# INTERSECTION THEORY CLASS 11

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CONTENTS



## 1. Rational equivalence on bundles

Last time I stated:

**Theorem.** Let $E$ be a vector bundle of rank $r = e + 1$ on a scheme $X$, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_k E$ is an isomorphism for all $k$.

(b) Each $\beta \in A_k\mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i}X \xrightarrow{\sim} A_k\mathbb{P}E.$$

$\theta_E : \oplus\alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^*\alpha_i$.

*Proof.* Here's the plan: $\pi^*$ surjective, $\theta_E$ surjective, $\theta_E$ injective, $\pi^*$ injective. So the proof is a delicate interplay between $E$ and $\mathbb{P}E$.

We'll make repeated use of something Rob stated, from the end of the first Chapter: the "excision exact sequence". Suppose $X$ is a scheme, $U$ an open set, and $Z$ the complement (a closed subset). Then the following sequence is exact:

$$A_k Z \to A_k X \to A_k U \to 0.$$

I'll now show surjectivity of $\pi^*$ and $\theta_E$. First reduction: it suffices to deal with the case where $E$ is the trivial bundle. Proof by the induction on the dimension of $X$. Here's the $\pi^*$ argument:

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Let $U$ be a dense open set where $E$ is trivial. Then its complement $Y$ is of dimension strictly smaller than $X$.

$$\begin{tikzcd} A_*Y \arrow[r] \arrow[d] & A_*X \arrow[r] \arrow[d] & A_*U \arrow[r] \arrow[d] & 0 \\ A_*(\pi^{-1}Y) \arrow[r] \arrow[d] & A_*E \arrow[r] \arrow[d, "?"] & A_*(\pi^{-1}U) \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

The two horizontal rows are exact. By the inductive hypothesis, the left column is exact. We're assuming we know the result for trivial vector bundles, so the right column is also exact. Then the central vertical row is exact, by a quick diagram chase.

The same argument works for $\theta_E$. Here's the exact sequence:

$$\begin{tikzcd} A_*Y \arrow[r] \arrow[d] & A_*X \arrow[r] \arrow[d] & A_*U \arrow[r] \arrow[d] & 0 \\ \oplus A_*(p^{-1}Y) \arrow[r] \arrow[d] & \oplus A_*\mathbb{P}E \arrow[r] \arrow[d, "?"] & \oplus A_*(p^{-1}U) \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

So let's show surjectivity of $\pi^*$ and $\theta_E$ in the case where $E$ is a trivial bundle. I'll show both by induction on the rank of $E$. In the case where the rank is $0$, both are clearly surjective. (In fact, $\pi^*$ is tautologically an isomorphism, and $\mathbb{P}E$ is the empty set, and the left side of $\theta_E$ is the empty direct sum!)

We assume the result for $E$ and prove it for $E \oplus \mathbf{1}$.

The surjectivity of $\pi^*$ in the trivial bundle was shown in Chapter 1, so for the sake of time I'll omit it. (The atomic statement that needs to be shown: $A_kX \to A_{k+1}(X \times \mathbb{A}^1)$ is surjective. Then by induction $A_kX \to A_{k+1}(X \times \mathbb{A}^n)$ is surjective.)

Recall that $\mathbb{P}(E \oplus \mathbf{1}) = \mathbb{P}E \coprod E$, where $\mathbb{P}E$ is a closed subset and $E$ is an open subset; let $i : \mathbb{P}E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the closed immersion, and $j : E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the open immersion. (In fact $\mathbb{P}E$ is a Cartier divisor, in class $\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)$; this was one of my definitions of $\mathcal{O}(1)$.) Let $q$ be the morphism $\mathbb{P}(E \oplus \mathbf{1}) \to X$. The excision exact sequence gives us:

$$\begin{tikzcd} A_k\mathbb{P}E \arrow[r, "i_*"] & A_k\mathbb{P}(E \oplus \mathbf{1}) \arrow[r] & A_kE \arrow[r] & 0 \\ & A_{k-r}X \arrow[u, "q^*"] \arrow[ur, "\pi^*"'] & & \end{tikzcd}$$

You may feel like drawing an arrow $A_{k-r}X \to A_kP$, but that's not right; the morphism is of course $A_{k-r}X \to A_{k-1}P$, as the fiber dimension of $A_k\mathbb{P}E \to A_{k-r}$ is $r - 1$.

*Remark:* For any $\alpha \in A_* X$, $c_1(\mathcal{O}_{\mathbb{P}(E\oplus \mathbf{1})}(1)) \cap q^*\alpha = i_* p^* \alpha$. Reason: I'll show this for any cycle $\alpha \in Z_* X$. Then we can interpret the left side as pulling the cycle back to $\mathbb{P}(E \oplus \mathbf{1})$, and intersecting with the Cartier divisor $\mathbb{P}E$. But that's exactly the same as the right side. (That's basically how we defined $c_1$ of a line bundle!)

Suppose $\beta \in A_*\mathbb{P}(E \oplus \mathbf{1})$. Then we can write $j^*\beta = \pi^*\alpha$ for some $\alpha \in A_* X$ (by surjectivity of $\pi^*$). Then $\beta - q^*\alpha$ maps to $0$ in $A_k E$, so it is in $A_k \mathbb{P}E$ by our excision exact sequence. Then by our inductive assumption that we already know surjectivity for smaller-dimensional schemes, we know:

$$\beta - q^*\alpha = i_* \left( \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^* \alpha_i \right)$$

for some $\alpha_i \in A_* X$. As $i^* \mathcal{O}_{\mathbb{P}(E\oplus \mathbf{1})} = \mathcal{O}_{\mathbb{P}E}(1)$:

$$\cdots = \beta - q^*\alpha = i_* \left( \sum_{i=0}^{e} i^* c_1(\mathcal{O}_{\mathbb{P}E\oplus \mathbf{1}}(1))^i \cap p^* \alpha_i \right)$$

Then by the projection formula we get

$$\begin{aligned} \cdots &= \beta - q^*\alpha = \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E\oplus \mathbf{1}}(1))^i \cap i_* p^* \alpha_i \\ &= \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E\oplus \mathbf{1}}(1))^i \cap c_1(\mathcal{O}_{\mathbb{P}E\oplus \mathbf{1}}(1) \cap q^* \alpha_i \end{aligned}$$

(the last step by using the remark). Thus we see that $\theta_{E\oplus \mathbf{1}}$ is surjective.

We next show that $\theta_E$ is an *isomorphism*. Suppose we have a relation

$$\sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^* \alpha_i = 0.$$

If the $\alpha_i$ are not all zero, then let $k$ be the largest integers with $\alpha_k \neq 0$. Then

$$p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e-k} \cap \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^* \alpha_i) = \alpha_k$$

by our Segre class theorem, giving a contradiction.

Finally, we'll show that $\pi^*$ is an isomorphism. I claim that as before, it suffices to do this for trivial bundles. The argument is by Noetherian induction again.

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, "?"] & \\ A_* Y \arrow[r] \arrow[d] & A_* X \arrow[r] \arrow[d] & A_* U \arrow[r] \arrow[d] & 0 \\ A_*(\pi^{-1} Y) \arrow[r] \arrow[d] & A_* E \arrow[r] \arrow[d] & A_*(\pi^{-1} U) \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

Now we'll do it by induction on the rank. So we want to show that $A_kX \hookrightarrow A_{k+1}X \times \mathbb{A}^1 \hookrightarrow A_{k+2}X \times \mathbb{A}^2 \hookrightarrow \cdots$: we just need to show the rank 1 case. Suppose $\alpha \in A_kX$ and $\pi^*\alpha \in A_{k+1}(X \times \mathbb{A}^1) = 0$. Consider $q^*\alpha \in A_{k+1}(X \times \mathbb{P}^1)$. As $\theta_E$ is an isomorphism, we have

$$q^*\alpha = i_*(p^*\alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap p^*\alpha_1).$$

(One nice thing about $\mathbb{P}^1$ is that $\mathcal{O}_{\mathbb{P}^1}(1)^2 = 0$: the intersection of two distinct points is empty!) Using our remark:

$$q^*\alpha = c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap q^*\alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1))^2 \cap p^*\alpha_1.$$

(Thus by injectivity of $\theta_E$ (which is uniqueness of $\alpha_0$ and $\alpha_1$) we have $\alpha_1 = 0$.) But the first part of the Segre class theorem stated that if we take a class $\alpha$ downstairs, pull it back to a projective bundle, and cap it with the right number of $\mathcal{O}(1)$'s (corresponding to the projective bundle), and push it forward, we'll get $\alpha$ again. Hence

$$\begin{aligned} \alpha &= q_*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap q^*\alpha) \\ &= c_1(\mathcal{O}_{\mathbb{P}^1}(1))^2 \cap q^*\alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1))^3 \cap p^*\alpha_1 \\ &= 0 \end{aligned}$$

□

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