion $W \rightarrow X$.

Something else to point out: the divisor of zeros and poles of a rational function r on a variety W is easy to understand if W is regular in codimension 1. It was a pain otherwise. Here's an alternate way of computing it.

Pull back the function r to \tilde{W} . Do the calculation there. Then take proper pushforward.

4. THINGS ROB WILL TELL YOU ABOUT ON WEDNESDAY

4.1. Excision exact sequence. Proposition. Let Y be a closed subscheme of X , and let $U = X - Y$. Let $i : Y \hookrightarrow X$ be the closed immersion (proper!) and $j : U \rightarrow X$ be the open immersion (flat!). Then

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \longrightarrow 0$$

is exact for all k .

(Aside: you certainly expect more on the left!)

Proof. We quickly check that

$$Z_k Y \xrightarrow{i_*} Z_k X \xrightarrow{j^*} Z_k U \longrightarrow 0$$

is exact. (Do it!) Hence we get exactness on the right in our desired sequence. We also get the composition of the two left arrows in our sequence is zero.

Next suppose $\alpha \in Z_k X$ and $j^* \alpha = 0$. That means $j^* \alpha = \sum_i \text{div } r_i$ where each $r_i \in R(W_i)^*$, where W_i are subvarieties of U . So r_i is also a rational function on $R(\overline{W}_i)$ where \overline{W}_i is the closure in X . To be clearer, call this rational function \bar{r}_i . Hence $j^*(\alpha - \sum [\text{div}(\bar{r}_i)]) = 0$ in $Z_k U$, and hence $j^*(\alpha - \sum [\text{div}(\bar{r}_i)]) \in Z_k Y$, and we're done. \square

Rob will also state:

Definition. $Y \rightarrow X$ is an *affine bundle of rank n* over X if there is an open covering $\cup U_\alpha$ of X such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^n \rightarrow U_i$. This is a flat morphism.

Proposition. Let $p : E \rightarrow X$ be an affine bundle of rank n . Then the flat pullback $p^* : A_k X \rightarrow A_{k+n} E$ is surjective for all k .

Proof omitted.

Immediate corollary: $A_k \mathbb{A}^n = 0$ for $k \neq n$.

He may not state the rest:

Exercise: Example 1.9.3 (a). Show that $A_k(\mathbb{P}^n)$ is generated by the class of a k -dimensional linear space. (Hint: use the excision exact sequence.)

Example 1.9.4: Let H be a reduced irreducible hypersurface of degree d in \mathbb{P}^n . Then $[H] = d[L]$ for L a hyperplane, and $A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d\mathbb{Z}$. Thus the codimension 1 Chow group is torsion. (Caution: where are you using reduced and irreducible?)

$Z_k X \otimes Z_l Y \rightarrow Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$.

Proposition.

- (a) if $\alpha \sim 0$ then $\alpha \times \beta \sim 0$. There are exterior products $A_k X \otimes A_l Y \rightarrow A_{k+l}(X \times Y)$.
- (b) If f and g are proper, then so is $f \times g$, and $(f \times g)_*(\alpha \times \beta) = f_* \alpha \times g_* \beta$. Hence exterior product respects proper pushforward.
- (c) If f and g are flat of relative dimensions m and n , (so $f \times g$ is flat of relative dimension $m + n$), then

$$(f \times g)^*(\alpha \times \beta) = f^* \alpha \times g^* \beta.$$

Hence exterior product respects flat pullback.

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