INTERSECTION THEORY CLASS 12

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CONTENTS

1. RATIONAL EQUIVALENCE ON BUNDLES

Last time we mostly proved:

Theorem. Let E be a vector bundle of rank $r = e + 1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_k E$ is an isomorphism for all k.

(b) Each $\beta \in A_k \mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^* \alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i}X \xrightarrow{\sim} A_k \mathbb{P}E.$$

$\theta_E : \oplus \alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^* \alpha_i$.

Proof. Our plan was to prove this in the following order: π^* surjective, θ_E surjective, θ_E injective, π^* injective. The proof is a delicate interplay between E and $\mathbb{P}E$. We had done all but the last step, and we had reduced the last step to the case where E is a trivial bundle, i.e. we wanted to show that $A_*X \hookrightarrow A_*(X \times \mathbb{A}^r)$. By induction, we needed to deal with the case where E had rank 1.

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We repeatedly used the "excision exact sequence". Suppose X is a scheme, U an open set, and Z the complement (a closed subset). Then the following sequence is exact:

$$A_k Z \to A_k X \to A_k U \to 0.$$

A construction we used throughout the proof was the following: Note that $\mathbb{P}(E \oplus \mathbf{1}) = \mathbb{P}E \coprod E$, where $\mathbb{P}E$ is a closed subset and E is an open subset; let $i : \mathbb{P}E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the closed immersion, and $j : E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the open immersion. (In fact $\mathbb{P}E$ is a Cartier divisor, in class $\mathcal{O}_{\mathbb{P}(E \oplus \mathbf{1})}(1)$; this was one of my definitions of $\mathcal{O}(1)$.) Let q be the morphism $\mathbb{P}(E \oplus \mathbf{1}) \to X$. The excision exact sequence gives us:

$$\begin{array}{ccccccc} A_k\mathbb{P}E & \xrightarrow{i_*} & A_k\mathbb{P}(E \oplus \mathbf{1}) & \longrightarrow & A_k E & \longrightarrow & 0 \\ & & {\scriptstyle q^*}\uparrow & \overset{\pi^*}{\nearrow} & & & \\ & & A_{k-r}X & & & & \end{array}$$

We showed the following useful *Remark:* For any $\alpha \in A_*X$, $c_1(\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)) \cap q^*\alpha = i_* p^* \alpha$.

So we want to show that $A_k X \hookrightarrow A_{k+1}(X \times \mathbb{A}^1) \hookrightarrow A_{k+2}(X \times \mathbb{A}^2) \hookrightarrow \cdots$. By induction we just need to show the rank 1 case: $A_k X \hookrightarrow A_{k+1}(X \times \mathbb{A}^1)$. Rather than starting this proof in the middle, I'll let you read it in the book; it is relative straightforward, compared to the rest of the argument.

1.1. **Intersecting with the zero-section of a vector bundle.** We can already intersect with the zero-section of a line bundle (i.e. an effective Cartier divisor); we get a map $A_k X \to A_{k-1} D$, which we've called the Gysin pullback.

Definition: Gysin pullback by zero section of a vector bundle. Let $s = s_E$ denote the zero section of a vector bundle E. s is a morphism from X to E with $\pi \circ s = id_X$. By part (a) of the Chern class theorem allows us to define *Gysin homomorphisms* $s^* : A_k E \to A_{k-r} X$, $r = \operatorname{rank} E$, by $s^*(\beta) := (\pi^*)^{-1}(\beta)$.

This ability to intersect with zero sections of vector bundles will be the basis for many important future constructions.

You should think of this as intersecting with the zero-section of a vector bundle. This should be a codimension r intersection. In fact there is "excess" intersection — the actual intersection is codimension 0 — but there is a class of the right dimension.

Proposition. Let $\beta \in A_k E$, and let $\overline{\beta}$ be any element of $A_k(\mathbb{P}(E \oplus \mathbf{1}))$ which restricts to β in $A_k E$. Then $s^*(\beta) = q_*(c_r(Q) \cap \overline{\beta})$ where q is the projection from $\mathbb{P}(E \oplus \mathbf{1})$ to X, and Q is the universal (rank r) quotient bundle of $q^*(E \oplus \mathbf{1})$.

Proof omitted (but is in book, and isn't too long). Note that c_r is the "top" Chern class.

Example If s is the zero section of a vector bundle E of rank r on X, then $s^* s_*(\alpha) = c_r(E) \cap \alpha$. This is a special case of the excess intersection formula.

2. Cones and Segre classes of subvarieties

2.1. **Introduction.** If X is a subvariety of a variety Y, the Segre class $s(X,Y)$ of X in Y is a class in A_*X defined as follows. $C = C_XY$ is the normal cone to X in Y, $\mathbb{P}C$ is the projectivized normal cone, p the projection from $\mathbb{P}C$ to X. I'll define the normal cone soon. Then

$$s(X,Y) = \sum_{k \geq 0} p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]).$$

Note that this is a class, *not* an operator.

In the case when X is a smooth subvariety of a smooth variety, C is the normal bundle. More generally, if Y is arbitrary, then X is a *local complete intersection* (hereafter *lci*) in Y (what Fulton calls a *regular imbedding*) if it is scheme-theoretically cut out by r equations, where r is the codimension of X in Y. (Example 1: X is a smooth subvariety of a smooth variety. Example 2: *any* Cartier divisor. Example 3: the union of the x and y axes in $\mathbb{A}^3$.) If X is a regular imbedding (=lci) in X, then X still has a normal bundle, defined as follows: if $\mathcal{I}$ is the ideal sheaf cutting out X, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle of rank r. This is the *conormal bundle*, and its dual is the normal bundle. (Warning: in differential geometry, if $X \hookrightarrow Y$, then X has a *tubular neighborhood* that looks like the normal bundle. In algebraic geometry, there are no such small neighborhoods, but in some sense it is even worse: in example 3, the total space $Y = \mathbb{A}^3$ is smooth, but the total space of the normal bundle — a vector bundle over a nodal curve — is singular.)

If X is regularly imbedded (=lci) in Y, then the definition of $s(X,Y)$ turns into

$$s(X,Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

More generally still, if X is arbitrarily horrible in arbitrarily horrible Y, it still has a *normal cone*. I'll define that shortly. Whatever it is, we'll have the same equation

$$s(X,Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

These Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X',Y')$ pushes forward to $s(X,Y)$. The coefficient of $[X]$ in $s(X,Y)$ is the multiplicity of Y along X. This magical invariance will be the main result of Chapter 4.

2.2. **Cones.** I'll now define *cone*. Let X be a scheme, and let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. This sounds complicated, but it isn't. It is defined so you can take $\underline{\mathrm{Proj}}(S^\cdot)$, and that this makes sense, and has a line bundle $\mathcal{O}(1)$.

Here's how it works: over any affine open set $\operatorname{Spec} R$ of X, $S^\cdot$ is a graded R-algebra, generated in degree 1. Then we can take $\underline{\mathrm{Proj}}$ of this graded R-algebra. The fact that the algebra is generated in degree 1 (by R_1 say) means that we have a surjective map of graded rings

$$\operatorname{Sym}^i R_1 \to \oplus_i R^i$$

which, upon applying $\underline{\text{Proj}}$, becomes

$$X' \hookrightarrow X \times \mathbb{P}(R^1)^\vee$$

where $\mathbb{P}(R^1)^\vee$ is an honest projective bundle. So the morphism $X' \to X$ is projective and has a line bundle called $\mathcal{O}(1)$. You can do this over each affine, and glue the result together, and the $\mathcal{O}(1)$'s also glue together.

Example 1: say let E be a vector bundle, and $S^i = \mathrm{Sym}^i(E^\vee)$. Then $\underline{\text{Proj}}\, S^\cdot = \mathbb{P}E$.

Example 2: Say $T^i = \mathrm{Sym}^i(E^\vee \oplus \mathbf{1}) = S^i \oplus S^{i-1}z$, so (better) $T^\cdot = S^\cdot[z]$. Then $\underline{\text{Proj}}\, T^\cdot = \mathbb{P}E$.

Example 3: The blow-up can be described in this way, and it will be good to know this. Suppose X is a subscheme of Y, cut out by ideal sheaf $\mathcal{I}$. (In our situation where all schemes are finite type, $\mathcal{I}$ is a coherent sheaf.) Then let $S^\cdot = \oplus_i \mathcal{I}^i$, where $\mathcal{I}$ is the ith power of the ideal $\mathcal{I}$. ($\mathcal{I}^0$ is defined to be $\mathcal{O}_X$.) Then $\mathrm{Bl}_X Y \cong \underline{\text{Proj}}\, S^\cdot$. A short calculation shows that the exceptional divisor class is $\mathcal{O}(-1)$. The *exceptional divisor* turns out to be $\underline{\text{Proj}} \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$. (Note that this is indeed a graded sheaf of algebras.) As $\oplus\mathcal{I}^n \to \oplus\mathcal{I}^n/\mathcal{I}^{n+1}$ is a surjective map of rings, this indeed describes a closed subscheme of the blow-up. (Remember this formula — it will come up again soon!)

Now I'll finally define *cone*. Let $S^\cdot$ be a sheaf of graded $\mathcal{O}_X$-algebras as before. Then $C = \underline{\text{Spec}}(S^\cdot)$ is a *cone*. (We can construct $\underline{\text{Spec}}(S^\cdot)$ of a sheaf of algebras in the same way as we can construct $\underline{\text{Proj}}$; in fact it is a logically prior construction.)

Remember that $\mathbb{P}(E \oplus \mathbf{1}) = E \coprod \mathbb{P}E$. The direct generalization is: $\underline{\text{Proj}}(S^\cdot[z]) = C \coprod \underline{\text{Proj}}(S^\cdot) = \underline{\text{Spec}}\, S^\cdot \coprod \underline{\text{Proj}}(S^\cdot)$. The argument is just the same. The right term is a Cartier divisor in class $\mathcal{O}_{\underline{\text{Proj}}(S^\cdot[z])}(1)$.

2.3. **Segre class of a cone.**

The *Segre class* of a cone C on X, denoted $s(C)$, is the class in $A_* X$ defined by the formula

$$s(C) = q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\text{Proj}}(C \oplus \mathbf{1})] \right).$$

This is very much the same definition as for vector bundles, *except* in the vector bundle case we get *operators* on Chow groups. In this case we get elements of Chow groups themselves: we are capping with a fundamental class!

Proposition (a) If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class of X, $r = \mathrm{rank}(E)$. $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. (I would write $s(E) = s(E) \cap [X]$, but the two uses of $s(E)$ are confusing!) This is basically our definition of Segre/Chern classes.

(b) Let $C_1, \ldots, C_t$ be the irreducible components of C, m_i the geometric multiplicities of C_i in C. Then $s(C) = \sum_{i=1}^t m_i s(C_i)$. (Note that the C_i are cones as well, so $s(C_i)$ makes sense.) In other words, we can compute the Segre class piece by piece.

Sketch of proof of (b). This is because each of the C_i is a cone. $[\underline{\mathrm{Proj}}(C \oplus \mathbf{1})] = \cup m_i [\underline{\mathrm{Proj}}(C_i \oplus \mathbf{1})]$. □

Example. For any cone C, $s(C \oplus \mathbf{1}) = s(C)$. (In the language of Dan's talk last week, the Segre class of a cone depends on its stable equivalence class.)

2.4. **The Segre class of a subscheme.** Let X be a closed subscheme of a scheme Y (not necessarily lci).

I told you that $\mathcal{I}/\mathcal{I}^2$ is the conormal bundle of a local complete intersection subscheme. In general, it is the conormal *sheaf*.

Consider $\sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}$. (Recall that $\underline{\mathrm{Proj}}$ of this sheaf gives us the exceptional divisor of the blow-up.) *Define* the normal cone $C = C_X Y$ by

$$C = \underline{\mathrm{Spec}} \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Define the *Segre class* of X in Y as the Segre class of the normal cone:

$$s(X, Y) = s(C_X Y) \in A_* X.$$

Proposition Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

(a) If f is proper, Y irreducible, and f maps each irreducible component of Y' onto Y then

$$g_*(s(X', Y')) = \deg(Y'/Y) s(X, Y).$$

(b) If f is flat, then

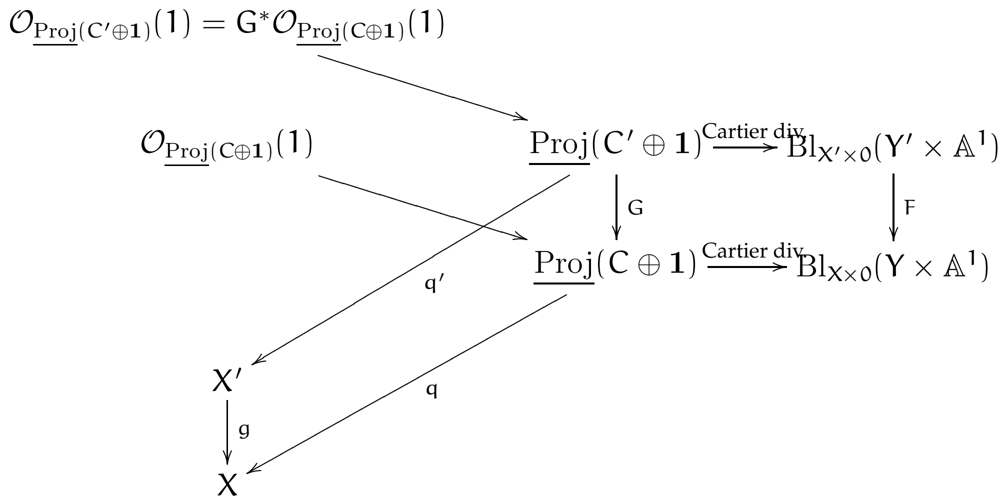
$$g^*(s(X', Y')) = s(X, Y).$$

Let me point out why I find this a remarkable result. X' is a priori some nasty scheme; even if it is nice, its codimension in Y' isn't necessarily the same as the codimension of X in Y. The argument is quite short, and shows that what we've proved already is quite sophisticated.

I will give the proof next time. Today I gave most of the proof, by describing the diagram around which everything revolves.

Let me assume that Y' is irreducible. (It's true in general, and I may deal with the general case later.)

Let me first write the diagram on the board, and then explain it.



Explanation: We blow up $Y \times \mathbb{A}^1$ along $X \times 0$, and similarly for Y' and X'. The exceptional divisor of $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1)$ is $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$, and similarly for Y' and X'. The universal property of blowing up $Y \times \mathbb{A}^1$ shows that there exists a unique morphism G from the top exceptional divisor to the bottom. Moreover, by construction, the exceptional divisor upstairs is the pullback of the exceptional divisor downstairs (that's the statement about the two $\mathcal{O}(1)$'s in the diagram). Let q be the morphism from the exceptional divisor $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$ to X, and similarly for q'. That square commutes: $q \circ G = g \circ q'$ (basically because that morphism G was defined by the universal property of blowing up).

We'll finish the proof next time (and I'll describe this diagram once again).

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