1. Linear Systems

General Setup: L is a line bundle on our space X. We choose some nontrivial subspace of sections $V \subset H^0(X, L)$ with basis $\{s_0, \ldots, s_r\}$.

General Goal: To understand the map to $\mathbb{P}^r$ induced by the sections, which 'should' be $x \mapsto [s_0(x); \ldots ; s_r(x)]$.

Possible problems:

(1) Does a choice of coordinate on our line bundle L alter the map?
   Nope, since projective space is nice under scalar action

(2) Projective space isn't supposed to have a point $[0; \ldots ; 0]$, so what do we do when $x \in \ker s_i$ for each s_i?
   This is a serious issue.

The good news about our second problem is that the offending set is as nice as we could want: it is naturally the (closed) subscheme of X which is cut out by the section s_i. This subscheme is the base locus of V, which we'll write as B. So we at least have a rational map $\varphi : X - B \to \mathbb{P}^r$.

Example. Let $X = \mathbb{P}^2$, and $L = O_{\mathbb{P}^2}(2)$. Some sections: $J := \{x^2, xy, y^2, xz, yz, z^2\}$. What's the base locus? It's nothing! Yeah! So we get a bona fide map $\mathbb{P}^2 \to \mathbb{P}^5$.

Example. Let's stick with our scheme and line bundle, but pick out some new sections: $R := \{x^2, xy, y^2, xz, yz\}$. Will we get lucky and not have a base locus again? No. This time $B = [0; 0; 1]$. So we get a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined away from $[0; 0; 1]$.

We're not happy with just having a rational map, since we think we should be able to 'fill in' the missing information.

Example. Let's return to the previous example for a minute, and see what's happening near the base locus. We'll let $[\alpha t; \beta t; 1]$ be a point near $[0; 0; 1]$ (of course, we won't let $\alpha = \beta = 0$ yet). Where does this point go?

$$[\alpha^2 t^2; \alpha\beta t^2; \beta t^2; \alpha t; \beta t] = [\alpha^2 t; \alpha\beta t; \beta^2 t; \alpha; \beta].$$

It seems that as we approach $[0; 0; 1]$ along the line $[\alpha t; \beta t; 1]$, the map is taking us to $[0; 0; 0; \alpha; \beta]$. Hmm...seems that we're getting an idea of what happens to a tangent to our base locus...what does that make us think of?

In fact, there is a way to extend the map we've been thinking about to the blowup of X along B, $\tilde{X} = Bl_B X$.

$$\begin{tikzcd} E \arrow[r, hook] & \tilde{X} = Bl_B X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ B \arrow[r, hook] & X \arrow[r, dashed] & \mathbb{P}^r \end{tikzcd}$$

How can we get our hands on f? We're going to pull back the line bundle L to $Bl_B X$ and tweak it remove the base locus. Fact: the bundle $\pi^* L - E$ has no base locus. Now we'll use this new line bundle to map to $\mathbb{P}^r$, and all will be well.

Example. Suppose our variety is $\mathbb{P}^1$, and we've chosen the bundle $O_{\mathbb{P}^1}(2)$, with sections $\{x^2, xy\}$. This gives a map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ away from $[0;1]$, though we 'should' know how to fix this map to make it a bona fide map. How do we resolve this? Let's follow the formula above: we'll blow up $\mathbb{P}^1$ along $[0;1]$ (this will just give us $[0;1] \hookrightarrow \mathbb{P}^1$ back again, since $[0;1]$ is cartier and cut out by $x = 0$), and twist our sections by $-E$, which in this case means divide by x.

$$\begin{tikzcd} {[0;1]} \arrow[r, hook] & \mathbb{P}^1 \arrow[d, "\pi"'] \arrow[dr, "{[x;y]}"] & \\ {[0;1]} \arrow[r, hook] & \mathbb{P}^1 \arrow[r, dashed, "{[x^2;xy]}"] & \mathbb{P}^2 \end{tikzcd}$$

Aren't we in intersection theory? So you might want to know why I'm talking about this in an intersection theory class. The answer is that we can use chern classes to say something about the degree of $f_*[\tilde{X}]$, which we will see is connected to the segre class of B in X and the degree of the map f. Yeah!

$$\begin{aligned} \deg_f \tilde{X} &:= \deg(\tilde{X}/f(\tilde{X})) \int_{\mathbb{P}^r} c_1(O_{\mathbb{P}^r}(1))^{\dim(X)} \cap [f(\tilde{X})] \\ &= \int_{\tilde{X}} c_1(f^*(O_{\mathbb{P}^r}(1)))^{\dim(X)}. \end{aligned}$$

Theorem (see Fulton, Prop 4.4)**.**

$$\deg_f \tilde{X} = \int_X c_1(L)^n - \int_B c(L)^n \cap s(B, X).$$

Sketch of proof. To make life easy, I'll write $\dim(X) = n$. We saw before that $f^*(O_{\mathbb{P}^r}(1)) = \pi^*(L) \otimes O(-E)$, so let's substitute:

$$\begin{aligned}
\deg_f \tilde{X} &= \int_{\tilde{X}} c_1(f^*(O_{\mathbb{P}^r}(1)))^n = \int_{\tilde{X}} (c_1(\pi^* L) - c_1(O(E)))^n \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_X c_1(L)^{n-i} \pi_* \left(c_1(O(E))^i \cap [\tilde{X}] \right) \\
&= \int_X c_1(L)^n - \int_X \sum_{i=1}^{n} \binom{n}{i} c_1(L)^{n-i} \cap (-1)^{i-1} \pi_* \left(E^i \right) \\
&= \int_X c_1(L)^n - \int_X \sum_{i=0}^{n} \binom{n}{i} c_1(L)^{n-i} \cap \sum_{k \geq 1} (-1)^{k-1} \pi_* \left(E^k \right) \\
&= \int_X c_1(L)^n - \int_B (1 + c_1(L))^n \cap s(B, X).
\end{aligned}$$

□

Useful reminder: We saw in class last day that, using our language of the day,

$$s(B, X) = \sum_{k \geq 1} (-1)^{k-1} \pi_*(E^k).$$

Useful reminder: The degree of $\alpha \in A_k X$ is 0 whenever $k > 0$.

Example. Let's return to the first example. According to the previous theorem we evaluate

$$\int_X c_1(O_{\mathbb{P}^2}(2))^2 = 4 \left(\int_X c_1(O_{\mathbb{P}^2}(1)) \right)^2 = 4.$$

Since this example has no base locus, we see that the degree of our map to $\mathbb{P}^5$ is 4.

Example. Let's return to the second example. Since the base locus is $[0;1]$ we see that $s(B, X) = [\mathrm{pt}]$. So our (extended) map has degree $4 - 1 = 3$.

Example. Cremona: Let's try $\mathbb{P}^2 \xrightarrow{[yz;xz;xy]} \mathbb{P}^2$. The base locus is the subscheme $yz = xz = xy = 0$, otherwise known as the three reduced points $[1;0;0]$, etc. Each has segre class of a point, so that we get the degree of the map $4 - 3 = 1$. It's a birational morphism! What's the inverse? Itself.

Example. Here's an example where the image variety has degree 1, so that we're left computing the degree of the map. We'll use $O_{\mathbb{P}^2}(2)$ again, with sections x^2, y^2, z^2. There's no base locus, so our theorem returns 4 for the degree of the map $\mathbb{P}^2 \xrightarrow{[x^2;y^2;z^2]} \mathbb{P}^2$.

Example. If we choose sections x^2, xy, y^2 of $O_{\mathbb{P}^2}(2)$ to map into $\mathbb{P}^2$, the base locus is the non-reduced point $[0; 0; 1]$. Since the image loses dimension (it sits on the conic $ac = b^2$), our theorem tells us that $4 = e_B X$.

2. Intersection Product

In this section I'm going to tie up some of the loose ends Ravi left for me on Monday. Recall that we're interested in finding a map $A_p(X) \otimes A_q(X) \xrightarrow{\times} A_{p+q-n}(X)$. The things left for me are to clear up that there's a map $Z_k(X) \otimes Z_l(Y) \xrightarrow{\times} Z_{k+l}(X \times Y)$ and that this map gives us a map on cycle classes.

For the first, we define the map by giving its action on subvarieties and extending by linearity. We take $[W] \times [V] \mapsto [W \times V]$. The studious student asks why this product lands in the right cycle class.

$$\begin{aligned} \dim W \times_k V &= \dim \left((W \times_k V) \times_k \bar{k} \right) = \dim \left((W \times_k \bar{k}) \times_{\bar{k}} (V \times_k \bar{k}) \right) \\ &= \dim \left(W \times_k \bar{k} \right) + \dim \left(V \times_k \bar{k} \right) = \dim W + \dim V. \end{aligned}$$

Here we have used two exercises from Chap 3 of Hartshorne and the following diagram

$$\begin{tikzcd}
 & & {} \arrow[dll, squiggly] \arrow[rr, squiggly] \arrow[dr, dotted] \arrow[dd, dotted] & & V \times_k \bar{k} \arrow[dl, squiggly] \arrow[dd] \\
W \times_k \bar{k} \arrow[rrr, squiggly] \arrow[dd] & & & \bar{k} \arrow[dd, dotted] & \\
 & & W \times_k V \arrow[rr] \arrow[dll] \arrow[dr, dotted] & & V \arrow[dl] \\
W \arrow[rrr] & & & k &
\end{tikzcd}$$

So we have left to justify that this gives us a map on cycle classes, and on the way we probably expect we get some result about push forwards and pull backs.

Theorem. *Let $\alpha \in A_k(X)$ and $\beta \in A_l(Y)$.*

- *If $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$.*
- *The product $f \times g$ of proper (resp., flat) maps is again proper (resp., flat), and $(f \times g)_* \alpha \times \beta = f_* \alpha \times g_* \beta$ (resp., $(f \times g)^* \alpha \times \beta = f^* \alpha \times g^* \beta$).*

Proof. Part 2 will follow once we split up $f \times g$ into the composition of $f \times mboxid$ and $\mathrm{id} \times g$. For part 1, assume that $\alpha \sim 0$, and reduce to the case where $W = Y$. Then $\alpha \times \beta$ is the pull back of α under the projection $X \times W \to X$. Since we can pull back classes under flat morphisms, we win. $\square$