INTERSECTION THEORY CLASS 3

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CONTENTS

The new times will be **Mondays 9–10:50** and **Wednesdays 10–10:50**. Because this is an advanced course, I won't have office hours; I'm happy to talk about it at any time. My 210 office hours are MW2:05–3 in case you want a specific time when I'll definitely be in my office.

1. LAST DAY

Some comments on last day:

I should have been clearer on what I meant by "numbers of zeros and poles of a rational function $r \in R(X)$ along a Weil divisor (codimension 1 subvariety)." I meant the function $\mathrm{ord}_V(r)$. I defined it as follows. If r is actually defined at the generic point of V, we have

$$\boxed{\mathrm{ord}_V(r) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(r))}$$

and then we define additively for quotients of two such: $\mathrm{ord}_V(r/s) = \mathrm{ord}_V(r) - \mathrm{ord}_V(s)$. Recall "length" is the one more than the length of the longest series of nested modules you can fit in a row, so the "length" of a vector space over K is its dimension.

The algebraic fact from Fulton shows that this function is well-defined. In doing the following exercise, *use* this definition.

Exercise. Consider y/x on $y^2 = x^3$. What is the order of this pole/zero?

I then defined the Chow group.

$$\boxed{Z_k X = \left\{ \sum n_i [V_i], \quad n_i \in \mathbb{Z} \right\}.}$$

is the group of k-cycles. A cycle is *positive* if all $n_i \geq 0$, some $n_i > 0$. (I may have forgotten to say this.)

Date: Friday, October 1, 2004.

The homotopies, or "rational equivalences" among k-cycles, were generated as follows. For any $(k+1)$-dimensional subvariety W of X, and any nonzero rational function $r \in R(W)^*$, define a K-cycle on X by

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_V(r)[V].$$

This generates a subgroup $\operatorname{Rat}_k X$, the subgroup of *cycles rationally equivalent to* 0.

Then $A_k(X) = Z_k X / \operatorname{Rat}_k X$.

2. Proper, projective, finite

2.1. Proper, projective, finite morphisms. Crash course in proper morphisms: A morphism $f : X \to Y$ is said to be *proper* if it is separated (true in our case of algebraic schemes), of finite type (true in our case), and *universally closed*. (Closed: takes closed sets to closed sets. Universally closed: for any $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.)

Some pictures: $f : \mathbb{A}^1 \to \operatorname{Spec} K$ is not proper. f is certainly separated and of finite type and closed, so what's the problem? Consider the fibered diagram:

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow f \\ \mathbb{P}^1 & \longrightarrow & \operatorname{Spec} K \end{array}$$

The projection on the left isn't closed: consider the graph of $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.

First approximation of how to think of proper morphisms, if you are a complex geometer: fibers are compact in the analytic topology. Warning: $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ isn't proper (it isn't closed), so I need to say something a bit more refined.

Key examples: *projective* morphisms are proper. As I said last day, a morphism $f : X \to Y$ is projective if Y can be covered by opens such that on each open U, $f^{-1}(X) \times_Y U \to U$ factors $f^{-1}(X) \times_Y U \hookrightarrow \mathbb{P}^k \times U \to U$ where the left morphism is a *closed immersion*.

Finite morphisms are projective, hence proper. A morphism is *finite* if for each affine open $U = \operatorname{Spec} S$, $f^{-1}(U)$ is affine $= \operatorname{Spec} R$, and the corresponding map of rings $S \to R$ is a finite ring extension, i.e. R is a finitely generated S-*module* (which is stronger than a finitely generated S-*algebra*!). I'll repeat the example from last time: parabola double-covering line. (How to recognize: finite implies each point of target has finite number of preimages. Reverse implication isn't true. finite = proper plus this property.) Another example: closed immersion.

Third (important) example: normalization (in good cases, such as those we'll consider). This requires a theorem in algebra! Normalization of an affine algebraic scheme $\operatorname{Spec} R$ is $\operatorname{Spec} \tilde{R}$, where $\tilde{R}$ is the normalization of R in its function field. Normalization of an algebraic scheme X in general is obtained by gluing. (Theorem: this is possible, and also independent of what affine cover you take of X.)

Crash course in normalization: given a variety W, define its normalization as follows. If A is affine, let $\tilde{A}$ be its integral closure in its function field $R(A) = R(W)$. We have $\operatorname{Spec}\tilde{A} \to \operatorname{Spec} A$. Do this for every open affine set of W. Fact: they all glue together. The result is called the normalization. Fact: The normalization map $\tilde{W} \to W$ is finite (algebra fact, Hartshorne Thm I.3.9A), hence proper. Hence: normalizations are regular in codimension 1. (Proof: all local rings are integrally closed; in particular true for dimension 1 rings = codimension 1 subvarieties; hence any dimension 1 local rings $(A, \mathfrak{m})$ is a discrete valuation ring, which (thanks to an earlier crash course) satisfies $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$, which is the definition of nonsingularity.

Finite, projective, and proper morphisms are preserved by base change: if f is one of them, then f' is too in the following fiber diagram:

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Z \end{array}$$

(They are also preserved by composition: f, g proper implies $g \circ f$ is too. Ditto for projective and finite.)

3. PROPER PUSHFORWARDS

3.1. For any subvariety V of X, let $W = f(V)$ be the image; it is closed (image of closed is closed for proper morphisms). I want to define f_*V. If $\dim W < \dim V$, define $f_*V = 0$. Otherwise, $R(V)$ is a finite field extension of $R(W)$ (both are field extensions of K of transcendence dgree $\dim V$). Set

$$\boxed{\deg(V/W) = [R(V) : R(W)].}$$

Define $f_*Z_kX \to Z_kY$ by

$$\boxed{f_*[V] = \deg(V/W)[W]}.$$

Note: If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_*f_*$.

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