INTERSECTION THEORY CLASS 2

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CONTENTS

The webpage http://math.stanford.edu/~vakil/245/ is up, and has last day's notes.

The new times *starting next week* will be **Mondays 9–10:50** and **Wednesdays 10–10:50**. So there *will be* a class on Friday.

To do: read the summaries of Chapters 1 and 2.

Looking over today's notes, I realize that what will be newest and most disconcerting for those who haven't seen schemes is the fact that we can localize at the generic point of a subvariety X of a scheme Y. What this means is that we are considering the ring of rational functions defined in a neighborhood of the generic point of X in Y; in other words, they are defined on a dense open subset of X. This is indeed a ring (you can add and multiply). The dimension of this ring is the difference of the dimensions of X and Y (or more precisely dimensions of X and "Y near X"). Recall that the points of Y correspond to irreducible subvarieties of Y; the "old-fashioned" ("before schemes") points are the *closed* points in the Zariski topology. So what are the points of $\operatorname{Spec} \mathcal{O}_{X,Y}$, or equivalently, what are the prime ideals of the ring $\mathcal{O}_{X,Y}$? They are the irreducible subvarieties of Y *containing* X. The *maximal* ideal of this local ring corresponds to X itself.

1. LAST DAY

1.1. Examples. I showed you some examples. For example: Parabola $x = y^2$ projected to t-line. $\mathbb{Q}[t] \mapsto \mathbb{Q}[x,y]/(x-y^2)$ via $t \to x$. (I'm letting my field be $\mathbb{Q}$ for the moment.) Intersecting parabola with a vertical line $x = \alpha$. We get the scheme

$$\operatorname{Spec} \mathbb{Q}[x,y]/(x-y^2, x-\alpha) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2-\alpha)$$

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which is length 2 over the base field $\mathbb{Q}$. If $\alpha = 1$, we get 2 points:

$$\mathbb{Q}[y]/(y^2 - \alpha) \cong (K[y]/(y+1)) \oplus (K[y]/(y-1))$$

If $\alpha = 0$, we get 1 point, with multiplicity 2:

$$\mathbb{Q}[y]/(y^2)$$

has only one maximal ideal. If $\alpha = 2$, we get 1 point with multiplicity 1, but this point has "degree 2 over $\mathbb{Q}$"; the residue field is a degree 2 extension of $\mathbb{Q}$.

1.2. Strategy. We're going to define Chow groups of a variety X as cycles modulo "homotopy" (called *rational equivalence*). Dimension k cycles are easy: they are dimension k subvarieties of X. More subtle is rational equivalence.

(1) Two points on $\mathbb{P}^1$ are defined to be rationaly equivalent.
(2) If $X \to Y$ is *flat* then there is a pullback. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) If $X \to Y$ is *proper* (new definition!) then we have a pushforward: $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

Just to be clear before we start: throughout this course we'll work over a field, to be denoted K. We'll consider schemes X that are sometimes called *algebraic schemes over* K. They are *schemes of finite type over* K. This means that you get them by gluing together a finite number of affine schemes of the form $\operatorname{Spec} K[x_1, \ldots, x_n]/I$. Mild generalization of algebraic variety. All morphisms between algebraic schemes are *separated* and *of finite type*. In this language, a variety is a reduced irreducible algebraic scheme. We'll end up localizing schemes: this leads to the notation of "essentially of finite type" = localizations of schemes/rings of finite type.

2. Zeros and poles

Given a *rational function* on an irreducible variety X, I'll define its order of pole or zero along a codimension 1 variety. (A *rational function* is a(n algebraic) function on some dense (Zariski-)open set.)

An irreducible codimension 1 variety is called a *Weil divisor*.

Example: $(x-1)^2(x^2-2)/(x-3)$ over $\mathbb{C}$. Over $\mathbb{Q}$. Weil divisors.

If X is generically nonsingular=smooth along Weil divisor, then "the same thing will work". More precicsely, in this case the local ring along the subvariety is dimension 1, with $\mathfrak{m}/\mathfrak{m}^2 = 1$, i.e. it is a *discrete valuation ring*, which I'll assume you've seen.

Discrete valuation rings are certain local rings $(A, \mathfrak{m})$. Here are some characterizations:

- an integral domain in which every ideal is principal over K
- a regular local ring of dimension 1
- a dimension 1 local ring that is integrally closed in its fraction field

- etc.

If generator of $\mathfrak{m}$ is π, then the ideals are all of the form (π^n) or 0. The corresponding scheme as 2 points; it is "the germ of a smooth curve".

Examples: $K[x, y]$, localized along divisor $x = 0$. We get rational functions of the form $f(x, y)/g(x, y)$ where x is not a factor of g. This is a local ring, and it is a DVR! Given any rational function, you can tell me the order of poles or zeros. (Ask: $(x^2 - 3y)/(x^2 + x^4 y)$?) Then this also works if x is replaced by some other irreducible polynomial, e.g. $x^2 - 3y$. This is nice and multiplicative.

So what if X is *singular* along that divisor ($\dim \mathfrak{m}/\mathfrak{m}^2 > 1$)? Example: $y^2 = x^3 - x^2$, the rational function y/x.

Exercise. Consider y/x on $y^2 = x^3$. What is the order of this pole/zero? (This will be homework, due date TBA.)

Patch 1: If V is a Weil divisor, and r is a rational function that gives an element of the local ring $\mathcal{O}_{V,X}$, then define

$$\boxed{\mathrm{ord}_V(r) = \dim_K \mathcal{O}_{V,X}/(r)}.$$

(What it means to be in the local ring, intuitively: at a general point of V it is defined. More precisely: there is an open set meeting V — not necessarily containing it — where the rational function is an actual function. For example, x/y on $\operatorname{Spec} K[x, y]$ is defined near the generic point of $x = 0$. Language of *generic points*.) Then given a general rational function, f, we can always write it as $f = r_1/r_2$, where r_1 and r_2 lie in the local ring.

(But we need to check that if we write f as a fraction in two different ways, then the answer is the same. That's true. More on that in a minute.)

Technical problem: If you have a dimension 1 local ring $(A, \mathfrak{m})$ with quotient field K, then A isn't necessarily a K-vector space. $\mathbb{Z}_{(p)}$, $p\mathbb{Z}_{(p)}$. (Exercise: Find an example in characteristic 0.)

Better:

$$\boxed{\mathrm{ord}_V(r) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(r))}.$$

Recall "length" is the one more than the length of the longest series of nested modules you can fit in a row. So the "length" of a vector space over K is its dimension.

Fact: ord **is well-defined (Appendix A.3):** If $ab = cd$ then $l(A/(a)) + l(A/(b)) = l(A/(c)) + l(A/(d))$. Hence this thing is well-defined.

Facts about facts. (I will pull facts out of Fulton's appendix as black boxes. But if you take a look at the appendix, you'll see that these results are very easy. The vast majority of proofs in A.1–A.5 are no longer than a few lines. With the exception of the section on determinantal identities — which we likely won't use in this course — I think almost no proof is longer than half a page. He even has a crash course in algebraic geometry in Appendix B.)

Fact: finiteness of zeros and poles (Appendix B.4.3). For a given r, there are only a finite number of Weil divisors V where $\operatorname{ord}_V(r) \neq 0$.

3. The Chow group

Let X be an algebraic scheme (again: finite type over field K). Recall: A k-cycle is a *finite* formal sum $\sum n_i[V_i]$, $n_i \in \mathbb{Z}$. A cycle is *positive* if all $n_i \geq 0$, some $n_i > 0$. (I forgot to mention this.) Call this $Z_k[X]$, the group of k-cycles.

$$\boxed{Z_k[X] = \left\{ \sum n_i[V_i], \quad n_i \in \mathbb{Z} \right\}.}$$

For any $(k+1)$-dimensional subvariety W of X, and any nonzero rational function $r \in R(W)^*$, define a K-cycle on X by

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_V(r)[V].$$

This generates a subgroup $\operatorname{Rat}_k X$, the subgroup of *cycles rationally equivalent to* 0.

(You can probably see where I'm going to go with this.) Define

$$\boxed{A_k(X) = Z_k[X]/\operatorname{Rat}_k[X]}$$

(Say visually.)

Note: this definition doesn't care about any nonreduced structure on X: $A_k[X] \equiv A_k[X^{\mathrm{red}}]$.

4. Proper pushforwards

Next day we'll see that rational equivalence pushes forward under proper maps. First:

4.1. Crash course in proper morphisms:. A morphism $f : X \to Y$ is said to be *proper* if it is separated (true in our case of algebraic schemes), of finite type (true in our case), and *universally closed*. (Closed: takes closed sets to closed sets. Universally closed: for any $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.) Key examples: *projective* morphisms are proper. A morphism $f : X \to Y$ is projective if Y can be covered by opens such that on each open U, $f^{-1}(X) \times_Y U \to U$ factors $f^{-1}(X) \times_Y U \hookrightarrow \mathbb{P}^k \times U \to U$ where the left morphism is a *closed immersion*.

Finite morphisms are projective, hence proper. A morphism is *finite* if for each affine open $U = \operatorname{Spec} S$, $f^{-1}(U)$ is affine $= \operatorname{Spec} R$, and the corresponding map of rings $S \to R$ is a finite ring extension, i.e. R is a finitely generated S-*module* (which is stronger than a finitely generated S-*algebra*!). Example: parabola double-covering line. (How to recognize: finite implies each point of target has finite number of preimages. Reverse implication isn't true. finite = proper plus this property.) Another example: closed immersion.

Finite, projective, and proper morphisms are preserved by base change: if f is one of them, then f' is too in the following fiber diagram:

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Z \end{array}$$

(They are also preserved by composition: f, g proper etc. implies $g \circ f$ is too.)

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