FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 21

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CONTENTS

This class will meet 8:40-9:55. Please be sure to be on the e-mail list so I can warn you which days class will take place.

Welcome back! Where we're going this quarter: last quarter, we established the objects of study: varieties or schemes. This quarter we'll be mostly concerned with important means of studying them: ~~vector bundles~~ quasicoherent sheaves and cohomology thereof. As a punchline for this quarter, I hope to say a lot of things about curves (Riemann surfaces) at the end of the quarter. However, in keeping with the attitude of last quarter, my goal isn't to make a beeline for the punchline. Instead we'll have a scorched-earth policy and try to cover everything between here and there relatively comprehensively. We start with ~~smoothness~~ nonsingularity of schemes. Then ~~vector bundles~~ locally free sheaves, quasicoherent sheaves and coherent sheaves. Then to ~~line bundles~~ invertible sheaves, and divisors. Then we'll interpret these for projective schemes in terms of graded modules. We'll investigate pushing forward and pulling back quasicoherent sheaves. We'll construct schemes using these notions, and for example define the notion of a projective morphism. We'll study differentials (e.g. the tangent bundle of smooth schemes, but also for singular things). Then we'll discuss cohomology (both Cech cohomology and derived functor cohomology). Then curves! The punch line for today: $\operatorname{Spec} \mathbb{Z}$ is a ~~smooth~~ nonsingular curve.

1. NONSINGULARITY ("SMOOTHNESS") OF NOETHERIAN SCHEMES

One natural notion we expect to see for geometric spaces is the notion of when an object is "smooth". In algebraic geometry, this notion, called *nonsingularity* (or *regularity*, although we won't use this term) is easy to define but a bit subtle in practice. We will soon define what it means for a scheme to be *nonsingular* (or *regular*) at a point. A point that is not nonsingular is (not surprisingly) called *singular* ("not smooth"). A scheme is said *nonsingular* if all its points are nonsingular, and singular if one of its points is singular.

Date: Friday, January 11, 2008.

The notion of nonsingularity is less useful than you might think. Grothendieck taught us that the more important notions are properties of morphisms, not of objects, and there is indeed a "relative notion" that applies to a morphism of schemes $f : X \to Y$ that is much better-behaved (corresponding to the notion of submersion in differential geometry). For this reason, the word "smooth" is reserved for these morphisms. We will discuss smooth morphisms in the spring quarter. However, nonsingularity is still useful, especially in (co)dimension 1, and we shall discuss this case (of *discrete valuation rings*) next day.

2. The Zariski tangent space

We begin by defining the notion of the tangent space of a scheme at a point. It will behave like the tangent space you know and love at "smooth" points, but will also make sense at other points. In other words, geometric intuition at the smooth points guides the definition, and then the definition guides the algebra at all points, which in turn lets us refine our geometric intuition.

This definition is short but surprising. The main difficulty is convincing yourself that it deserves to be called the tangent space. I've always found this tricky to explain, and that is because we want to show that it agrees with our intuition; but unfortunately, our intuition is worse than we realize. So I'm just going to define it for you, and later try to convince you that it is reasonable.

Suppose $\mathfrak{p}$ is a prime ideal of a ring A, so $[\mathfrak{p}]$ is a point of $\operatorname{Spec} A$. Then $[\mathfrak{p}A_\mathfrak{p}]$ is a point of the scheme $\operatorname{Spec} A_\mathfrak{p}$. For convenience, we let $\mathfrak{m} := \mathfrak{p}A_\mathfrak{p} \subset A_\mathfrak{p} =: B$. Let $k = B/\mathfrak{m}$ be the residue field. Then $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field k: it is an B-module, and elements of $\mathfrak{m}$ acts like 0. This is defined to be the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

Note that this definition is intrinsic. It doesn't depend on any specific description of the ring itself (such as the choice of generators over a field k, which is equivalent to the choice of embedding in affine space). Notice that in some sense, the cotangent space is more algebraically natural than the tangent space. There is a moral reason for this: the cotangent space is more naturally determined in terms of functions on a space, and we are very much thinking about schemes in terms of "functions on them". This will come up later.

I'll give two of plausibility arguments that this is a reasonable definition. Hopefully one will catch your fancy.

In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$$(fg)' = f'g + g'f.$$

Translation: a derivation is a map $\mathfrak{m} \to k$. But $\mathfrak{m}^2 \to 0$, as if $f(p) = g(p) = 0$, then

$$(fg)'(p) = f'(p)g(p) + g'(p)f(p) = 0.$$

Thus we have a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, i.e. an element of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

2.A. EXERCISE. Check that this is reversible, i.e. that any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ gives a derivation. In other words, verify that the Leibniz rule holds.

Here is a second vaguer motivation that this definition is plausible for the cotangent space of the origin of $\mathbb{A}^n$. Functions on $\mathbb{A}^n$ should restrict to a linear function on the tangent space. What function does $x^2 + xy + x + y$ restrict to "near the origin"? You will naturally answer: $x + y$. Thus we "pick off the linear terms". Hence $\mathfrak{m}/\mathfrak{m}^2$ are the linear functionals on the tangent space, so $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space. In particular, you should picture functions vanishing at a point (lying in $\mathfrak{m}$) as giving functions on the tangent space in this obvious a way.

2.1. *Old-fashioned example.* Here is an example to help tie this down to earth. Computing the Zariski-tangent space is actually quite hands-on, because you can compute it just as you did when you learned multivariable calculus. In $\mathbb{A}^3$, we have a curve cut out by $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. (You know enough to check that this is a curve, but it is not important to do so.) What is the tangent line near the origin? (Is it even smooth there?) Answer: the first surface looks like $x + y = 0$ and the second surface looks like $x - 2y + z = 0$. The curve has tangent line cut out by $x + y = 0$ and $x - 2y + z = 0$. It is smooth (in the analytic sense). In multivariable calculus, the students do a page of calculus to get the answer, because we aren't allowed to tell them to just pick out the linear terms.

Let's make explicit the fact that we are using. If A is a ring, $\mathfrak{m}$ is a maximal ideal, and $f \in \mathfrak{m}$ is a function vanishing at the point $[\mathfrak{m}] \in \operatorname{Spec} A$, then the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $\mathfrak{m}$ is cut out in the Zariski tangent space of $\operatorname{Spec} A$ (at $\mathfrak{m}$) by the single linear equation $f \pmod{\mathfrak{m}^2}$. The next exercise will force you think this through.

2.B. IMPORTANT EXERCISE ("KRULL'S PRINCIPAL IDEAL THEOREM FOR THE ZARISKI TANGENT SPACE"). Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the Zariski tangent space of A/f is cut out in the Zariski tangent space of A by $f \pmod{\mathfrak{m}^2}$. (Note: we can quotient by f and localize at $\mathfrak{m}$ in either order, as quotienting and localizing "commute".) Hence the dimension of the Zariski tangent space of $\operatorname{Spec} A$ at $[\mathfrak{m}]$ is the dimension of the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $[\mathfrak{m}]$, or one less.

Here is another example to see this principle in action: $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ cuts out a curve, which obviously passes through the origin. If I asked my multivariable calculus students to calculate the tangent line to the curve at the origin, they would do a reams of calculations which would boil down to picking off the linear terms. They would end up with the equations $x + y = 0$ and $x + y = 0$, which cuts out a plane, not a line. They would be disturbed, and I would explain that this is because the curve isn't smooth at a point, and their techniques don't work. We on the other hand bravely declare that the cotangent space is cut out by $x + y = 0$, and (will soon) *define* this as a singular point. (Intuitively, the curve near the origin is very close to lying in the plane $x + y = 0$.) Notice: the cotangent space jumped up in dimension from what it was "supposed to be", not down. We'll see that this is not a coincidence soon, in Theorem 3.1.

Here is a nice consequence of the notion of Zariski tangent space.

2.2. Problem. Consider the ring $A = k[x, y, z]/(xy - z^2)$. Show that (x, z) is not a principal ideal.

As $\dim A = 2$ (by Krull's Principal Ideal Theorem), and $A/(x, z) \cong k[y]$ has dimension 1, we see that this ideal is height 1 (as codimension is the difference of dimensions for finitely generated k-domains). Our geometric picture is that $\operatorname{Spec} A$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x+y)/2)^2 - ((x-y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$), and that (x, z) is a ruling of the cone. (See Figure 1 for a sketch.) This suggests that we look at the cone point.

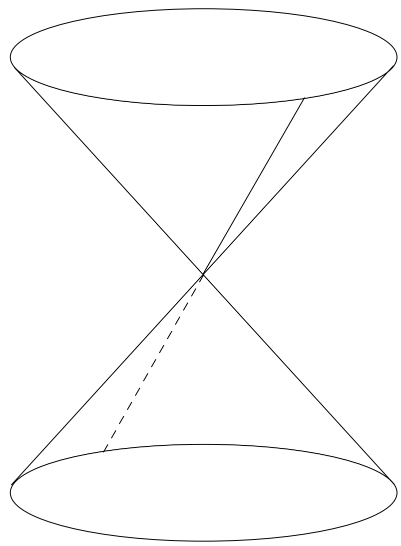

FIGURE 1. $V(x, z) \subset \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is a ruling on a cone; $(x, z)^2$ is not (x, z)-primary.

Solution. Let $\mathfrak{m} = (x, y, z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} A$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} A/(x, z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} A/(f)$ must have Zariski tangent space of dimension at least 2 at the origin by Exercise 2.B.

2.C. EXERCISE. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a codimension 1 ideal that is not principal. (See Figure 2 for a sketch.)

2.3. Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent

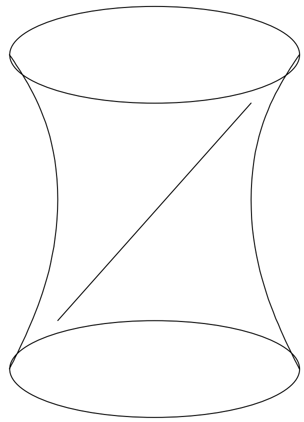

FIGURE 2. The ruling $V(x,z)$ on $V(wz - xy) \subset \mathbb{P}^3$.

space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed local-ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$, from which we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought.

Here are some exercises to give you practice with the Zariski tangent space.

2.D. USEFUL EXERCISE (THE JACOBIAN CRITERION FOR COMPUTING THE ZARISKI TANGENT SPACE). Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms, see Example 2.1 .) You might be alarmed: what does $\frac{\partial f}{\partial x_1}$ mean?! Do you need deltas and epsilons? No! Just define derivatives formally, e.g.

$$\frac{\partial}{\partial x_1}(x_1^2 + x_1 x_2 + x_2^2) = 2x_1 + x_2.$$

(Hint: Do this first when p is the origin, and consider linear terms, just as in Example 2.1. Note for future reference that you have not yet used the algebraic closure of k. Then in the general case (with k algebraically closed), "translate p to the origin."

2.E. LESS IMPORTANT EXERCISE ("HIGHER-ORDER DATA"). In an earlier exercise, you computed the equations cutting out the three coordinate axes of $\mathbb{A}^3_k$. (Call this scheme X.) Your ideal should have had three generators. Show that the ideal can't be generated by fewer than three elements. (Hint: working modulo $\mathfrak{m} = (x, y, z)$ won't give any useful information, so work modulo $\mathfrak{m}^2$.)

2.F. EXERCISE. Suppose X is a k-scheme. Describe a natural bijection $\operatorname{Mor}_k(\operatorname{Spec} k[\epsilon]/(\epsilon^2), X)$ to the data of a point with residue field is k, necessarily a closed point.

2.G. EXERCISE. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2 + 4)$. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[\sqrt{-2}] \cong \mathbb{Z}[x]/(x^2 + 2)$.

3. THE LOCAL DIMENSION IS AT MOST THE DIMENSION OF THE TANGENT SPACE

We are ready to define nonsingularity. The key idea is contained in the title of this section.

3.1. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

If equality holds, we say that A is a **regular local ring**. If a Noetherian ring A is regular at all of its primes, we say that A is a **regular ring**.

A locally Noetherian scheme X is **regular** or **nonsingular** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point).

Proof of Theorem 3.1: Note that $\mathfrak{m}$ is finitely generated (as A is Noetherian), so $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $(A/\mathfrak{m} = k)$-module, hence finite-dimensional. Say $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Choose a basis of $\mathfrak{m}/\mathfrak{m}^2$, and lift them to elements $f_1, \ldots, f_n$ of $\mathfrak{m}$. Then by Nakayama's lemma (version 4), $(f_1, \ldots, f_n) = \mathfrak{m}$.

We need here one fancy fact that I forgot to say last quarter. Krull's Principal Ideal Theorem states that the codimension of any irreducible component of the locus cut out by one equation is at most one. There is a generalization to an arbitrary number of equations: if A is a Noetherian ring, then any irreducible component of $V(f_1, \ldots, f_n)$ has codimension at most n. The proof isn't much harder than Krull, but I haven't given it to you. Sorry! You can read a proof in Eisenbud (Theorem 10.2, p. 235).

3.A. EXERCISE. Prove this if A is an irreducible variety over a field. (Hint: this isn't that hard. Use the fact that codimension is the difference of dimensions in this happy case.)

In our case, $V((f_1, \ldots, f_n)) = V(\mathfrak{m})$ is just the point $[\mathfrak{m}]$, so the codimension of $\mathfrak{m}$ is at most n. Thus the longest chain of prime ideals contained in $\mathfrak{m}$ is at most $n+1$. But this is also the longest chain of prime ideals in A (as $\mathfrak{m}$ is the unique maximal ideal), so $n \geq \dim A$. $\square$

3.B. EXERCISE. Show that if A is a Noetherian local ring, then A has finite dimension. (Noetherian rings in general could have infinite dimension, as we saw in an earlier exercise.)

In the case of finite type schemes over an algebraically closed field k, the Jacobian criterion (Exercise 2.D) gives a hands-on method for checking for singularity at closed points.

3.C. EXERCISE. Suppose k is algebraically closed. Show that the singular *closed* points of the hypersurface $f(x_1, \ldots, x_n) = 0$ in $\mathbb{A}_k^n$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

3.D. EXERCISE. Suppose k is algebraically closed. Show that $\mathbb{A}_k^1$ and $\mathbb{A}_k^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}_k^2$ are; this is trickier for $\mathbb{A}_k^3$.) Show that $\mathbb{P}_k^1$ and $\mathbb{P}_k^2$ are nonsingular. (This holds even if k isn't algebraically closed, and in higher dimension.)

Let's apply this technology to an arithmetic situation.

3.E. EASY EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

Here are some fun comments: What is the derivative of 35 at the prime 5? Answer: $35 \pmod{25}$, so 35 has the same "slope" as 10. What is the derivative of 9, which doesn't vanish at 5? Answer: the notion of derivative doesn't apply there. You'd think that you'd want to subtract its value at 5, but you can't subtract "$4 \pmod 5$" from the integer 9. Also, $35 \pmod{25}$ you might *think* you want to restate as $7 \pmod 5$, by dividing by 5, but that's morally wrong — you're dividing by a particular choice of generator 5 of the maximal ideal of $\mathbb{Z}_5$ (the 5-adics); in this case, one appears to be staring you in the face, but in general that won't be true. Follow-up fun: you can talk about the derivative of a function only for functions vanishing at a point. And you can talk about the second derivative of a function only for functions that vanish, and whose first derivative vanishes. For example, 75 has second derivative $75 \pmod{125}$ at 5. It's pretty flat.

3.F. EXERCISE. (This exercise is for those who know about the primes of the Gaussian integers $\mathbb{Z}[i]$.) Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and x^2+1 is not 0, so $\mathbb{Z}[x]/(x^2+1)$ has dimension 1 by Krull's Principal Ideal Theorem. Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve.

3.G. EXERCISE. Show that there is one singular point of $\operatorname{Spec} \mathbb{Z}[5i]$, and describe it.

Let's return to more geometric examples.

3.H. EXERCISE (THE EULER TEST FOR PROJECTIVE HYPERSURFACES). There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Suppose k is algebraically closed. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general, not just the closed points. I don't want to prove this, and I won't use it.)

3.I. EXERCISE. Suppose that k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$, so be careful when using the Euler test.

3.J. EXERCISE. Find all the singular closed points of the following plane curves. Here we work over an algebraically closed field.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

(I haven't given a precise definition of a node, etc.)

3.K. EXERCISE. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 22

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CONTENTS

Last day, we discussed the Zariski tangent space, and saw that it was often quite computable. We proved the key inequality $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$ for Noetherian local rings $(A, \mathfrak{m})$. When equality holds, we said that the ring was regular (or nonsingular), and we defined the notion of (non)singularity for locally Noetherian schemes.

1. DISCRETE VALUATION RINGS: DIMENSION 1 NOETHERIAN REGULAR LOCAL RINGS

The case of dimension 1 is important, because if you understand how primes behave that are separated by dimension 1, then you can use induction to prove facts in arbitrary dimension. This is one reason why Krull's Principal Ideal Theorem is so useful.

A dimension 1 Noetherian regular local ring can be thought of as a "germ of a smooth curve" (see Figure 1). Two examples to keep in mind are $k[x]_{(x)} = \{f(x)/g(x) : x \not| g(x)\}$ and $\mathbb{Z}_{(5)} = \{a/b : 5 \not| b\}$.

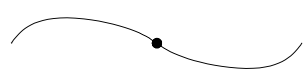

FIGURE 1. A germ of a curve

The purpose of this section is to give a long series of equivalent definitions of these rings. We will eventually have seven equivalent definitions, (a) through (g).

1.1. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then the following are equivalent.*

(a) $(A, \mathfrak{m})$ *is regular.*
(b) $\mathfrak{m}$ *is principal.*

Date: Monday, January 14, 2008.

Here is why (a) implies (b). If A is regular, then $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional. Choose any element $t \in \mathfrak{m} - \mathfrak{m}^2$. Then t generates $\mathfrak{m}/\mathfrak{m}^2$, so generates $\mathfrak{m}$ by Nakayama's lemma. We call such an element a **uniformizer**. Warning: we need to know that $\mathfrak{m}$ is finitely generated to invoke Nakayama — but fortunately we do, thanks to the Noetherian hypothesis.

Conversely, if $\mathfrak{m}$ is generated by one element t over A, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by one element t over $A/\mathfrak{m} = k$. Note that $t \notin \mathfrak{m}^2$, as otherwise $\mathfrak{m} = \mathfrak{m}^2$ and hece $\mathfrak{m} = 0$ by Nakayama's Lemma. □

We will soon use a useful fact, and we may as well prove it in much more generality than we need, because the proof is so short.

1.2. *Proposition. — If $(A, \mathfrak{m})$ is a Noetherian local ring, then $\cap_i \mathfrak{m}^i = 0$.*

The geometric intuition for this is that any function that is analytically zero at a point (vanishes to all orders) actually vanishes at that point.

It is tempting to argue that $\mathfrak{m}(\cap_i \mathfrak{m}^i) = \cap_i \mathfrak{m}^i$, and then to use Nakayama's lemma to argue that $\cap_i \mathfrak{m}^i = 0$. Unfortunately, it is not obvious that this first equality is true: product does not commute with infinite intersections in general.

Proof. Let $I = \cap_i \mathfrak{m}^i$. We wish to show that $I \subset \mathfrak{m}I$; then as $\mathfrak{m}I \subset I$, we have $I = \mathfrak{m}I$, and hence by Nakayama's Lemma, $I = 0$. Fix a primary decomposition of $\mathfrak{m}I$. It suffices to show that $\mathfrak{q}$ contains I for any $\mathfrak{q}$ in this primary decomposition, as then I is contained in all the primary ideals in the decomposition of $\mathfrak{m}I$, and hence $\mathfrak{m}I$. Let $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

If $\mathfrak{p} \neq \mathfrak{m}$, then choose $x \in \mathfrak{m} - \mathfrak{p}$. Now x is not nilpotent in $A/\mathfrak{q}$, and hence is not a zero-divisor. (Recall that $\mathfrak{q}$ is primary if and only if in $A/\mathfrak{q}$, each zero-divisor is nilpotent.) But $xI \subset \mathfrak{m}I \subset \mathfrak{q}$, so $I \subset \mathfrak{q}$.

On the other hand, if $\mathfrak{p} = \mathfrak{m}$, then as $\mathfrak{m}$ is finitely generated, and each generator is in $\sqrt{\mathfrak{q}} = \mathfrak{m}$, there is some a such that $\mathfrak{m}^a \subset \mathfrak{q}$. But $I \subset \mathfrak{m}^a$, so we are done. □

1.3. *Proposition. — Suppose $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1 (i.e. satisfying (a) above). Then A is an integral domain.*

Proof. Suppose $xy = 0$, and $x, y \neq 0$. Then by Proposition 1.2, $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ for some $i \geq 0$, so $x = at^i$ for some $a \notin \mathfrak{m}$. Similarly, $y = bt^j$ for some $j \geq 0$ and $b \notin \mathfrak{m}$. As $a, b \notin \mathfrak{m}$, a and b are invertible. Hence $xy = 0$ implies $t^{i+j} = 0$. But as nilpotents don't affect dimension,

$$\dim A = \dim A/(t) = \dim A/\mathfrak{m} = \dim k = 0, \tag{1}$$

contradicting $\dim A = 1$. □

1.4. *Theorem. — Suppose $(A, \mathfrak{m})$ is a Noetherian local ring of dimension 1. Then (a) and (b) are equivalent to:*

(c) *all ideals are of the form* $\mathfrak{m}^n$ *or* 0.

Proof. Assume (a): suppose $(A, \mathfrak{m}, k)$ is a Noetherian regular local ring of dimension 1. Then I claim that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any n. Otherwise, by Nakayama's lemma, $\mathfrak{m}^n = 0$, from which $t^n = 0$. But A is a domain, so $t = 0$, from which $A = A/\mathfrak{m}$ is a field, which can't have dimension 1, contradiction.

I next claim that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is dimension 1. Reason: $\mathfrak{m}^n = (t^n)$. So $\mathfrak{m}^n$ is generated as as a A-module by one element, and $\mathfrak{m}^n/(\mathfrak{m}\mathfrak{m}^n)$ is generated as a $(A/\mathfrak{m} = k)$-module by 1 element (non-zero by the previous paragraph), so it is a one-dimensional vector space.

So we have a chain of ideals $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ with $\cap \mathfrak{m}^i = (0)$ (Proposition 1.2). We want to say that there is no room for any ideal besides these, because "each pair is "separated by dimension 1", and there is "no room at the end". Proof: suppose $I \subset A$ is an ideal. If $I \neq (0)$, then there is some n such that $I \subset \mathfrak{m}^n$ but $I \not\subset \mathfrak{m}^{n+1}$. Choose some $u \in I - \mathfrak{m}^{n+1}$. Then $(u) \subset I$. But u generates $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, hence by Nakayama it generates $\mathfrak{m}^n$, so we have $\mathfrak{m}^n \subset I \subset \mathfrak{m}^n$, so we are done. Conclusion: in a Noetherian local ring of dimension 1, regularity implies all ideals are of the form $\mathfrak{m}^n$ or (0).

We now show that (c) implies (a). Assume (a) is false: suppose we have a dimension 1 Noetherian local domain that is not regular, so $\mathfrak{m}/\mathfrak{m}^2$ has dimension at least 2. Choose any $u \in \mathfrak{m} - \mathfrak{m}^2$. Then $(u, \mathfrak{m}^2)$ is an ideal, but $\mathfrak{m} \subsetneq (u, \mathfrak{m}^2) \subsetneq \mathfrak{m}^2$. ☐

1.A. EASY EXERCISE. Suppose $(A, \mathfrak{m})$ is a Noetherian dimension 1 local ring. Show that (a)–(c) above are equivalent to:

(d) A *is a principal ideal domain.*

1.5. Discrete valuation rings. We next define the notion of a discrete valuation ring. Suppose K is a field. A **discrete valuation** on K is a **surjective homomorphism** $\nu : K^* \to \mathbb{Z}$ (homomorphism: $\nu(xy) = \nu(x) + \nu(y)$) satisfying

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

except if $x + y = 0$ (in which case the left side is undefined). (Such a valuation is called **non-archimedean**, although we will not use that term.) It is often convenient to say $\nu(0) = \infty$. More generally, a **valuation** is a surjective homomorphism $\nu : K^* \to G$ to a totally ordered group G, although this isn't so important to us. (Not every valuation is discrete. Consider the ring of Puisseux series over a field k, $K = \cup_{n \geq 1} k((x^{1/n}))$, with $\nu : K^* \to \mathbb{Q}$ given by $\nu(x^q) = q$.)

Examples.

(i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.
(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f."

(iii) $K = k(x)$, $\nu(f)$ is the negative of the degree. This is really the same as (ii), with x replaced by $1/x$.

Then $0 \cup \{x \in K^* : \nu(x) \geq 0\}$ is a ring, which we denote $\mathcal{O}_\nu$. It is called the **valuation ring** of ν.

1.B. EXERCISE. Describe the valuation rings in the three examples above. (You will notice that they are familiar-looking dimension 1 Noetherian local rings. What a coincidence!)

1.C. EXERCISE. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

An integral domain A is called a **discrete valuation ring** (or **DVR**) if there exists a discrete valuation ν on its fraction field $K = FF(A)$ for which $\mathcal{O}_\nu = A$.

Now if A is a Noetherian regular local ring of dimension 1, and t is a uniformizer (a generator of $\mathfrak{m}$ as an ideal, or equivalently of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space) then any non-zero element r of A lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$, so $r = t^n u$ where u is a unit (as t^n generates $\mathfrak{m}^n$ by Nakayama, and so does r), so $FF(A) = A_t = A[1/t]$. So any element of $FF(A)$ can be written uniquely as ut^n where u is a unit and $n \in \mathbb{Z}$. Thus we can define a valuation $\nu(ut^n) = n$.

1.D. EXERCISE. Show that ν is a discrete valuation.

Thus (a)–(d) implies (e).

Conversely, suppose $(A, \mathfrak{m})$ is a discrete valuation ring.

1.E. EXERCISE. Show that $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1. (Hint: Show that the ideals are all of the form (0) or $I_n = \{r \in A : \nu(r) \geq n\}$, and I_1 is the only prime of the second sort. Then we get Noetherianness, and dimension 1. Show that I_1/I_2 is generated by the image of any element of $I_1 - I_2$.)

Hence we have proved:

1.6. *Theorem. — An integral domain* A *is a Noetherian local ring of dimension* 1 *satisfying (a)–(d) if and only if*

(e) A *is a discrete valuation ring.*

1.F. EXERCISE. Show that there is only one discrete valuation on a discrete valuation ring.

Thus any Noetherian regular local ring of dimension 1 comes with a unique valuation on its fraction field. If the valuation of an element is $n > 0$, we say that the element has a **zero of order** n. If the valuation is $-n < 0$, we say that the element has a **pole of order** n. We'll come back to this shortly, after dealing with (f) and (g).

1.7. *Theorem. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* 1*. Then (a)–(e) are equivalent to:*

(f) A *is a unique factorization domain,*
(g) A *is integrally closed in its fraction field* $K = FF(A)$*.*

Proof. (a)–(e) clearly imply (f), because we have the following stupid unique factorization: each non-zero element of r can be written uniquely as ut^n where $n \in \mathbb{Z}^{\geq 0}$ and u is a unit. Also, (f) implies (b), by an earlier easy Proposition, that in a unique factorization domain all codimension 1 prime ideals are principal. (In fact, we could just have (b) $\Longleftrightarrow$ (f) from the harder Proposition we proved, which showed that this property characterizes unique factorization domains.)

(f) implies (g), because unique factorization domains are integrally closed in their fraction fields (an earlier exercise).

It remains to check that (g) implies (a)–(e). We'll show that (g) implies (b).

Suppose $(A, \mathfrak{m})$ is a Noetherian local domain of dimension 1, integrally closed in its fraction field $K = FF(A)$. Choose any nonzero $r \in \mathfrak{m}$. Then $S = A/(r)$ is a Noetherian local ring of dimension 0 — its only prime is the image of $\mathfrak{m}$, which we denote $\mathfrak{n}$ to avoid confusion. Then $\mathfrak{n}$ is finitely generated, and each generator is nilpotent (the intersection of all the prime ideals in any ring are the nilpotents). Then $\mathfrak{n}^N = 0$, where N is the maximum of the nilpotence order of the finite set of generators. Hence there is some n such that $\mathfrak{n}^n = 0$ but $\mathfrak{n}^{n-1} \neq 0$.

Thus in A, $\mathfrak{m}^n \subseteq (r)$ but $\mathfrak{m}^{n-1} \not\subset (r)$. Choose $s \in \mathfrak{m}^{n-1} - (r)$. Consider $x = r/s$. Then $x^{-1} \notin A$, so as A is integrally closed, x^{-1} is not integral over A.

Now $x^{-1}\mathfrak{m} \not\subset \mathfrak{m}$ (or else $x^{-1}\mathfrak{m} \subset \mathfrak{m}$ would imply that $\mathfrak{m}$ is a faithful $A[x^{-1}]$-module, contradicting an Exercise from the Nakayama section that I promised we'd use). But $x^{-1}\mathfrak{m} \subset A$. Thus $x^{-1}\mathfrak{m} = A$, from which $\mathfrak{m} = xA$, so $\mathfrak{m}$ is principal. ☐

(At some point I'd like a different proof using a more familiar version of Nakayama, rather than this version which people might not remember.)

1.8. Geometry of normal Noetherian schemes. Suppose X is a locally Noetherian scheme. Then for any regular codimension 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1), we have a discrete valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a **zero of order** $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a **pole of order** $-\nu(f)$. (We aren't yet allowed to discuss order of vanishing at a point that is not regular codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.)

So we can finally make precise the fact that the function $(x-2)^2x/(x-3)^4$ on $\mathbb{A}^1_{\mathbb{C}}$ has a double zero at $x = 2$ and a quadruple pole at $x = 3$. Furthermore, we can say that 75/34 has a double zero at 5, and a single pole at 2! (What are the zeros and poles of $x^3(x+y)/(x^2+xy)^3$ on $\mathbb{A}^2$?)

1.G. EXERCISE. Suppose X is an integral Noetherian scheme, and $f \in FF(X)^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} A$. If $f = f_1/f_2$, where $f_i \in A$, prove the result for f_i.)

Suppose A is an Noetherian integrally closed domain. Then it is *regular in codimension* 1 (translation: all its codimension at most 1 points are regular). If A is dimension 1, then obviously A is nonsingular.

For example, $\operatorname{Spec} \mathbb{Z}[i]$ is nonsingular, because it is dimension 1 (proved earlier — e.g. it is integral over $\operatorname{Spec} \mathbb{Z}$), and $\mathbb{Z}[i]$ is a unique factorization domain. Hence $\mathbb{Z}[i]$ is normal, so all its closed (codimension 1) points are nonsingular. Its generic point is also nonsingular, as $\mathbb{Z}[i]$ is a domain.

Remark. A (Noetherian) scheme can be singular in codimension 2 and still be normal. For example, you have shown that the cone $x^2 + y^2 = z^2$ in $\mathbb{A}^3$ is normal (an earlier exercise), but it is clearly singular at the origin (the Zariski tangent space is visibly three-dimensional).

But singularities of normal schemes are not so bad. For example, we've already seen Hartogs' Theorem for Noetherian normal schemes, which states that you could extend functions over codimension 2 sets.

Remark: We know that for Noetherian rings we have implications

$$\text{unique factorization domain} \Longrightarrow \text{integrally closed} \Longrightarrow \text{regular in codimension 1}.$$

Hence for locally Noetherian schemes, we have similar implications:

$$\text{regular in codimension 1} \Longrightarrow \text{normal} \Longrightarrow \text{factorial}.$$

Here are two examples to show you that these inclusions are strict.

1.H. EXERCISE. Let A be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (Informally, we allow all polynomials that don't include a non-zero multiple of the monomial x.) Show that A is not integrally closed (hint: consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering A_{x^2} and A_y).

1.I. EXERCISE. You have checked that $k[w,x,y,z]/(wz-xy)$ is integrally closed (at least if k is algebraically closed of characteristic not 2, an earlier exercise). Show that it is not a unique factorization domain. (One possibility is to do this "directly". This might be hard to do rigorously — how do you know that x is irreducible in $k[w,x,y,z]/(wz-xy)$? Another possibility, faster but less intuitive, is to use the intermediate result that in a unique factorization domain, any height 1 prime is principal, and considering the exercise from last day that the cone over a ruling is not principal.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 23

RAVI VAKIL

CONTENTS

1. VALUATIVE CRITERIA FOR SEPARATEDNESS AND PROPERNESS

We now come to a topic that I regret bringing up. It is useful in practice, although to be honest, I've never used it myself in any meaningful way, and we will not use it later in this course. In fairness, I should say that many people love this fact, and the reason I felt compelled to discuss is was that I feared I would be cast out of the algebraic geometric if I didn't talk about it. But in retrospect I think you shouldn't see it soon after seeing separatedness the first time. In particular, you probably should just ignore this section.

In good circumstances, it is possible to verify separatedness by checking only maps from spectra of discrete valuations rings.

There are two reasons you might like it (even if you never use it). First, it gives useful intuition for what separated morphisms look like. Second, given that we understand schemes by maps to them (the Yoneda philosophy), we might expect to understand morphisms by mapping certain maps of schemes to them, and this is how you can interpret the diagram you'll see soon.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We'll then give a more general version for more general readers.

1.1. *Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then* f *is separated if and only if the following condition holds. For any discrete valuation ring* A *with function field* K*, and any diagram of the form*

(1)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ {\scriptstyle \textit{open imm.}} \big\downarrow & & \big\downarrow {\scriptstyle f} \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

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(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K$), there is at most one morphism $\operatorname{Spec} A \to X$ such that the diagram

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \text{open imm.} \downarrow & \overset{\leq 1}{\nearrow} & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

commutes.

A useful thing to take away from this statement is the intuition behind it. We think of $\operatorname{Spec} A$ as a "germ of a curve", and $\operatorname{Spec} K$ as the "germ minus the origin". Then this says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. In the case where Y is a field, you can think of this as saying that limits of one-parameter families are unique (if they exist).

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k: we take $\operatorname{Spec} A$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin. (I drew pictures here, which I have not yet latexed up: the map of the line with doubled origin to a point; the map of the line with the doubled origin to a line; and the map of the line with doubled origin to itself. In the first two cases, we could see the valuative criterion failing. In the last case, it did not fail.)

1.A. EXERCISE. Make this precise: show that map of the line with doubled origin over k to $\operatorname{Spec} k$ fails the valuative criterion for separatedness.

1.2. Note on moduli spaces and the valuative criterion of separatedness. I said a little more about separatedness of moduli spaces, for those familiar such objects. Suppose we are interested in a moduli space of a certain kind of object. That means that there is a scheme M with a "universal family" of such objects over M, such that there is a bijection between families of such objects over an arbitrary scheme S, and morphisms $S \to B$. (One direction of this map is as follows: given a morphism $S \to B$, we get a family of objects over S by pulling back the universal family over B.) The separatedness of the moduli space (over the base field, for example, if there is one) can be interpreted as follows. Fix a valuation ring A (or even discrete valuation ring, if our moduli space of of finite type) with fraction field K. We interpret Spec intuitively as a germ of a curve, and we interpret $\operatorname{Spec} K$ as the germ minus the "origin" (an analogue of a small punctured disk). Then we have a family of objects over $\operatorname{Spec} K$ (or over the punctured disk), or equivalently a map $\operatorname{Spec} K \to M$, and the moduli space is separated if there is *at most one way* to fill in the family over the origin, i.e. a family over $\operatorname{Spec} A$.

$\star$ The rest of this section should be ignored upon first reading.

Proof. (This proof is more telegraphic than I'd like. I may fill it out more later. Because we won't be using this result later in the course, you should feel free to skip it, but you may want to skim it.) One direction is fairly straightforward. Suppose $f : X \to Y$ is separated, and such a diagram (1) were given. suppose g_1 and g_2 were two morphisms $\operatorname{Spec} A \to X$ making (2) commute. Then $g = (g_1, g_2) : \operatorname{Spec} A \to X \times_Y X$ is a morphism, with $g(\operatorname{Spec} K)$ contained in the diagonal. Hence as $\operatorname{Spec} K$ is dense in $\operatorname{Spec} A$, and g is continuous, $g(\operatorname{Spec} A)$ is contained in the closure of the diagonal. As the diagonal is closed (the separated hypotheses), $g(\operatorname{Spec} A)$ is also contained *set-theoretically* in the diagonal. As $\operatorname{Spec} A$ is reduced, g factors through the induced reduced subscheme structure of the diagonal. Hence g factors through the diagonal:

$$\operatorname{Spec} A \longrightarrow X \xrightarrow{\delta} X \times_Y X,$$

which means $g_1 = g_2$ by our earlier exercise about maps from a reduced schemed to a separated scheme.

Suppose conversely that f is not separated, i.e. that the diagonal $\Delta \subset X \times_Y X$ is not closed. Note that $X \times_Y X$ is Noetherian (X is Noetherian, and $X \times_Y X \to X$ is finite type as it is obtained by base change from the finite type $X \to Y$), As Δ isn't a closed subset, there is a point in $\overline{\Delta} - \Delta$ and another point in Δ so that the first (say z) is in the closure of the second (say a). (I believe we checked earlier in our discussion of Chevalley's theorem that for Noetherian schemes, a subset is closed if and only if it is closed under specialization.) By the Noetherian condition, there is a maximal chain of closed subsets

$$\overline{a} \subset \overline{b} \subset \cdots \subset \overline{z}$$

(where $a, \ldots, z$ are the generic points). Thus we can find two "adjacent" points (say p and q, so $\operatorname{codim}_{\overline{q}} p = 1$) such that $q \in \Delta$ and $p \notin \Delta$. Let Q be the scheme obtained by giving the induced reduced subscheme structure to $\overline{q}$. Then p is a codimension 1 point on Q; let $A' = \mathcal{O}_{Q,p}$ be the local ring of Q at p. Then A' is a Noetherian local domain of dimension 1. Let A'' be the normalization of A. Choose any point p'' of $\operatorname{Spec} A''$ mapping to p; such a point exists because the normalization morphism $\operatorname{Spec} A \to \operatorname{Spec} A'$ is surjective (normalization is an integral extension, hence surjective by the Going-up theorem). Now A'' is Noetherian (I need to explain why... if $R \hookrightarrow R'$ is an integral extension of rings, then R is Noetherian if and only if R'' is Noetherian, by the going down theorem...). Let A be the localization of A'' at p''. Then A is a normal Noetherian local domain of dimension 1, and hence a discrete valuation ring. Let K be its fraction field. Then $\operatorname{Spec} A \to X \times_Y X$ does not factor through the diagonal, but $\operatorname{Spec} K \to X \times_Y X$ does, and we are done. □

With a more powerful invocation of commutative algebra, we can prove a valuative criterion with much less restrictive hypotheses.

1.3. *Theorem: Valuative criterion of separatedness. — Suppose $f : X \to Y$ is a quasiseparated morphism. Then f is separated if and only if the following condition holds. For any valuation ring A with function field K, and any diagram of the form* (1), *there is at most one morphism $\operatorname{Spec} A \to X$ such that the diagram* (2) *commutes.*

Because I've already proved something useful that we'll never use, I feel no urge to prove this harder fact. The proof of one direction, that separated implies that the criterion holds, is identical. The other direction is similar: get P and Q. Then use an algebra fact.

There is a valuative criterion for properness too. I've never used it personally, but it *is* useful, both directly, and also philosophically. I'll make statements, and then discuss some philosophy.

1.4. *Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes). — Suppose $f : X \to Y$ is a morphism of finite type of locally Noetherian schemes. Then f is proper if and only if the following condition holds. For any discrete valuation ring A with function field K, and or any diagram of the form*

(3)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K$), there is exactly *one morphism $\operatorname{Spec} A \to X$ such that the diagram*

(4)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

commutes.

Recall that the valuative criterion for separatedness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameter families always exist, and are unique.

I discussed the moduli interpretation of this criterion.

1.B. EXERCISE. Use the valuative criterion of properness to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper if A is Noetherian. (This is a difficult way to prove this fact!)

1.5. *Theorem (Valuative criterion of properness). — Suppose $f : X \to Y$ is a quasiseparated, finite type (hence quasicompact) morphism. Then f is proper if and only if the following condition holds. For any valuation ring R with function field K, and or any diagram of the form* (3), *there is* exactly *one morphism $\operatorname{Spec} R \to X$ such that the diagram* (4) *commutes.*

Uses: (1) intuition. (2) moduli idea: exactly one way to fill it in (stable curves). (3) motivates the definition of properness for stacks.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 24

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CONTENTS

Quasicoherent and coherent sheaves are natural generalizations of the notion of a vector bundle. In order to help motivate them, we first discuss vector bundles, and how they can be interpreted in terms of locally free shaves.

In a nutshell, a **free sheaf** on X is an $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{\oplus I}$ where the sum is over some index set I. A **locally free sheaf** X is an $\mathcal{O}_X$-module locally isomorphic to a free sheaf. This corresponds to the notion of a vector bundle. A **quasicoherent sheaf on** X may be defined as an $\mathcal{O}_X$-module which may be locally written as the cokernel of a map of free sheaves. These definitions are useful for ringed spaces in general. We will instead start with two other definitions of quasicoherent sheaf which better highlight the parallel between this notion and that of modules over a ring, and make it easy to work with a scheme by considering an affine cover.

1. VECTOR BUNDLES AND LOCALLY FREE SHEAVES

As motivation, we discuss vector bundles on real manifolds. Examples to keep in mind are the tangent bundle to a manifold, and the Möbius strip over a circle.

Arithmetically-minded readers shouldn't tune out! Fractional ideals of the ring of integers in a number field will turn out to be an example of a "line bundle on a smooth curve".

A *rank* n *vector bundle on a manifold* M is a fibration $\pi : V \to M$ with the structure of an n-dimensional real vector space on $\pi^{-1}(x)$ for each point $x \in M$, such that for every $x \in M$, there is an open neighborhood U and a homeomorphism

$$\phi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

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over U (so that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xleftrightarrow{\cong} & U \times \mathbb{R}^n \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \text{projection to first factor}} & \\ & U & \end{array} \tag{1}$$

commutes) that is an isomorphism of vector spaces over each $y \in U$.

An isomorphism (1) is called a **trivialization over** U.

In this definition, n is called the **rank** of the vector bundle. A rank 1 vector bundle is called a **line bundle**. (It is sometimes convenient to be agnostic about the rank of the vector bundle, so it can have different ranks on different connected components. It is also sometimes convenient to consider infinite-rank vector bundles.)

1.1. Transition functions. Given trivializations over U_1 and U_2, over their intersection, the two trivializations must be related by an element T_{ij} of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$. If $\{U_i\}$ is a cover of M, and we are given trivializations over each U_i, then the $\{T_{ij}\}$ must satisfy the *cocycle condition*:

$$f_{ij}|_{U_i \cap U_j \cap U_k} \circ f_{jk}|_{U_i \cap U_j \cap U_k} = f_{ik}|_{U_i \cap U_j \cap U_K}. \tag{2}$$

Note that this implies $T_{ij} = T_{ji}^{-1}$. The data of the T_{ij} are called **transition functions** for the trivialization.

Conversely, given the data of a cover $\{U_i\}$ and transition functions T_{ij} (an element of $GL(n)$ with entries that are functions on $U_i \cap U_j$), we can recover the vector bundle (up to unique isomorphism) by "gluing together the $U_i \times \mathbb{R}^n$ along over $U_i \cap U_j$ using f_{ij}".

1.2. Sheaf of sections. Fix a rank n vector bundle $V \to M$. The sheaf of sections $\mathcal{F}$ of V is an $\mathcal{O}_M$-module — given any open set U, we can multiply a section over U by a function on U and get another section.

Moreover, given a U and a trivialization, the sections over U are naturally identified with n-tuples of functions of U.

$$\begin{array}{c} U \times \mathbb{R}^n \\ {\scriptstyle \pi} \Big\downarrow \Big\uparrow {\scriptstyle \mathbf{f} = \text{an } n\text{-tuple of functions}} \\ U \end{array}$$

Thus given a trivialization, over each open set U_i, we have an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. We say that $\mathcal{F}$ is a **locally free sheaf of rank** n. (As stated earlier, a sheaf $\mathcal{F}$ is **free of rank** n if $\mathcal{F} \cong \mathcal{O}^{\oplus n}$.)

1.3. Transition functions for the sheaf of sections. Suppose we have a vector bundle on M, along with a trivialization over an open cover U_i. Suppose we have a section of the vector bundle over M. (This discussion will apply with M replaced by any open subset.) Then over each U_i, the section corresponds to an n-tuple functions over U_i, say $\mathbf{f}_i$.

1.A. EXERCISE. Show that over $U_i \cap U_j$, the vector-valued function $\mathbf{f}_i$ is related to $\mathbf{f}_j$ by the transition functions:

$$T_{ij}\mathbf{f}_i = \mathbf{f}_j$$

Given a locally free sheaf $\mathcal{F}$ with rank n, and a trivializing neighborhood of $\mathcal{F}$ (an open cover $\{U_i\}$ such that over each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ as $\mathcal{O}$-modules), we have transition functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (2). Thus in conclusion the data of a locally free sheaf of rank n is equivalent to the data of a vector bundle of rank n.

A rank 1 locally free sheaf is called an **invertible sheaf**. We'll see later why it is called invertible; but it is still a somewhat heinous term for something so fundamental.

1.4. Locally free sheaves on schemes.

Suitably motivated, we now become rigorous and precise. We can generalize the notion of locally free sheaves to schemes without change. A **locally free sheaf of rank n on a scheme** X is an $\mathcal{O}_X$-module $\mathcal{F}$ that is locally trivial of rank n. Precisely, there is an open cover $\{U_i\}$ of X such that for each U_i, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. A locally free sheaf may be described in terms of transition functions: the data of a cover $\{U_i\}$ of X, and functions $T_{ij} \in GL(n, \mathcal{O}(U_i \cap U_j))$ satisfying the cocycle condition (2). As before, given this data, we can find the sections over any open set U. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij}\vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

You should think of these "as" vector bundles, but just keep in mind that they are not the "same", just equivalent notions. We will later define the "total space" of the vector bundle $V \to X$ (a scheme over X) in terms of the sheaf version of Spec (precisely, $\underline{\operatorname{Spec}} \operatorname{Sym} \mathbf{V}^\bullet$). But the locally free sheaf perspective will prove to be more useful. As one example: the definition of a locally free sheaf is much shorter than that of a vector bundle.

As in our motivating discussion, it is sometimes convenient to let the rank vary among connected components, or to consider infinite rank locally free sheaves.

1.5. Useful constructions.

We now give some useful constructions in the form of a series of exercises. Most will later generalize readily to quasicoherent sheaves.

1.B. EXERCISE. Suppose s is a section of a locally free sheaf $\mathcal{F}$ on a scheme X. Define the notion of the **subscheme cut out by** $s = 0$. (Hint: given a trivialization over an open set

U, s corresponds to a number of functions $f_1, \ldots$ on U; on U, take the scheme cut out by these functions.)

1.C. EXERCISE. Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves on X of rank m and n respectively. Show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf of rank mn.

1.D. EXERCISE. If $\mathcal{E}$ is a locally free sheaf of rank n, show that $\mathcal{E}^\vee := \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{O})$ is also a locally free sheaf of rank n. This is called the **dual** of $\mathcal{E}$. Given transition functions for $\mathcal{E}$, describe transition functions for $\mathcal{E}^\vee$. (Note that if $\mathcal{E}$ is rank 1 (i.e. invertible), the transition functions of the dual are the inverse of the transition functions of the original.) Show that $\mathcal{E} \cong \mathcal{E}^{\vee\vee}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

1.E. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a locally free sheaf. (Here $\otimes$ is tensor product as $\mathcal{O}_X$-modules, defined last quarter) If $\mathcal{F}$ is an invertible sheaf, show that $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

1.F. EXERCISE. Recall that tensor products tend to be only right-exact in general. Show that tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of $\mathcal{O}_X$-modules, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.

1.G. EXERCISE. If $\mathcal{E}$ is a locally free sheaf, and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}) \cong \underline{\mathrm{Hom}}(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G})$.

1.H. EXERCISE AND IMPORTANT DEFINITION. Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the **Picard group** of X, and is denoted $\mathrm{Pic}\, X$. (For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.)

1.6. Random concluding remarks.

We define **rational and regular sections of a locally free sheaf** on a scheme X.

1.I. LESS IMPORTANT EXERCISE. Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' theorem": sections defined away from a set of codimension at least 2 extend over that set.

1.7. *Remark.* Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

1.J. EXERCISE (FOR ARITHMETICALLY-MINDED PEOPLE ONLY — I WON'T DEFINE MY TERMS). Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf. Show that two that yield the same invertible sheaf differ by a principal ideal. The *class group* is defined to be the group of fractional ideals modulo the principal ideals. This exercises shows that the class group is (isomorphic to) the Picard group. (This discussion applies to the ring integers in any global field.)

1.8. The problem with locally free sheaves.

Recall that $\mathcal{O}_X$-modules form an abelian category: we can talk about kernels, cokernels, and so forth, and we can do homological algebra. Similarly, vector spaces form an abelian category. But locally free sheaves (i.e. vector bundles), along with reasonably natural maps between them (those that arise as maps of $\mathcal{O}_X$-modules), don't form an abelian category. As a motivating example in the category of differentiable manifolds, consider the map of the trivial line bundle on $\mathbb{R}$ (with co-ordinate t) to itself, corresponding to multiplying by the co-ordinate t. Then this map jumps rank, and if you try to define a kernel or cokernel you will get yourself confused.

This problem is resolved by enlarging our notion of nice $\mathcal{O}_X$-modules in a natural way, to quasicoherent sheaves.

$\mathcal{O}_X$-modules	$\supset$	quasicoherent sheaves	$\supset$	locally free sheaves
abelian category		abelian category		not an abelian category

Similarly, finite rank locally free sheaves will sit in a nice smaller abelian category, that of *coherent sheaves*.

quasicoherent sheaves	$\supset$	coherent sheaves	$\supset$	finite rank locally free sheaves
abelian category		abelian category		not an abelian category

2. TOWARD QUASICOHERENT SHEAVES: THE DISTINGUISHED AFFINE BASE

Schemes generalize and geometrize the notion of "ring". It is now time to define the corresponding analogue of "module", which is a quasicoherent sheaf.

One version of this notion is that of an $\mathcal{O}_X$-module. They form an abelian category, with tensor products.

We want a better one — a subcategory of $\mathcal{O}_X$-modules. Because these are the analogues of modules, we're going to define them in terms of affine open sets of the scheme. So let's think a bit harder about the structure of affine open sets on a general scheme X. I'm going to define what I'll call the *distinguished affine base* of the Zariski topology. This won't be a base in the sense that you're used to. (For experts: it is a first example of a *Grothendieck topology*.)

The open sets are the affine open subsets of X. We've already observed that this forms a base. But forget about that.

We like distinguished open sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, and we don't really understand open immersions of one random affine open subset in another. So we just remember the "nice" inclusions.

Definition. The **distinguished affine base** of a scheme X is the data of the affine open sets and the distinguished inclusions.

In other words, we are remembering only some of the open sets (the affine open sets), and only some of the morphisms between them (the distinguished morphisms). For experts: if you think of a topology as a category (the category of open sets), we have described a subcategory.

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we're going: we'll show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up, we can recover stalks as follows. (We will be implicitly using only the following fact. We have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key fact that given any two affine open sets $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine open sets that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. This is a *cofinal* condition.)

The stalk $\mathcal{F}_x$ is the direct limit $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in U. We compare this to $\varinjlim(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this! It corresponds to the fact that the basic elements are cofinal in this directed system.)

2.A. EXERCISE. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

2.1. *Theorem.* —

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.*
(c) *An $\mathcal{O}_X$-module "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

This proof is identical to our argument showing that sheaves are (essentially) the same as sheaves on a base, using the "sheaf of compatible germs" construction. The main reason for repeating it is to let you see that all that is needed is for the open sets to form a cofinal system (or better, that the category of open sets and inclusions we are considering is cofinal).

For experts: (a) and (b) are describing an equivalence of categories between sheaves on the Zariski topology of X and sheaves on the distinguished affine base of X.

Proof. (a) Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define the sheaf of compatible germs

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_x \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. This is an isomorphism on stalks, and hence an isomorphism by an Exercise from last quarter.

2.B. EXERCISE. Prove (b).

2.C. EXERCISE. Prove (c). □

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 25

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CONTENTS

We began by recalling the distinguished affine base.

Definition. The **distinguished affine base** of a scheme X is the data of the affine open sets and the distinguished inclusions.

0.1. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on a distinguished affine base uniquely determines a morphism of sheaves.*
(c) *An $\mathcal{O}_X$-module "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

1. QUASICOHERENT SHEAVES

We now define the notion of *quasicoherent sheaf*. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". We will give two equivalent definitions; each definition is useful in different circumstances. The first just involves the distinguished topology.

The first definition is more directly "sheafy". Given an A-module M, we defined a sheaf $\tilde{M}$ on $\operatorname{Spec} A$ long ago — the sections over $D(f)$ were M_f.

Definition A. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} A$,

$$\mathcal{F}|_{\operatorname{Spec} A} \cong \Gamma(\widetilde{\operatorname{Spec} A, \mathcal{F}}).$$

Date: Wednesday, January 23, 2008.

(The "wide tilde" is supposed to cover the entire right side $\Gamma(\operatorname{Spec} A, \mathcal{F})$.) This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

Hence by this definition, the sheaves on $\operatorname{Spec} A$ correspond to A-modules. Given an A-module M, we get a sheaf $\tilde{M}$. Given a sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, we get an A-module $\Gamma(X, \mathcal{F})$. These operations are inverse to each other. So in the same way as schemes are obtained by gluing together rings, quasicoherent sheaves are obtained by gluing together modules over those rings.

The second definition really focuses on the distinguished affine base, and is reminiscent of the Affine Covering Lemma.

Definition B. Suppose $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \subset X$ is a distinguished open set. Let $\phi : \Gamma(\operatorname{Spec} A, \mathcal{F}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F})$ be the restriction map. The source of ϕ is an A-module, and the target is an A_f-module, so by the universal property of localization, ϕ naturally factors as:

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} A, \mathcal{F}) & \xrightarrow{\quad\phi\quad} & \Gamma(\operatorname{Spec} A_f, \mathcal{F}) \\ & \searrow \qquad \nearrow_{\alpha} & \\ & \Gamma(\operatorname{Spec} A, \mathcal{F})_f & \end{array}$$

An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for each such distinguished $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, α is an isomorphism.

Thus a quasicoherent sheaf is the data of one module for each affine open subset (a module over the corresponding ring), such that the module over a distinguished open set $\operatorname{Spec} A_f$ is given by localizing the module over $\operatorname{Spec} A$. This will be an easy criterion to check.

1.1. *Proposition. — Definitions A and B are the same.*

Proof. Clearly Definition A implies Definition B. (Recall that the definition of $\tilde{M}$ was in terms of the distinguished topology on $\operatorname{Spec} A$.) We now show that Definition B implies Definition A. By Definition B, the sections over any distinguished open $\operatorname{Spec} A_f$ of $\mathcal{M}$ on $\operatorname{Spec} A$ is precisely $\Gamma(\operatorname{Spec} A, \mathcal{M})_f$, i.e. the sections of $\widetilde{\Gamma(\operatorname{Spec} A, \mathcal{M})}$ over $\operatorname{Spec} A_f$, and the restriction maps agree. Thus the two sheaves agree. □

We like Definition B because it says that to define a quasicoherent $\mathcal{O}_X$-module is that we just need to know what it is on all affine open sets, and that it behaves well under inverting a single element.

One reason we like Definition A is that it works well in gluing arguments, as in the proof of the following fact.

1.2. *Proposition (quasicoherence is an affine-local notion). — Let* X *be a scheme, and* $\mathcal{F}$ *an* $\mathcal{O}_X$*-module. Then let* P *be the property of affine open sets that* $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{\Gamma(\operatorname{Spec} A, \mathcal{F})}$*. Then* P *is an affine-local property.*

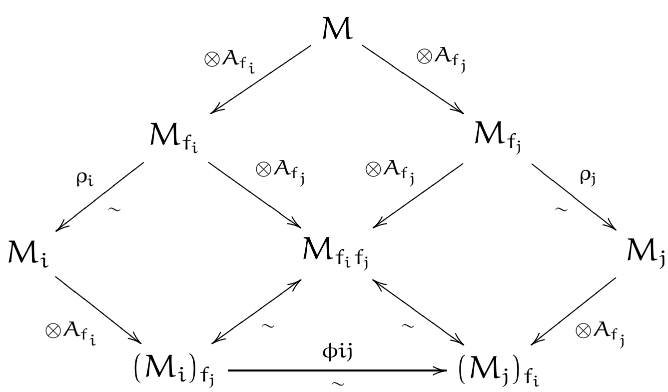
Before we prove this, we give an exercise to show its utility.

1.A. EXERCISE. Show that locally free sheaves are quasicoherent.

Proof. By the Affine Communication Lemma, we must check two things. Clearly if $\operatorname{Spec} A$ has property P, then so does the distinguished open $\operatorname{Spec} A_f$: if M is an A-module, then $\tilde{M}|_{\operatorname{Spec} A_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} A_f}$-modules (both sides agree on the level of distinguished open sets and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma. Suppose we have modules $M_1, \dots, M_n$, where M_i is an A_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $A_{f_i f_j}$-modules, satisfying the cocycle condition. We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} A_{f_i}$, or equivalently, isomorphisms $\rho_i : \Gamma(D(f_i), \tilde{M}) \to M_i$, so that the bottom triangle of

(1)



commutes.

We already know that M should be the sections of $\mathcal{F}$ over $\operatorname{Spec} A$, as $\mathcal{F}$ is a sheaf. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. Then

$$0 \to M \to M_1 \times \cdots \times M_n \to M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the latter morphism is the "difference" morphism). So M is a kernel of a morphism of A-modules, hence an A-module. We are left to show that $M_i \cong M_{f_i}$ (and that this isomorphism satisfies (1)).

For convenience we assume $i = 1$. Localization is exact, so

(2)

$$0 \longrightarrow M_{f_1} \longrightarrow M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \longrightarrow M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

is an exact sequence of A_{f_1}-modules.

We now identify many of the modules appearing in (2) in terms of M_1. First of all, f_1 is invertible in A_{f_1}, so $(M_1)_{f_1}$ is canonically M_1. Also, $(M_j)_{f_1} \cong (M_1)_{f_j}$ via ϕ_{ij}. Hence if

$i, j \neq 1$, $(M_{ij})_{f_1} \cong (M_1)_{f_i f_j}$ via ϕ_{1i} and ϕ_{1j} (here the cocycle condition is implicitly used). Furthermore, $(M_{1i})_{f_1} \cong (M_1)_{f_i}$ via ϕ_{1i}. Thus we can write (2) as
(3)

$$0 \longrightarrow M_{f_1} \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \xrightarrow{\alpha} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots (M_1)_{f_{n-1} f_n}$$

By assumption, $\mathcal{F}|_{\operatorname{Spec} A_{f_1}}$ is quasicoherent, so by considering the cover of

$$\operatorname{Spec} A_{f_1} = \operatorname{Spec} A_{f_1} \cup \operatorname{Spec} A_{f_1 f_2} \cup \operatorname{Spec} A_{f_1 f_3} \cup \cdots \cup \operatorname{Spec} A_{f_1 f_n}$$

(which indeed has a "redundant" first term), and identifying sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1}$ in terms of sections over the open sets in the cover and their pairwise overlaps, we have an exact sequence of A_{f_1}-modules

$$0 \longrightarrow M_1 \longrightarrow M_1 \times (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \xrightarrow{\beta} (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \times (M_1)_{f_2 f_3} \times \cdots (M_1)_{f_{n-1} f_n}$$

which is very similar to (3). Indeed, the final map β of the above sequence is the same as the map α of (3), so $\ker \alpha = \ker \beta$, i.e. we have an isomorphism $M_1 \cong M_{f_1}$.

Finally, the triangle of (1) is commutative, as each vertex of the triangle can be identified as the sections of $\mathcal{F}$ over $\operatorname{Spec} A_{f_1 f_2}$. □

1.B. IMPORTANT EXERCISE. Suppose X is a quasicompact and quasiseparated scheme (i.e. covered by a finite number of affine open sets, the pairwise intersection of which is also covered by a finite number of affine open sets). Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & \searrow^{\otimes_A A_f} \quad \nearrow^{\sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_A A_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

1.C. LESS IMPORTANT EXERCISE. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes. The key idea is that infinite direct sums do not commute with localization.)

1.D. IMPORTANT EXERCISE (COROLLARY TO EXERCISE 1.B). Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X. Show that $\pi_*\mathcal{F}$ is a quasicoherent sheaf on Y.

1.E. UNIMPORTANT EXERCISE (NOT EVERY $\mathcal{O}_X$-MODULE IS A QUASICOHERENT SHEAF).
(a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$ and the usual $k[t]$-module structure. Show that this is an $\mathcal{O}_X$-module that s not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X.

2. QUASICOHERENT SHEAVES FORM AN ABELIAN CATEGORY

The category of A-modules is an abelian category. Indeed, this is our motivating example of our notion of abelian category. Similarly, quasicoherent sheaves form an abelian category. I'll explain how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. As a metaphor, there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it forms a subgroup.

You can look at back at the definition of an abelian category, and you'll see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in your subcategory
(ii) your subcategory is closed under finite sums
(iii) your subcategory is closed under kernels and cokernels

In our case of

$$\{\text{quasicoherent sheaves}\} \subset \{\mathcal{O}_X\text{-modules}\},$$

the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and over $\operatorname{Spec} A$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$, so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii). Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α (exactness can be checked stalk-locally). Thus the quasicoherent sheaves indeed form an abelian category.

2.A. EXERCISE. Show that a sequence of quasicoherent sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ on X is exact if and only if it is exact on each open set in an affine cover of X. (In particular, taking sections over an affine open $\operatorname{Spec} A$ is an exact functor from the category of quasicoherent sheaves on X to the category of A-modules. Recall that taking sections is only left-exact in general.) In particular, we may check injectivity or surjectivity of a morphism of quasicoherent sheaves by checking on an affine cover.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and we have exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} A, \mathcal{F}) = 0$. This will indeed be the case.)

2.B. EXERCISE (CONNECTION TO ANOTHER DEFINITION). Show that an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is quasicoherent if and only if there exists an open cover by U_i such that on each U_i, $\mathcal{F}|_{U_i}$ is isomorphic to the cokernel of a map of two free sheaves:

$$\mathcal{O}_{U_i}^{\oplus I} \to \mathcal{O}_{U_i}^{\oplus J} \to \mathcal{F}|_{U_i} \to 0$$

is exact. We have thus connected our definitions to the definition given at the very start of the chapter.

We then began to discuss module-like constructions for quasicoherent sheaves, and I've left these for the next day's notes, so all of our discussion on that topic is in one place.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 26

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CONTENTS

1. MODULE-LIKE CONSTRUCTIONS

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves.

As an important example, we consider tensor products.

1.A. EXERCISE. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a quasicoherent sheaf described by the following information: If $\operatorname{Spec} A$ is an affine open, and $\Gamma(\operatorname{Spec} A, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} A, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} A_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_A N \to (M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$. (We are using the algebraic fact that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of. This is a feature we will use often: constructions involving quasicoherent sheaves that involve sheafification for general sheaves don't require sheafification when considered on the distinguished affine base. Along with the fact that injectivity, surjectivity, kernels and so on may be computed on affine opens, this is the reason that it is particularly convenient to think about quasicoherent sheaves in terms of affine open sets.

1.B. EASY EXERCISE. Show that the stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.

Date: Friday, January 25, 2008.

Given a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, we have a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is an invertible sheaf, this section is denoted st.

1.1. Tensor algebra constructions.

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}(\sigma)} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$ where σ is a permutation of $\{1, \dots, n\}$. It is a "skew-commutative" A-algebra. It is most correct to write $T_A^*(M)$, $\operatorname{Sym}_A^*(M)$, and $\wedge_A^*(M)$, but the "base ring" A is usually omitted for convenience. (Better: both Sym and $\wedge$ are defined by universal properties. For example, $\operatorname{Sym}_A^n(M)$ is universal among modules such that any map of A-modules $M^{\otimes n} \to N$ that is symmetric in the n entries factors uniquely through $\operatorname{Sym}_A^n(M)$.)

1.C. EXERCISE. Suppose $\mathcal{F}$ is a quasicoherent sheaf. Define the quasicoherent sheaves $\operatorname{Sym}^n \mathcal{F}$ and $\wedge^n \mathcal{F}$. (One possibility: describe them on each affine open set.) If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^* \mathcal{F}$, the sheaf of commutative algebras $\operatorname{Sym}^* \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^* \mathcal{F}$.

1.D. EXERCISE (POSSIBLE HELP FOR LATER PROBLEMS). Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of locally free sheaves on X. Suppose $U = \operatorname{Spec} A$ is an affine open set where $\mathcal{F}'$, $\mathcal{F}''$ are free, say $\mathcal{F}'|_{\operatorname{Spec} A} = \tilde{A^a}$, $\mathcal{F}''|_{\operatorname{Spec} A} = \tilde{A^b}$. Show that $\mathcal{F}$ is also free, and that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ can be interpreted as coming from the tautological exact sequence $0 \to A^a \to A^{a+b} \to A^b \to 0$. Show that given such an open cover, the transition matrices of $\mathcal{F}$ may be interpreted as block upper-diagonal matrices, where thet top $a \times a$ block are transition matrices for $\mathcal{F}'$, and the bottom $b \times b$ blocks are transition matrices for $\mathcal{F}''$.

1.E. IMPORTANT EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'').$$

(Possible hint for this, and Exercise 1.G: It suffices to consider a small enough affine open set $\operatorname{Spec} A$, where $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are free, and to show that your construction behaves well with respect to localization at an element $f \in A$. In such an open set, the sequence is $0 \to A^p \to A^{p+q} \to A^q \to 0$ by the Exercise 1.D. Let $e_1, \dots, e_n$ be the standard basis of A^n, and $f_1, \dots, f_q$ be the the standard basis of A^q. Let $e_1', \dots, e_p'$ be denote the images of $e_1, \dots, e_p$ in A^{p+q}. Let $f_1', \dots, f_q'$ be any lifts of $f_1, \dots, f_q$ to A^{p+q}. Note that f_i' is well-defined modulo $e_1', \dots, e_p'$. Note that

$$\operatorname{Sym}^s \mathcal{F}|_{\operatorname{Spec} A} \cong \oplus_{i=0}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}.$$

Show that $\mathcal{F}^p := \oplus_{i=p}^s \operatorname{Sym}^i \mathcal{F}'|_{\operatorname{Spec} A} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \operatorname{Sym}^{s-i} \mathcal{F}''|_{\operatorname{Spec} A}$ gives a well-defined (locally free) subsheaf that is independent of the choices made, e.g. of the basis $e_1, \dots, e_p$ (this is in $GL_p(A)$), $f_1, \dots, f_q$ (this is in $GL_q(A)$), and the lifts $f_1', \dots, f_q'$.)

1.F. EXERCISE. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the **determinant (line) bundle** or (perhaps better) **determinant locally free sheaf**. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

1.G. EXERCISE. Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with subquotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$. In fact we only need that $\mathcal{F}''$ is locally free.

1.H. EXERCISE (DETERMINANT LINE BUNDLES BEHAVE WELL IN EXACT SEQUENCES). Suppose $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of finite rank locally free sheaves on X. Show that "the alternating product of determinant bundles is trivial":

$$\det(\mathcal{F}_1) \otimes \det(\mathcal{F}_2)^\vee \otimes \det(\mathcal{F}_3) \otimes \det(\mathcal{F}_4)^\vee \otimes \cdots \cong \mathcal{O}_X.$$

1.2. Torsion-free sheaves (a stalk-local condition). Recall that an A-module M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **torsion-free** if $\mathcal{F}_p$ is a torsion-free $\mathcal{O}_{X,p}$-module for all p.

1.I. EXERCISE. Show that if M is a torsion-free A-module, then so is any localization of M. Hence show that $\tilde{M}$ is a torsion free sheaf on $\operatorname{Spec} A$.

1.J. UNIMPORTANT EXERCISE (TORSION-FREENESS IS NOT AFFINE LOCAL FOR STUPID REASONS). Find an example on a two-point space showing that $M := A$ might not be torsion-free on $\operatorname{Spec} A$ even though $\mathcal{O}_{\operatorname{Spec} A} = \tilde{M}$ is torsion-free.

2. Finiteness conditions on quasicoherent sheaves: finite type quasicoherent sheaves, and coherent sheaves

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be **finitely generated**. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too. It could be finitely presented — it could have a finite number of generators with a finite number of relations: there exists a **finite presentation**

$$A^q \to A^p \to M \to 0.$$

The third notion is frankly a bit surprising, and I'll justify it soon. We say that an A-module M is **coherent** if (i) it is finitely generated, and (ii) whenenver we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

2.1. *Proposition. — If A is Noetherian, then these three definitions are the same.*

Before proving this, we take this as an excuse to develop some algebraic background.

2.2. *Noetherian conditions for modules.* If A is any ring, not necessarily Noetherian, we say an A-module is Noetherian if it satisfies the ascending chain condition for submodules. Thus for example A is a Noetherian ring if and only if it is a Noetherian A-module.

2.A. Exercise. Show that if M is a Noetherian A-module, then any submodule of M is a finitely generated A-module.

2.B. Exercise. If $0 \to M' \to M \to M'' \to 0$ is exact, show that M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''. Possible further hint: prove that if $M' \longrightarrow M \xrightarrow{\phi} M''$ is exact, and $N, N' \subset M$, and $N \cap M' = N' \cap M'$ and $\phi(N) = \phi(N')$, then $N = N'$.)

2.C. Exercise. Show that if A is a Noetherian ring, then A^n is a Noetherian A-module.

2.D. Exercise. Show that if A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module. Hence by Exercise 2.A, any submodule of a finitely generated module over a Noetherian ring is finitely generated.

Proof of Proposition 2.1. As we observed earlier, coherent implies finitely presented implies finitely generated. So suppose M is finitely generated. Take any $A^p \xrightarrow{\alpha} M$. Then $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated by Exercise 2.D. Thus M is coherent. □

Hence most normal people can think of these three notions as the same thing.

2.3. *Proposition. — The coherent A-modules form an abelian subcategory of the category of A-modules.*

The proof in general is given in §3 in a series of short exercises.

Proof if A is Noetherian. Recall from our discussion a few classes ago that we must check three things:

(i) The 0-sheaf is coherent.
(ii) The category of coherent modules is closed under finite sums.
(iii) The category of coherent modules is closed under kernels and cokernels

The first two are clear. For (iii), suppose that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring, Exercise 2.D). □

2.E. Easy Exercise (only important for non-Noetherian people). Show A is coherent as an A-module if and only if the notion of finitely presented agrees with the notion of coherent.

2.F. Exercise. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (The "coherent" case is the tricky one.)

2.G. Exercise. If $(f_1, \dots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

Definition. A quasicoherent sheaf $\mathcal{F}$ is **finite type** (resp. **coherent**) if for every affine open $\operatorname{Spec} A$, $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a finitely generated (resp. coherent) A-module.

Thanks to the affine communication lemma, and the two previous exercises 2.F and 2.G, it suffices to check this on the open sets in a single affine cover.

I want to say a few words on the notion of coherence. I see Proposition 2.3 as a good motivation for this definition: it gives a small (in a non-technical sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition (as opposed to working in a Noetherian world and always thinking that coherent equals finite type). Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this. Then Oka's theorem states that the structure sheaf is coherent, and this is very hard.

The second sort of people who should care are the sort of arithmetic people who sometimes are forced to consider non-Noetherian rings. For example, the ring of *adeles* is non-Noetherian.

Warning: it is common in the later literature to incorrectly define coherent as finitely generated. Please only use the correct definition, as the wrong definition only causes confusion. I will try to be scrupulous about this. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

3. COHERENT MODULES OVER NON-NOETHERIAN RINGS ⋆⋆

This section is intended for people who might work with non-Noetherian rings, or who otherwise might want to understand coherent sheaves in a more general setting. Read this only if you really want to!

Suppose A is a ring. Recall that an A-module M is *finitely generated* if there is a surjection $A^n \to M \to 0$. It is *finitely presented* if there is a presentation $A^m \to A^n \to M \to 0$. And M is *coherent* if (i) M is finitely generated, and (ii) every map $A^n \to M$ has a finitely generated kernel. The reason we like this third definition is that coherent modules form an abelian category.

Here are some quite accessible exercises working out why these notions behave well. Some repeat earlier discussion in order to keep this section self-contained.

3.A. EXERCISE. Show that coherent implies finitely presented implies finitely generated. (This was discussed in the previous section.)

3.B. EXERCISE. Show that 0 is coherent.

Suppose for problems 3.C–3.I that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules. In thise series of problems, we will show that if two of (1) are coherent, the third is as well, whiich will prove very useful.

Hint $\star$. Here is a *hint* which applies to several of the problems: try to write

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \longrightarrow & A^{p+q} & \longrightarrow & A^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

and possibly use the Snake Lemma.

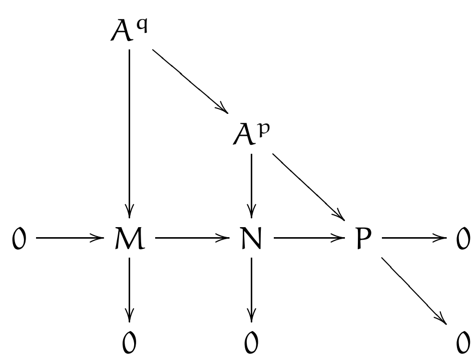
3.C. EXERCISE. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

3.D. EXERCISE. Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

3.E. EXERCISE. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

3.F. EXERCISE. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

3.G. EXERCISE. Show that N, P coherent implies M coherent. Hint for (i):



(You will only need left-exactness of (1).)

3.H. EXERCISE. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): $\star$.)

3.I. EXERCISE. Show that M, P coherent implies N coherent. (Hint: $\star$.)

3.J. EXERCISE. Show that a finite direct sum of coherent modules is coherent.

3.K. EXERCISE. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

3.L. EXERCISE. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

3.M. EXERCISE. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

3.N. EXERCISE. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

3.O. EXERCISE. If M is finitely presented and N is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

3.P. EXERCISE. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

3.Q. EXERCISE. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (Hint: localization is exact.) (This exercise appeared earlier as Exercise 2.F.)

3.R. EXERCISE. Suppose $(f_1, \ldots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

3.S. EXERCISE. Suppose $(f_1, \ldots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

3.T. EXERCISE. Show that the ring $A := k[x_1, x_2, \ldots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \ldots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

4. PLEASANT PROPERTIES OF FINITE TYPE AND COHERENT SHEAVES

4.A. EXERCISE. Suppose $\mathcal{F}$ is a coherent sheaf on X, and $\mathcal{G}$ is a quasicoherent sheaf on X. Show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$. (Hint: Describe it on affine open sets, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M, N)_f \cong$

$\mathrm{Hom}_{A_f}(M_f, N_f)$, take a presentation $A^q \to A^p \to M \to 0$, and apply $\mathrm{Hom}(\cdot, N)$ and localize. You will use the fact that p and q are finite.) If further $\mathcal{G}$ is coherent, show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is also coherent. Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

Recall that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{F}$, and is denoted $\mathcal{F}^\vee$.

4.B. USEFUL EXERCISE: GEOMETRIC NAKAYAMA. Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $x \in U \subset X$ is a neighborhood of x in X and $a_1, \dots, a_n \in \mathcal{F}(U)$ so that the images $\overline{a}_1, \dots, \overline{a}_n \in \mathcal{F}_x$ generate $\mathcal{F} \otimes k(x)$, then there is a neighborhood $x \subset V \subset U$ of x so that $a_1|_V, \dots, a_n|_V$ generate $\mathcal{F}|_V$. In particular, if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$.

4.C. LESS IMPORTANT EXERCISE. Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible": these sheaves are the invertible elements of the "monoid of finite type sheaves".

4.1. The support of a finite type sheaf is closed. Recall the definition of *support* of a section of a sheaf, and of a sheaf.

Suppose $\mathcal{F}$ is a sheaf of abelian groups (resp. $\mathcal{O}_X$-module) on a topological space X (resp. ringed space $(X, \mathcal{O}_X)$). Define the **support** of a section s of $\mathcal{F}$ to be

$$\mathrm{Supp}\, s = \{p \in X : s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

I think of this as saying where s "lives". Define the **support** of $\mathcal{F}$ as

$$\mathrm{Supp}\, \mathcal{F} = \{p \in X : \mathcal{F}_p \neq 0\}.$$

It is the union of "all the supports of sections on various open sets". I think of this as saying where $\mathcal{F}$ "lives". *Caution.* This is where the *germ(s)* are nonzero, not where the *value(s)* are nonzero.

Support is a stalk-local notion, and hence behaves well with respect to restriction to open sets, or to stalks.

4.D. EXERCISE. The support of a finite type quasicoherent sheaf on a scheme X is a closed subset. (Hint: Reduce to the case X affine. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$. Warning: this example won't work if $\oplus$ is replaced by $\prod$, so be careful!)

4.2. Rank of a finite type sheaf at a point.

The **rank** $\mathcal{F}$ of a finite type sheaf at a point p is $\dim_k \mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ where $\mathfrak{m}$ is the maximal ideal corresponding to p. More explicitly, on any affine set $\mathrm{Spec}\, A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\mathrm{Spec}\, A) = M$, then the rank is $\dim_{FF(A/\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. The rank is finite because of the finite type hypothesis. By Nakayama's lemma (again using the finite type condition), this is the minimal number of generators of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

If $\mathcal{F}$ is quasicoherent (not necessarily finite type), then $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ can be interpreted as the fiber of the sheaf at the point. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$.

4.E. EXERCISE. Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.)

4.F. EXERCISE. Show that $\operatorname{rank}(\mathcal{F})$ is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

Note that this definition of rank is consistent with the notion of rank of a locally free sheaf. In the locally free case, the rank is a (locally) constant function of the point. The converse is sometimes true, as is shown in Exercise 4.G below.

4.G. IMPORTANT HARD EXERCISE. (a) If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 4.D. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus (as rank is uppersemicontinuous, Exercise 4.F) coherent sheaves are locally free on a dense open set. Hint: Reduce to the case where X is affine, say $\operatorname{Spec} A$, so the closed points are dense. Then show it in a neighborhood of a closed point $[\mathfrak{m}]$. Choose $m_1, \dots, m_n$ generators of $M/\mathfrak{m}M$, and lift them to elements of M. Then they generate M, by Nakayama's Lemma. Let $\phi : A^n \to M$ with $(r_1, \dots, r_n) \mapsto \sum r_i m_i$. Let K be the cokernel, which is finitely generated. Then $K_\mathfrak{m} = 0$ (because $\otimes A_\mathfrak{m}$ is right-exact), so there is an $f \in A$ such that $K_f = 0$ (take the product of the annihilators of a finite generating set of K). Replace A by A_f. We now have that $\operatorname{coker} \phi = 0$, and we want to prove $\ker \phi = 0$. Otherwise, say $(r_1, \dots, r_n)$ is in the kernel, with $r_1 \neq 0$. As $r_1 \neq 0$, there is some $\mathfrak{p}$ where $r_1 \notin \mathfrak{p}$ — here we use the reduced hypothesis. Then r_1 is invertible in $A_\mathfrak{p}$, so $M_\mathfrak{p}$ has fewer than n generators, contradicting the constancy of rank.
(b) Show that part (a) can be false without the condition of X being reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type sheaves, or even quasicoherent sheaves. I drew some pictures in class, but I haven't figured out yet how to latex them up.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 27

RAVI VAKIL

CONTENTS

1. QUASICOHERENT SHEAVES OF IDEALS, AND CLOSED SUBSCHEMES

The point of this section is that closed subschemes correspond precisely to quasicoherent sheaves of ideals.

Recall that if $i : X \hookrightarrow Y$ is a closed immersion, then we have a surjection of sheaves on Y: $\mathcal{O}_Y \twoheadrightarrow i_*\mathcal{O}_X$. (The i_* is often omitted, as we are considering the sheaf on X as being a sheaf on Y.) The sheaf $i_*\mathcal{O}_X$ is quasicoherent on Y; this is in some sense the definition of "closed subscheme". The kernel $\mathcal{I}_{X/Y}$ is a "sheaf of ideals" in Y: for each open subset of Y, the sections form an ideal in the ring of functions of Y. As quasicoherent sheaves on Y form an abelian category, $\mathcal{I}_{X/Y}$ is a *quasicoherent sheaf of ideals.*

Conversely, a quasicoherent sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$ defines a closed subscheme. This was stated in slightly different language in Exercise 1. Precisely, $\mathcal{I}$ is quasicoherent precisely if, for each distinguished open $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, $\mathcal{I}(\operatorname{Spec} A_f) = \mathcal{I}(\operatorname{Spec} A)_f$ (Definition B of quasicoherent sheaves), and this was one criterion for when ideals in affine open sets define a closed subscheme (Exercise 1). (An example of a non-quasicoherent sheaf of ideals was given in an earlier Exercise.)

We call

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0 \tag{1}$$

the **closed subscheme exact sequence** corresponding to $X \hookrightarrow Y$.

Date: Monday, January 28, 2008. Mild correction Feb. 19 (thanks Nathan!).

2. INVERTIBLE SHEAVES (LINE BUNDLES) AND DIVISORS

We next develop some mechanism of understanding invertible sheaves (line bundles) on a given scheme X. Recall that $\operatorname{Pic} X$ is the group of invertible sheaves on X. Our goal will be to develop convenient and powerful ways of describing and working with invertible sheaves.

We begin by describing invertible sheaves on projective space (over a field). We then discuss sheaves of ideals that happen to be invertible (effective Cartier divisors). Partially motivated by this insight that invertible sheaves correspond to "codimension 1 information", we will discuss the theory of Weil divisors, and use this to actually compute $\operatorname{Pic} X$ in a number of circumstances.

3. SOME LINE BUNDLES ON PROJECTIVE SPACE

We now describe a family of invertible sheaves on projective space over a field k.

As a warm-up, we begin with the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(1)$ on $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. (The subscript $\mathbb{P}^1_k$ refers to the space on which the sheaf lives, and is often omitted when it is clear from the context.) We describe the invertible sheaf $\mathcal{O}(1)$ using transition functions. It is trivial on the usual affine open sets $U_0 = D(x_0) = \operatorname{Spec} k[x_{1/0}]$ and $U_1 = D(x_1 = \operatorname{Spec} k[x_{0/1}]$. (We continue to use the convention $x_{i/j}$ for describing coordinates on patches of projective space.) Thus the data of a section over U_0 is a polynomial in $x_{1/0}$. The transition function from U_0 to U_1 is multiplication by $x_{0/1} = x_{1/0}^{-1}$. The transition function from U_1 to U_0 is hence multiplication by $x_{1/0} = x_{0/1}^{-1}$.

This information is summarized below:

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$	$U_1 = \operatorname{Spec} k[x_{0/1}]$
trivialization and transition functions	$k[x_{1/0}] \underset{\times x_{1/0} = x_{0/1}^{-1}}{\overset{\times x_{0/1} = x_{1/0}^{-1}}{\rightleftarrows}} k[x_{0/1}]$	

To test our understanding, let's compute the global sections of $\mathcal{O}(1)$. This will be analogous to our hands-on calculation that $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k$. A global section is a polynomial $f(x_{1/0}) \in k[x_{1/0}]$ and a polynomial $g(x_{0/1}) in k[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1} = g(x_{0/1})$. A little thought will show that f must be linear: $f(x_{1/0}) = a x_{1/0} + b$, and hence $f(x_{0/1}) = a + b x_{0/1}$. Thus

$$\dim \Gamma(\mathbb{P}^1_k, \mathcal{O}(1)) = 2 \neq 1 = \dim \Gamma(\mathbb{P}^1_k, \mathcal{O}).$$

Thus $\mathcal{O}(1)$ is not isomorphic to $\mathcal{O}$, and we have constructed our first (proved) example of a nontrivial line bundle!

We next define more generally $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$. It is defined in the same way, except that the transition functions are the nth powers of those for $\mathcal{O}(1)$.

open cover	$U_0 = \operatorname{Spec} k[x_{1/0}]$ $\quad$ $U_1 = \operatorname{Spec} k[x_{0/1}]$
trivialization and transition functions	$k[x_{1/0}] \underset{\times x^n_{1/0} = x^{-n}_{0/1}}{\overset{\times x^n_{0/1} = x^{-n}_{1/0}}{\rightleftarrows}} k[x_{0/1}]$

In particular, thanks to the explicit transition functions, we see that $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ (with the obvious meaning if n is negative: $(\mathcal{O}(1)^{\otimes(-n)})^\vee$). Clearly also $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m + n)$.

3.A. IMPORTANT EXERCISE. Show that $\Gamma(\mathbb{P}^1, \mathcal{O}(n)) = n + 1$ if $n \geq 0$, and 0 otherwise.

Long ago, we warned that sheafification was necessary when tensoring $\mathcal{O}_X$-modules: if $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules on a ringed space, then it is not necessarily true that $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{G}(X) \cong (\mathcal{F} \otimes \mathcal{G})(X)$. We now have an example: let $X = \mathbb{P}^1_k$, $\mathcal{F} = \mathcal{O}(1)$, $\mathcal{G} = \mathcal{O}(-1)$.

3.B. EXERCISE. Show that if $m \neq n$, then $\mathcal{O}(m) \not\cong \mathcal{O}(n)$. Hence conclude that we have an injection of groups $\mathbb{Z} \hookrightarrow \operatorname{Pic} \mathbb{P}^1_k$ given by $n \mapsto \mathcal{O}(n)$.

It is useful to identify the global sections of $\mathcal{O}(n)$ with the homogeneous polynomials of degree n in x_0 and x_1, i.e. with the degree n part of $k[x_0, x_1]$. Can you see this from your solution to Exercise 3.A? We will see that this identification is natural in many ways. For example, we will later see that the definition of $\mathcal{O}(n)$ doesn't depend on a choice of affine cover, and this polynomial description is also independent of cover. As an immediate check of the usefulness of this point of view, ask yourself: where does the section $x_0^3 - x_0 x_1^2$ of $\mathcal{O}(3)$ vanish? The section $x_0 + x_1$ of $\mathcal{O}(1)$ can be multiplied by the section x_0^2 of $\mathcal{O}(2)$ to get a section of $\mathcal{O}(3)$. Which one? Where does the rational section $x_0^4(x_1 + x_0)/x_1^7$ of $\mathcal{O}(-2)$ have zeros and poles, and to what order? (We will rigorously define the meaning of zeros and poles shortly, but you should already be able to intuitively answer these questions.)

We now define the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(n)$ on the projective space $\mathbb{P}^m_k$. On the usual affine open set $U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) = \operatorname{Spec} A_i$, it is trivial, so sections (as an A_i-module) are isomorphic to A_i. The transition function from U_i to U_j is multiplication by $x^n_{i/j} = x^{-n}_{j/i}$. Note that these transition functions clearly satisfy the cocycle condition.

$$U_i = \operatorname{Spec} k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) \qquad U_j = \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1)$$

$$k[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1) \underset{\times x^n_{j/i} = x^{-n}_{i/j}}{\overset{\times x^n_{i/j} = x^{-n}_{j/i}}{\rightleftarrows}} \operatorname{Spec} k[x_{0/j}, \dots, x_{m/j}]/(x_{j/j} - 1)$$

3.C. ESSENTIAL EXERCISE. Show that $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n)) = \binom{m+n}{n}$.

You will notice that, as in the $\mathbb{P}^1$ case, sections of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ are naturally identified with forms degree m polynomials in our $n+1$ variables. Thus $x+y+2z$ is a section of $\mathcal{O}(1)$ on $\mathbb{P}^2$. It isn't a function, but I can say where this section vanishes — precisely where $x+y+2z=0$.

Also, notice that for fixed n, $\binom{m+n}{n}$ is a polynomial in m of degree n *for* $m \geq 0$ (or better: for $m \geq -n-1$). This should be telling you that this function "wants to be a polynomial" but has not succeeded. We will later define $h^0(\mathbb{P}^n_k, \mathcal{O}(m)) := \Gamma(\mathbb{P}^n_k, \mathcal{O}(m))$, and later still we will define higher cohomology groups, and we will define the *Euler characteristic* $\chi(\mathbb{P}^n_k, \mathcal{O}(m)) := \sum_{i=0}^{\infty} (-1)^i h^i(\mathbb{P}^n_k, \mathcal{O}(m))$ (cohomology will vanish in degree higher than n). We will discover the moral that the Euler characteristic is better-behaved than h^0, and so we should now suspect (and later prove) that this polynomial is in fact the Euler characteristic, and the reason that it agrees with h^0 for $m \geq 0$ because all the other cohomology groups should vanish.

We finally note that we can define $\mathcal{O}(n)$ on $\mathbb{P}^m_A$ for any ring A: the above definition applies without change.

4. EFFECTIVE CARTIER DIVISORS "=" INVERTIBLE IDEAL SHEAVES

In the previous section, we produced a number of interesting invertible sheaves on projective space by explicitly giving transition functions. We now give a completely different means of describing invertible sheaves on a scheme.

Suppose $D \hookrightarrow X$ is a closed subscheme such that corresponding ideal sheaf $\mathcal{I}$ is an invertible sheaf. Then D is called an *effective Cartier divisor*. Suppose D is an effective Cartier divisor. Then $\mathcal{I}$ is locally trivial; suppose U is a trivializing affine open set $\operatorname{Spec} A$. Then the closed subscheme exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

corresponds to

$$0 \to I \to A \to A/I \to 0$$

with $I \cong A$ as an A-module. Thus I is generated by a single element, say a, and this exact sequence starts as

$$0 \longrightarrow A \xrightarrow{\times a} A$$

As multiplication by a is injective, a is not a zero-divisor. We conclude that D is locally cut out by a single equation, that is not a zero-divisor. This was the definition of effective Cartier divisor given before. This argument is clearly reversible, so we now have a quick new definition of effective Cartier divisor (that $\mathcal{I}$ is invertible).

4.A. EASY EXERCISE. Show that a is unique up to multiplication by a unit.

In the case where X is locally Noetherian, and we can use the language of associated points, we can restate this definition as: D is locally cut out by a single equation, not vanishing at any associated point of X.

We now define an invertible sheaf corresponding to D. The seemingly obvious definition would be to take $\mathcal{I}_D$, but instead we define the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor to be the *dual*: $\mathcal{I}_D^\vee$. The ideal sheaf itself is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section s_D: Tensoring $0 \to \mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$ gives us $\mathcal{O} \to \mathcal{I}^\vee$. (Easy unimportant fact to check: instead of tensoring $\mathcal{I} \to \mathcal{O}$ with $\mathcal{I}^\vee$, we could have dualized $\mathcal{I} \to \mathcal{O}$, and we would get the same section.)

4.B. SURPRISINGLY TRICKY EXERCISE. Recall that a section of a locally free sheaf on X cuts out a closed subscheme of X. Show that this section s_D cuts out D.

This construction has a converse.

4.C. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero divisor (make sense of this!). Show that $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$. (Again, if X is locally Noetherian, "not locally a zero divisor" translate to "does not vanish at an associated point".)

4.D. EXERCISE. Suppose $\mathcal{I}$ and $\mathcal{J}$ are invertible ideal sheaves (hence corresponding to effective Cartier divisors, say D and D' respectively). Show that $\mathcal{I}\mathcal{J}$ is an invertible ideal sheaf. (First make sense of this notation!) We define the corresponding Cartier divisor to be $D + D'$. Verify that $\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$.

Thus the effective Cartier divisors form a semigroup. Thus we have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zero-divisors (and hence also to the Picard group of invertible sheaves).

Hence we can get a bunch of invertible sheaves, by taking differences of these two. In fact we "usually get them all"! It is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way. For example, we will see soon that there are none if the scheme is nonsingular or even factorial. We will see later that there are none if X is quasiprojective. over a field.

We thus have an important correspondence between *effective Cartier divisors* (closed subschemes whose ideal sheaves are invertible, or equivalently locally cut out by one non-zero-divisor, or in the locally Noetherian case locally cut out by one equation not vanishing at an associated point) and ordered pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero-divisor (or in the locally Noetherian case, not vanishing at an associated point). This is an isomorphism of semigroups.

An effective Cartier divisor is pure codimension 1 by Krull's Principal Ideal Theorem. This correspondence of "invertible sheaf with section" with "codimension one information" is a powerful theme that we will explore further in the next section.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 28 AND 29

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CONTENTS

1. INVERTIBLE SHEAVES AND WEIL DIVISORS

In the previous section, we saw a link between line bundles and codimension 1 information. We now continue this theme. The notion of Weil divisors will give a great way of understanding and classifying line bundles, at least on Noetherian normal schemes. Some of what we discuss will apply in more general circumstances, and the expert is invited to consider generalizations.

For the rest of this section, we consider only *Noetherian schemes*. We do this because we want to discuss codimension 1 subsets, and also have decomposition into irreducibles components.

Define a **Weil divisor** as a formal sum of codimension 1 irreducible closed subsets of X. In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ codimension } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

For example, if X is a curve (such as the Spec of a Dedekind domain), the Weil divisors are linear combination of points.

We say that $[Y]$ is an **irreducible** (Weil) divisor. A Weil divisor is said to be **effective** if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The **support** of a Weil divisor D is the subset $\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\operatorname{Weil} X \to \operatorname{Weil} U$, where $\sum n_Y [Y] \mapsto \sum_{Y \cap U \neq \varnothing} n_Y [Y \cap U]$.

Suppose now that X is *regular in codimension* 1 (and Noetherian). We add this hypothesis because we will use properties of discrete valuation rings. Suppose that $\mathcal{L}$ is an invertible

Date: Wednesday, January 30 and Friday, February 1, 2008.

sheaf, and s a rational section not vanishing on any irreducible component of X. (Rational sections are given by a section over a dense open subset of X, with the obvious equivalence.) Then s determines a Weil divisor

$$\operatorname{div}(s) := \sum_Y \operatorname{val}_Y(s)[Y]$$

called the **divisor of zeros and poles**. To determine the valuation $\operatorname{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a function on U, which thus has a valuation. Any two such trivializations differ by a unit, so this valuation is well-defined. ($\operatorname{val}_Y(s) = 0$ for all but finitely many Y, by an earlier exercise.) This map gives a group homomorphism

$$\operatorname{div} : \{(\mathcal{L}, s)\} \to \operatorname{Weil} X.$$

A unit has no poles or zeros, so this descends to a group homomorphism

(1) $$\operatorname{div} : \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^* \to \operatorname{Weil} X.$$

1.A. EXERCISE. (a) *(divisors of rational functions)* Verify that on $\mathbb{A}^1_k$, $\operatorname{div}(x^3/(x+1)) = 3[(x)] - [(x+1)] = 3[0] - [-1]$.
(b) *(divisor of a rational sections of a nontrivial invertible sheaf)* On $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" $x^2/(x+y)$. Figure out what this means, and calculate $\operatorname{div}(x^2/(x+y))$.

We want to classify all invertible sheaves on X, and this homomorphism (1) will be the key. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often even an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group of all line bundles. Let's put this strategy into action.

1.1. *Proposition. — If X is normal and Noetherian then the map* div *is injective.*

Proof. Suppose $\operatorname{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' lemma for invertible sheaves, s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\mathcal{O}_X \to \mathcal{L}$ (given by $1 \mapsto s$). □

Motivated by this, we try to find the inverse map to div.

1.2. Important Definition. Suppose D is a Weil divisor. If $U \subset X$ is an open subscheme, recall that $\mathrm{FF}(U)$ is the field of total fractions of U, i.e. the product of the stalks at the minimal primes of U (in this case that X is normal). If U is irreducible, this is the function field. Define $\mathrm{FF}(U)^*$ to be those rational functions not vanishing at any generic point of

U, that is, not vanishing on any irreducible component of U. Define the sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{s \in FF(U)^* : \operatorname{div} s + D|_U \geq 0\}.$$

The subscript will often be omitted when it is clear from the context. Define a rational section s_D of $\mathcal{O}_X(D)$ corresponding to $1 \in FF(U)^*$.

It may seem more reasonable to consider those s such that $\operatorname{div} s \geq D|_U$. The reason for the convention we use is the following exercise.

1.B. IMPORTANT EXERCISE. Show that $\operatorname{div} s_D = D$.

We connect this to the important example of projective space that we have recently studied:

1.C. IMPORTANT EXERCISE. Let $D = \{x_0 = 0\}$ be a hyperplane divisor on $\mathbb{P}^n_k$. Show that $\mathcal{O}(nD) \cong \mathcal{O}(n)$. (For this reason, $\mathcal{O}(1)$ is sometimes called the **hyperplane class** in $\operatorname{Pic} X$.)

1.3. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf, and s is a rational section not vanishing on any irreducible component of X. Then there is an isomorphism $(\mathcal{L}, s) \cong (\mathcal{O}(\operatorname{div} s), t)$, where t is the canonical rational section described above.*

Proof. We first describe the isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Over open subscheme $U \subset X$, we have a bijection $\Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{O}(\operatorname{div} s))$ given by $s' \mapsto s'/s$, with inverse obviously given by $t' \mapsto st'$. Clearly under this bijection, s corresponds to the section 1 in $FF(U)^*$; this is the section we are calling t. □

We denote the subgroup of $\operatorname{Weil} X$ corresponding to divisors of rational *functions* the subgroup of **principal divisors**, which we denote $\operatorname{Prin} X$. Define the **class group** of X, $\operatorname{Cl} X$, by $\operatorname{Weil} X / \operatorname{Prin} X$. If X is normal, then by taking the quotient of the inclusion (1) by $\operatorname{Prin} X$, we have the inclusion $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$. This is summarized in the convenient diagram

$$\begin{array}{ccccc}
 & & \operatorname{div} : \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^* & \hookrightarrow & \operatorname{Weil} X \\
 & & \downarrow {\scriptstyle /\{(\mathcal{O}_X, s)\}} & & \downarrow {\scriptstyle /\operatorname{Prin} X} \\
\operatorname{Pic} X & = & \{\mathcal{L}\} & \hookrightarrow & \operatorname{Cl} X
\end{array}$$

This diagram is very important, and although it is short to say, it takes some time to internalize. (If X is Noetherian and regular in codimension 1 but not necessarily normal, then we have a similar diagram, except the horizontal maps are not necessarily inclusions.)

We can now compute of $\operatorname{Pic} X$ in a number of interesting cases!

1.D. EXERCISE. Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$. (One direction is easy: we have already shown that Unique Factorization Domains are integrally closed in their fraction fields. Also, an earlier exercise showed that all codimension 1 primes of a Unique Factorization Domain are principal, so that implies that $\mathrm{Cl}\operatorname{Spec} A = 0$. It remains to show that if A is integrally closed and $\mathrm{Cl}\operatorname{Spec} A = 0$, then all codimension 1 prime ideals are principal, as this characterizes Unique Factorization Domains. Hartogs' lemma may arise in your argument.) This is the third important characterization of unique factorization domains promised long ago.

Hence $\mathrm{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\mathrm{Pic}(\mathbb{A}^n_k) = 0}$. Geometers will find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles".

Removing subset of X of codimension greater 1 doesn't change the Class group, as it doesn't change the Weil divisor group or the principal divisors.

Removing a subset of codimension 1 changes the Weil divisor group in a controllable way. For example, suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Weil}\, X \longrightarrow \mathrm{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we lose exactness on the left, and get:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}(X - Z) \longrightarrow 0. \tag{2}$$

1.E. EASY EXERCISE. Suppose $X \hookrightarrow \mathbb{A}^n$ is an open subset. Show that $\mathrm{Pic}\, X = \{0\}$.

For example, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \to \mathrm{Cl}\, \mathbb{P}^n_k \to \mathrm{Cl}\, \mathbb{A}^n_k \to 0$$

from which $\mathrm{Cl}\, \mathbb{P}^n_k = \mathbb{Z}[Z]$ (which is $\mathbb{Z}$ or 0), and $\mathrm{Pic}\, \mathbb{P}^n_k$ is a subgroup of this.

By Exercise 1.C, $[Z] \to \mathcal{O}(1)$. Hence $\mathrm{Pic}\, \mathbb{P}^n_k \hookrightarrow \mathrm{Cl}\, \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\mathrm{Pic}\, \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The **degree** of an invertible sheaf on $\mathbb{P}^n$ is defined using this: $\deg \mathcal{O}(d) := d$.

More generally:

1.4. *Proposition. — If X is Noetherian and factorial (all stalks are unique factorization domains) then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\mathrm{Pic}\, X \to \mathrm{Cl}\, X$ is an isomorphism.*

Proof. It will suffice to show that $[Y]$ is effective Cartier if Y is any irreducible divisor. Our goal is to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$, where we take the function 1. Next, we find an open set U containing an arbitrary $x \in Y$, and a function on U. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is codimension 1 and hence principal (by an earlier Exercise). Let $f \in FF(A)$ be a generator. Then f is regular at x. f has a finite number of zeros and poles, and through x there is only one 0, notably $[Y]$. Let U be X minus all the others zeros and poles. □

I will now mention a bunch of other examples of class groups and Picard groups you can calculate.

First, notice that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. For example, if X is something crazy, and $Y \cong \mathbb{P}^1$, then we're happy, because we understand invertible sheaves on $\mathbb{P}^1$. Effective Cartier divisors sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate and this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

1.5. Fun with hypersurface complements.

1.F. EXERCISE: A TORSION PICARD GROUP. Show that Y is an irreducible degree d hypersurface of $\mathbb{P}^n$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$.)

As a very explicit example, we can consider the plane minus a conic ($n = d = 2$).

The next two exercises explore its consequences, and provide us with some examples we have been waiting for.

1.G. EXERCISE. Keeping the same notation, assume $d > 1$ (so $\operatorname{Pic}(\mathbb{P}^n - Y) \neq 0$), and let $H_0, \dots, H_n$ be the $n+1$ coordinate hyperplanes on $\mathbb{P}^n$. Show that $\mathbb{P}^n - Y$ is affine, and $\mathbb{P}^n - Y - H_i$ is a distinguished open subset of it. Show that the $\mathbb{P}^n - Y - H_i$ form an open cover of $\mathbb{P}^n - Y$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y - H_i) = 0$. Then by Exercise 1.D, each $\mathbb{P}^n - Y - H_i$ is the Spec of a unique factorization domain, but $\mathbb{P}^n - Y$ is not. Thus the property of being a unique factorization domain is not an affine-local property — it satisfies only one of the two hypotheses of the affine communication lemma.

1.H. EXERCISE. Keeping the same notation as the previous exercise, show that on $\mathbb{P}^n - Y$, H_i (restricted to this open set) is an effective Cartier divisor that is not cut out by a single equation. (Hint: Otherwise it would give a trivial element of the class group.)

1.6. Quadric surfaces.

1.I. EXERCISE. Let $X = \operatorname{Proj} k[w,x,y,z]/(wz - xy)$, a smooth quadric surface (Figure 1). Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if L and M are two lines in different rulings (e.g. $L = V(w,x)$ and $M = V(w,y)$), then $X - L - M \cong \mathbb{A}^2$. This will give you a surjection

$\mathbb{Z}\oplus\mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This is a bit longer to do, but enlightening.)

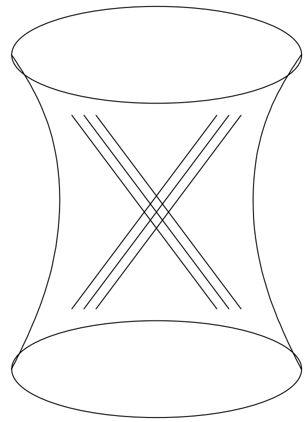

FIGURE 1. Finding all line bundles on the quadric surface

1.J. EXERCISE. Let $X = \operatorname{Spec} k[w,x,y,z]/(xy-z^2)$, a cone. show that $\operatorname{Pic} X = 1$, and $\operatorname{Cl} X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\operatorname{Cl} X$ by showing that its complement is isomorphic to $\mathbb{A}^2_k$. Show that $2[Z] = \operatorname{div}(x)$ (and hence principal), and that Z is not principal (an example we did when learning how to use the Zariski tangent space).

1.7. Nagata's Lemma $\star\star$.

I mentioned earlier that I only know a few ways of checking that a ring is a unique factorization domain. Nagata's Lemma is the last, and least useful.

1.K. EXERCISE. Prove Nagata's Lemma: Suppose A is a Noetherian domain, $x \in A$ an element such that (x) is prime and $A[1/x]$ is a unique factorization domain. Then A is a unique factorization domain. (Hint: Exercise 1.D. Use the short exact sequence $[(x)] \to \operatorname{Cl} \operatorname{Spec} A \to \operatorname{Cl} A[1/x] \to 0$ (2) to show that $\operatorname{Cl} \operatorname{Spec} A = 0$. Show that $A[1/x]$ is integrally closed, then show that A is integrally closed as follows. Suppose $T^n + a_{n-1}T^{n-1} + \cdots + a_0 = 0$, where $a_i \in A$, and $T \in FF(A)$. Then by integral closure of $A[1/x]$, we have that $T = r/x^m$, where if $m > 0$, then $r \notin x$. Then we quickly get a contradiction if $m > 0$.)

This leads to a remarkable algebra fact. Suppose k is an algebraically closed field of characteristic not 2. Let $A = k[x_1, \dots, x_n]/(x_1^2 + \cdots + x_m^2)$ where $m \leq n$. When $m \leq 2$, we get some special behavior. (If $m = 0$, we get affine space; if $m = 1$, we get a non-reduced scheme; if $m = 2$, we get a reducible scheme that is the union of two affine spaces.) If $m \geq 3$, we have verified that $\operatorname{Spec} A$ is normal, in an earlier exercise.

In fact, if $m \geq 3$, then A is a unique factorization domain *unless* $m = 4$. The failure at 4 comes from the geometry of the quadric surface: we have checked that in $\operatorname{Spec} k[w,x,y,z]/(wx-$

yz), there is a codimension 1 prime ideal — the cone over a line in a ruling — that is not principal.

We already understand success at 3: $A = k[x, y, z, w_1, \dots, w_{n-3}]/(x^2 + y^2 - z^2)$ is a unique factorization domain, as it is normal and has class group 0 (as verified above).

1.L. EXERCISE (THE CASE $m \geq 5$). Suppose that k is algebraically closed of characteristic not 2. Show that if $m \geq 3$, then $A = k[a, b, x_1, \dots, x_n]/(ab - x_1^2 - \cdots - x_m^2)$ is a unique factorization domain, by using the Nagata's Lemma with $x = a$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 30

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CONTENTS

Today, we will discuss the relationship between quasicoherent sheaves on projective A-schemes and graded modules.

1. THE QUASICOHERENT SHEAF CORRESPONDING TO A GRADED MODULE

We now describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from the data of $\mathbb{Z}^{\geq 0}$-graded ring $S_\bullet$, with $S_0 = A$, and S_+ finitely generated as an A-module. The resulting scheme is denoted $\operatorname{Proj} S_\bullet$.

Let $X = \operatorname{Proj} S_\bullet$. Suppose $M_\bullet$ is a graded $S_\bullet$ module, *graded by* $\mathbb{Z}$. (While reading the next section, you may wonder why we don't grade by $\mathbb{Z}^+$. You'll see that it doesn't really matter either way. The reason to prefer a $\mathbb{Z}$-grading is when we produce an $M_\bullet$ from a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$.) We define the quasicoherent sheaf $\widetilde{M_\bullet}$ as follows. For each f of positive degree, we define a quasicoherent sheaf $\widetilde{M_\bullet(f)}$ on the distinguished open $D(f) = \{p : f(p) \neq 0\}$ by

$$\widetilde{M_\bullet(f)} := \widetilde{(M_f)_0}.$$

The subscript 0 here means "the 0-graded piece". We have obvious isomorphisms of the restriction of $\widetilde{M_\bullet(f)}$ and $\widetilde{M_\bullet(g)}$ to $D(fg)$, satisfying the cocycle conditions. (Think through this yourself, to be sure you agree with the word "obvious"!) Then by an earlier problem set problem telling how to glue sheaves, these sheaves glue together to a single sheaf on $\widetilde{M_\bullet}$ on X. We then discard the temporary notation $\widetilde{M_\bullet(f)}$.

This is clearly quasicoherent, because it is quasicoherent on each $D(f)$, and quasicoherence is local.

Date: Monday, February 4, 2008.

1.A. EXERCISE. Show that the stalk of $\widetilde{M_\bullet}$ at a point corresponding to homogeneous prime $\mathfrak{p} \subset S_\bullet$ is isomorphic to the 0th graded piece of $(M_\bullet)_\mathfrak{p}$.

1.B. UNIMPORTANT EXERCISE. Use the previous exercise to give an alternate definition of $\widetilde{M_\bullet}$ in terms of "compatible stalks".

Given a map of graded modules $\phi : M_\bullet \to N_\bullet$, we we get an induced map of sheaves $\widetilde{M_\bullet} \to \widetilde{N_\bullet}$. Explicitly, over $D(f)$, the map $M_\bullet \to N_\bullet$ induces $M_\bullet[1/f] \to N_\bullet[1/f]$, which induces $\phi_f : (M_\bullet[1/f])_0 \to (N_\bullet[1/f])_0$; and this behaves well with respect to restriction to smaller distinguished open sets, i.e. the following diagram commutes.

$$\begin{array}{ccc} (M_\bullet[1/f])_0 & \xrightarrow{\phi_f} & (N_\bullet[1/f])_0 \\ \downarrow & & \downarrow \\ (M_\bullet[1/(fg)])_0 & \xrightarrow{\phi_{fg}} & (N_\bullet[1/(fg)])_0. \end{array}$$

Thus $\sim$ is a functor from the category of graded $S_\bullet$-modules to the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$. We shall see that this isn't quite an isomorphism, but it is close. The relationship is akin to that between presheaves and sheaves, and the sheafification functor.

1.C. EASY EXERCISE. Show that $\sim$ is an exact functor.

1.D. EXERCISE. Show that if $M_\bullet$ and $M'_\bullet$ agree in high enough degrees, then $\widetilde{M_\bullet} \cong \widetilde{M'_\bullet}$. Thus the map from graded $S_\bullet$-modules to quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ is not a bijection.

1.E. EXERCISE. Describe a map of S_0-modules $M_0 \to \Gamma(\widetilde{M_\bullet}, X)$. (This foreshadows the "saturation map" that takes a graded module to its saturation.)

1.F. EXERCISE. Show that $\widetilde{M_\bullet} \otimes \widetilde{N_\bullet} \cong \widetilde{M_\bullet \otimes_{S_\bullet} N_\bullet}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished open sets.)

1.1. Graded ideals of $S_\bullet$ give closed subschemes of $\operatorname{Proj} S_\bullet$. Recall that a graded ideal $I_\bullet \subset S_\bullet$ yields a closed subscheme. $\operatorname{Proj} S_\bullet / I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$.

For example, suppose $S_\bullet = k[w,x,y,z]$, so $\operatorname{Proj} S_\bullet \cong \mathbb{P}^3$. The ideal $I_\bullet = (wz - xy, x^2 - wy, y^2 - xz)$ yields our old friend, the twisted cubic.

1.G. EXERCISE. Show that if the functor $\sim$ is applied to the exact sequence of graded $S_\bullet$-modules

$$0 \to I_\bullet \to S_\bullet \to S_\bullet / I_\bullet \to 0$$

we obtain the closed subscheme exact sequence for $\operatorname{Proj} S_\bullet/I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$.

We will soon see (§4.E) that all closed subschemes of $\operatorname{Proj} S_\bullet$ arise in this way.

2. Invertible sheaves (line bundles) on projective A-schemes

Suppose that $S_\bullet$ is generated in degree 1. By an earlier exercise, this is not a huge assumption, as we can change the grading by some multiple to arrange that this is the case. Suppose $M_\bullet$ is a graded $S_\bullet$-module. Define the graded module $M(n)_\bullet$ so that $M(n)_m := M_{n+m}$. Thus the quasicoherent sheaf $\widetilde{M(n)_\bullet}$ satisfies

$$\Gamma(D(f), \widetilde{M(n)_\bullet}) = \widetilde{(M_f)_n}$$

where here the subscript means we take the nth graded piece. (These subscripts are admittedly confusing!)

2.A. Important exercise. If $S_\bullet$ is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_\bullet}(n)$ is an invertible sheaf.

2.B. Exercise. If $S_\bullet = k[x_0, \dots, x_m]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^m_k$, show that this definition of $\mathcal{O}(n)$ agrees with our earlier definition involving transition functions.

If $\mathcal{F}$ is a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, define $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n)$. This is often called *twisting $\mathcal{F}$ by $\mathcal{O}(n)$*. More generally, if $\mathcal{L}$ is an invertible sheaf, then $\mathcal{F} \otimes \mathcal{L}$ is often called "twisting $\mathcal{F}$ by $\mathcal{L}$".

2.C. Exercise. Show that $\widetilde{M_\bullet}(n) \cong \widetilde{M(n)_\bullet}$.

2.D. Exercise. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

2.1. *Unimportant remark.* Even if $S_\bullet$ is not generated in degree 1, then by Exercise , $S_{d\bullet}$ is generated in degree 1 for some d. In this case, we may define the invertible sheaves $\mathcal{O}(dn)$ for $n \in \mathbb{Z}$. This does *not* mean that we *can't* define $\mathcal{O}(1)$; this depends on $S_\bullet$. For example, if $S_\bullet$ is the polynomial ring $k[x, y]$ with the usual grading, except without linear terms, then $S_{2\bullet}$ and $S_{3\bullet}$ are both generated in degree 1, meaning that we may define $\mathcal{O}(2)$ and $\mathcal{O}(3)$. There is good reason to call their "difference" $\mathcal{O}(1)$.

3. Generation by global sections, and Serre's Theorem

3.1. Generated by global sections. Suppose X is a scheme, and $\mathcal{F}$ is a $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *generated by global sections at a point* p if we can find $\phi : \mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective at the stalk of p: $\phi_p : \mathcal{O}_p^{\oplus I} \to \mathcal{F}_p$ is surjective. (Some what more precisely, the stalk of $\mathcal{F}$ at p is generated by global sections of $\mathcal{F}$. The global sections in question are the images of the 1's in $|I|$ factors of $\mathcal{O}_p^{\oplus I}$.) We say that $\mathcal{F}$ is *generated by global sections* or *globally generated* if it is generated by global sections at all p, or equivalently, if we can we can find $\mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective. (By our earlier result that we can check surjectivity at stalks, so this is the same as saying that it is surjective at all stalks.) If I can be taken to be finite, we say that $\mathcal{F}$ is generated by a finite number of global sections. We'll see soon why we care.

3.A. EASY EXERCISE. If quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections at a point p, then so is $\mathcal{F} \otimes \mathcal{G}$. (This exercise is less important, but is good practice.)

3.B. EASY EXERCISE. If $\mathcal{F}$ is a finite type sheaf, show that $\mathcal{F}$ is generated by global sections at p if and only if "the fiber of $\mathcal{F}$ is generated by global sections at p", i.e. the map from global sections to the fiber $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$. (Hint: Geometric Nakayama.)

3.C. EASY EXERCISE. An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. We'll soon discuss classifying maps to projective space in terms of invertible sheaves generated by global sections, and we'll see then why we care about such notions.

3.D. EASY EXERCISE. If $\mathcal{F}$ is finite type, and X is quasicompact, show that $\mathcal{F}$ is generated by global sections if and only if it is generated by a *finite number* of global sections.

3.2. *Lemma. — Suppose $\mathcal{F}$ is a finite type sheaf on X. Then the set of points where $\mathcal{F}$ is generated by global sections is an open set.*

Proof. Suppose $\mathcal{F}$ is generated by global sections at a point p. Then it is generated by a finite number of global sections, say m. This gives a morphism $\phi : \mathcal{O}^{\oplus m} \to \mathcal{F}$, hence $\operatorname{im}\phi \hookrightarrow \mathcal{F}$. The support of the (finite type) cokernel sheaf is a closed subset not containing p. □

3.E. IMPORTANT EXERCISE (AN IMPORTANT THEOREM OF SERRE). Suppose S_0 is a Noetherian ring, and $S_\bullet$ is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_\bullet$. Show that for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

I'm going to sketch how you should tackle this exercise, after first telling you the reason we will care.

3.3. *Corollary. — Any coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S_\bullet$ can be presented as:*

$$\oplus_{\mathit{finite}}\mathcal{O}(-n) \to \mathcal{F} \to 0.$$

We're going to use this a lot! One clue of how this might be useful: we can use this to build a resolution of $\mathcal{F}$:

$$\cdots \to \oplus \mathcal{O}(-n_2) \to \oplus \mathcal{O}(-n_1) \to \mathcal{F} \to 0.$$

We understand the $\mathcal{O}(n)$'s pretty well, so we can use this to prove things about coherent sheaves (such as vector bundles) in general.

This Corollary is false for quasicoherent sheaves in general; consider $\oplus_{m \leq 0} \mathcal{O}(m)$.

Proof. Suppose we have m global sections $s_1, \ldots, s_m$ of $\mathcal{F}(n)$ that generate $\mathcal{F}(n)$. This gives a map

$$\oplus_m \mathcal{O} \longrightarrow \mathcal{F}(n)$$

given by $(f_1, \ldots, f_m) \mapsto f_1 s_1 + \cdots + f_m s_m$ on any open set. Because these global sections generate $\mathcal{F}$, this is a surjection. Tensoring with $\mathcal{O}(-n)$ (which is exact, as tensoring with any locally free is exact) gives the desired result. $\square$

Here is now a hint/sketch for the Serre exercise 3.E.

Suppose $\deg f = 1$. Say $\mathcal{F}|_{D(f)} = \tilde{M}$, where M is a $(S_\bullet[1/f])_0$-module, generated by m_1, $\ldots$, m_n. As these elements generate the module, they clearly generate all the stalks over all the points of $D(f)$. They are sections over this ("big") distinguished open set $D(f)$. It would be wonderful if we knew that they had to be restrictions of global sections, i.e. that there was a global section m_i' that restricted to m_i on $D(f)$. If that were always true, then we would cover X with a finite number of each of these $D(f)$'s, and for each of them, we would take the finite number of generators of the corresponding module. Sadly this is not true.

However, we will see that $f^N m$ "extends", where m is any of the m_i's, and N is sufficiently large. We will see this by (easily) checking first that $f^N m$ extends over another distinguished open $D(g)$ (i.e. that there is a section of $\mathcal{F}(N)$ over $D(g)$ that restricts to $f^N m$ on $D(g) \cap D(f) = D(fg)$).

So we're done, right? Wrong — we still don't that these extensions on various open sets glue together, and in fact they might not! More precisely: we don't know that the extension over $D(g)$ and over some other $D(g')$ agree on the overlap $D(g) \cap D(g') = D(gg')$. But after multiplying both extensions by $f^{N'}$ for large enough N', we will see that they agree on the overlap. By quasicompactness, we need to to extend over only a finite number of $D(g)$'s, and to make sure extensions agree over the finite number of pairs of such $D(g)$'s, so we will be done.

Let's now begin to make this precise. We first investigate what happens on $D(g) = \operatorname{Spec} A$, where the degree of g is also 1. Say $\mathcal{F}|_{D(g)} \cong \tilde{N}$. Let $f' = f/g$ be "the function corresponding to f on $D(g)$". We have a section over $D(f')$ on the affine scheme $D(g)$, i.e. an element of $N_{f'}$, i.e. something of the form $n/(f')^N$ for some $n \in N$. So then if we multiply it by f'^N, we can certainly extend it! So if we multiply by a big enough power of f, m certainly extends over any $D(g)$.

As described earlier, the only problem is, we can't guarantee that the extensions over $D(g)$ and $D(g')$ agree on the overlap (and hence glue to a single extensions). Let's check on the intersection $D(g) \cap D(g') = D(gg')$. Say $m = n/(f')^N = n'/(f')^{N'}$ where we can take $N = N'$ (by increasing N or N' if necessary). We certainly may not have $n = n'$, but by the (concrete) definition of localization, after multiplying with enough f''s, they become the same.

In conclusion: after multiplying with enough f's, our sections over $D(f)$ extend over each $D(g)$. After multiplying by even more, they will all agree on the overlaps of any two such distinguished affine. Exercise 3.E is to make this precise.

4. Every quasicoherent sheaf on a projective A-scheme arises from a graded module

We have gotten lots of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ from graded $S_\bullet$-modules. We'll now see that we can get them all in this way.

We want to figure out how to "undo" the $\sim$ construction. When you do the Exercise computing the space of global sections of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$, you will suspect that in good situations,

$$M_n \cong \Gamma(\operatorname{Proj} S_\bullet, \tilde{M}(n)).$$

Motivated by this, we define

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_\bullet, \mathcal{F}(n)).$$

Then $\Gamma_\bullet(\mathcal{F})$ is a graded $S_\bullet$-module, and we can dream that $\Gamma_\bullet(\mathcal{F})^\sim \cong \mathcal{F}$. We will see that this is indeed the case!

4.A. Exercise. Show that $\Gamma_\bullet$ gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ to the category of graded $S_\bullet$-modules. In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$, describe the natural map $\Gamma_\bullet(\mathcal{F}) \to \Gamma_\bullet(\mathcal{G})$, and show that such natural maps respect the identity and composition.

Note that $\sim$ and $\Gamma_\bullet$ cannot be inverses, as $\sim$ can turn two different graded modules into the same quasicoherent sheaf (see for example Exercise 1.D).

Our initial goal is to show that there is a natural isomorphism $\widetilde{\Gamma_\bullet(\mathcal{F})} \to \mathcal{F}$, and that there is a natural map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. The latter map is called the **saturation map**, although this language isn't important to us. We will show something better: that $\sim$ and $\Gamma_\bullet$ are adjoint.

We start by describing the saturation map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. We describe it in degree n. Given an element m_n, we seek an element of $\Gamma(\operatorname{Proj} S_\bullet, \widetilde{M_\bullet}(n)) = \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M_{(n+\bullet)}})$. By shifting the grading of $M_\bullet$ by n, we can assume $n = 0$. For each $D(f)$, we certainly have an element of $(M[1/f])_0$ (namely m), and they agree on overlaps, so the map is clear.

4.B. EXERCISE. Show that this canonical map need not be injective, nor need it be surjective. (Hint: $S_\bullet = k[x]$, $M_\bullet = k[x]/x^2$ or $M_\bullet = \{$ polynomials with no constant terms $\}$.)

The natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ is more subtle, but will have the advantage of being an isomorphism.

4.C. EXERCISE. Describe the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f}\mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'}m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

4.D. LONGER EXERCISE. Show that the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (Exercise 3.E). Then show that it is injective.

4.1. *Corollary. — Every quasicoherent sheaf arises from this tilde construction.*

4.E. EXERCISE. Show that each closed subscheme of $\operatorname{Proj} S_\bullet$ arises from a graded ideal $I_\bullet \subset S_\bullet$. (Hint: Suppose Z is a closed subscheme of $\operatorname{Proj} S_\bullet$. Consider the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\operatorname{Proj} S_\bullet} \to \mathcal{O}_Z \to 0$. Apply $\Gamma_\bullet$, and then $\sim$.)

4.F. EXERCISE ($\Gamma_\bullet$ AND $\sim$ ARE ADJOINT FUNCTORS, PART 1). Prove part of the statement that $\Gamma_\bullet$ and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_\bullet, \Gamma_\bullet(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_\bullet}, \mathcal{F})$. For the map from left to right, start with a morphism $M_\bullet \to \Gamma_\bullet(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_\bullet} \to \widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category".

4.G. EXERCISE ($\Gamma_\bullet$ AND $\sim$ ARE ADJOINT FUNCTORS, PART 2) $\star$. Show that $\Gamma_\bullet$ and $\sim$ are adjoint.

4.2. Saturated $S_\bullet$-modules. We end with a remark: different graded $S_\bullet$-modules give the same quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, but the results of this section show that there is a "best" (saturated) graded module for each quasicoherent sheaf, and there is a map from each graded module to its "best" version, $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. A module for which this is an isomorphism (a "best" module) is called *saturated*. We won't use this term later.

This "saturation" map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$ is analogous to the sheafification map, taking presheaves to sheaves. For example, the saturation of the saturation equals the saturation.

There is a bijection between saturated quasicoherent sheaves of ideals on $\operatorname{Proj} S_\bullet$ and closed subschemes of $\operatorname{Proj} S_\bullet$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 31

RAVI VAKIL

CONTENTS

1. PUSHFORWARDS AND PULLBACKS OF QUASICOHERENT SHEAVES

There are two things you can do with modules and a ring homomorphism $B \to A$. If M is an A-module, you can create an B-module M_B by simply treating it as an B-module. If N is an B-module, you can create an A-module $N \otimes_B A$.

These notions behave well with respect to localization (in a way that we will soon make precise), and hence work (often) in the category of quasicoherent sheaves (and indeed always in the category of modules over ringed spaces, see Remark 3.7, although this will not concern us here). The two functors are adjoint:

$$\operatorname{Hom}_A(A \otimes_B N, M) \cong \operatorname{Hom}_B(N, M_B)$$

(where this isomorphism of groups is functorial in both arguments), and we will see that this remains true on the scheme level.

One of these constructions will turn into our old friend pushforward. The other will be a relative of pullback, whom I'm reluctant to call an "old friend".

2. PUSHFORWARDS OF QUASICOHERENT SHEAVES

The main message of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

2.A. EXERCISE. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished open sets!)

Date: Wednesday, February 6, 2008.

In particular, $f_*\tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent. The following result, proved in an earlier Exercise, generalizes this statement.

2.1. *Theorem. — Suppose* $f : X \to Y$ *is a quasicompact quasiseparated morphism, and* $\mathcal{F}$ *is a quasicoherent sheaf on* X*. Then* $f_*\mathcal{F}$ *is a quasicoherent sheaf on* Y.

2.B. EXERCISE. Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_*\mathcal{F}$ is not quasicoherent. (Possible answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$, where the ith entry is the function on the ith component of X. The key point here is that infinite direct products do not commute with localization.)

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, given by $k \mapsto k[t]$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated k-module. Under especially good situations, coherent sheaves do push forward. For example:

2.C. EXERCISE. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k, \mathcal{F})$ is a finite-dimensional k-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_\bullet$, then $\Gamma(\operatorname{Proj} S_\bullet, \mathcal{F})$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". We will first need to define "projective morphism"! This notion is a generalization of $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

More generally, pushforwards of coherent sheaves by proper morphisms are also coherent sheaves. I'd like to give a proof of this, at least in the notes, at some point.

3. PULLBACK OF QUASICOHERENT SHEAVES

I find the notion of the pullback of a quasicoherent sheaf to be confusing on first (and second) glance. I will try to introduce it in two ways. One is directly in terms of thinking of quasicoherent sheaves in terms of modules over rings corresponding to affine open sets, and is suitable for direct computation. The other is elegant and functorial in terms of adjoints, and applies to ringed spaces in general. Both perspectives have advantages and disadvantages, and it is worth having some experience working with both.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

3.1. Construction/description of the pullback. Let us now define the pullback functor precisely. Suppose $X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y. We will describe the pullback quasicoherent sheaf $f^*\mathcal{G}$ on X by describing it as a sheaf on a variant of the distinguished affine base. In our base, we will permit only those affine open sets $U \subset X$ such that $f(U)$ is contained in an affine open set of Y. The distinguished restriction map will force this sheaf to be quasicoherent.

Suppose $U \subset X$, $V \subset Y$ are affine open sets, with $f(U) \subset V$, $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Suppose $\mathcal{F}|_V \cong \tilde{N}$. Then define $\Gamma(U, f_V^*\mathcal{F}) := A \otimes_B N \otimes_B$. Our main goal will be to show that this is independent of our choice of V.

We begin as follows: we fix an affine open subset $V \subset Y$, and use it to define sections over any affine open subset $U \subset f^{-1}(V)$. We show that this gives us a quasicoherent sheaf $f_V^*\mathcal{G}$ on $f^{-1}(V)$, by showing that these sections behave well with respect to distinguished restrictions. First, note that if $D(f) \subset U$ is a distinguished open set, then

$$\Gamma(D(f), f_V^*\mathcal{F}) = N \otimes_B A_f \cong (N \otimes_B A) \otimes_A A_f = \Gamma(U, f_V^*\mathcal{F}) \otimes_A A_f.$$

Define the restriction map $\Gamma(U, f_V^*\mathcal{F}) \to \Gamma(D(f), f_V^*\mathcal{F})$ by

$$\Gamma(f_V^*\mathcal{F}, U) \to \Gamma(f_V^*\mathcal{F}, U) \otimes_A A_f \tag{1}$$

(with $\alpha \mapsto \alpha \otimes 1$ of course). Thus on the *distinguished affine topology of* $\operatorname{Spec} A$ we have defined a quasicoherent sheaf.

To sum up: we have defined a quasicoherent sheaf on $f^{-1}(V)$, where V is an affine open subset of Y.

We want to show that this construction, as V varies over all affine open subsets of Y, glues into a single quasicoherent sheaf on X.

3.A. EXERCISE. Do this. (Possible hint: possibly use the idea behind the affine covering lemma. Begin by showing that the sheaf on $f^{-1}(\operatorname{Spec} A)$ restricted to the preimage of the distinguished open subset $f^{-1}(\operatorname{Spec} A_g)$ is canonically isomorphic to the sheaf on $f^{-1}(\operatorname{Spec} A_g)$. Another possible hint: figure out what the stalks should be, and define it as a sheaf of compatible germs.)

Hence we have described a quasicoherent sheaf $f^*\mathcal{G}$ on X whose behavior on affines mapping to affines was as promised.

3.2. *Theorem. —*

(1) *The pullback of the structure sheaf is the structure sheaf.*
(2) *The pullback of a finite type sheaf is finite type. Hence if $f : X \to Y$ is a morphism of locally Noetherian schemes, then the pullback of a coherent sheaf is coherent. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
(3) *The pullback of a locally free sheaf of rank r is another such. (In particular, the pullback of an invertible sheaf is invertible.)*

(4) *(functoriality in the morphism)* $\pi_1^*\pi_2^*\mathcal{F} \cong (\pi_2 \circ \pi_1)^*\mathcal{F}$

(5) *(functoriality in the quasicoherent sheaf) If $\pi : X \to Y$, then π^* is a functor from the category of quasicoherent sheaves on Y to the category of quasicoherent sheaves on Y. (Hence as a section of a sheaf $\mathcal{F}$ on Y is the data of a map $\mathcal{O}_Y \to \mathcal{F}$, by (1) and (6), if $s : \mathcal{O}_Y \to \mathcal{F}$ is a section of $\mathcal{F}$ then there is a natural section $\pi^* s : \mathcal{O}_X \to \pi^*\mathcal{F}$ of $\pi^*\mathcal{F}$. The pullback of the locus where s vanishes is the locus where the pulled-back section $\pi^* s$ vanishes.)*

(6) *(stalks) If $\pi : X \to Y$, $\pi(x) = y$, then there is an isomorphism* $(\pi^*\mathcal{F})_x \xrightarrow{\sim} \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$.

(7) *(fibers) Pullback of fibers are given as follows: if $\pi : X \to Y$, where $\pi(x) = y$, then*

$$\pi^*\mathcal{F}/\mathfrak{m}_{X,x}\pi^*\mathcal{F} \cong (\mathcal{F}/\mathfrak{m}_{Y,y}\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$

(8) *(tensor product)* $\pi^*(\mathcal{F} \otimes \mathcal{G}) = \pi^*\mathcal{F} \otimes \pi^*\mathcal{G}$

(9) *pullback is a right-exact functor*

All of the above are interconnected in obvious ways.

In fact much more is true, that you should be able to prove on a moment's notice, such as for example that the pullback of the symmetric power of a locally free sheaf is naturally isomorphic to the symmetric power of the pullback, and similarly for wedge powers and tensor powers.

Most of these are left to the reader. It is convenient to do right-exactness early; it is related to right-exactness of $\otimes$. For the tensor product fact, show that $(M \otimes_S R) \otimes (N \otimes_S R) \cong (M \otimes N) \otimes_S R$, and that this behaves well with respect to localization. The proof of the fiber fact is as follows. $(S, \mathfrak{n}) \to (R, \mathfrak{m})$.

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ S/\mathfrak{n} & \longrightarrow & R/\mathfrak{m} \end{array}$$

$(N \otimes_S R) \otimes_R (R/\mathfrak{m}) \cong (N \otimes_S (S/\mathfrak{n})) \otimes_{S/\mathfrak{n}} (R/\mathfrak{m})$ as both sides are isomorphic to $N \otimes_S (R/\mathfrak{m})$.

3.B. EXERCISE. Prove the Theorem.

3.C. UNIMPORTANT EXERCISE. Verify that the following is a example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

3.3. *Remark.* After proving the theorem, you'll see the importance of right-exactness. Given $\pi : X \to Y$, if the functor π^* from quasicoherent sheaves on Y to quasicoherent sheaves on X is also left-exact (hence exact), we will say that π is a *flat* morphism. This is an incredibly important notion, and we will come back to it later, next quarter.

3.4. A second definition, that doesn't always apply. Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, so π_* is a functor from quasicoherent sheaves on X to quasicoherent sheaves on Y. Then π^* and π_* are adjoints. More precisely:

3.5. Three more "definitions". Pullback is left-adjoint of the pushforward. If it exists, then it is unique up to unique isomorphism by Yoneda nonsense. One can thus take this as a definition of pullback, at least if π is quasicompact and quasiseparated. This defines the pullback up to unique isomorphism. The problem with this is that pullbacks should exist even without these hypotheses on π. And in any case, any proof by universal property requires an explicit construction as well, so we are led once again to our earlier constructive definition.

3.6. *Theorem. — Suppose* $\pi : X \to Y$ *is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward. More precisely,* $\mathrm{Hom}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{F})$.

More precisely still, we describe natural homomorphisms that are functorial in both arguments. We show that it is a bijection of sets, but it is fairly straightforward to verify that it is an isomorphism of groups. Not surprisingly, we will use adjointness for modules.

Proof. Let's unpack the right side. What's an element of $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$? For every affine V in Y, we get an element of $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V)))$, and this behaves well with respect to distinguished open sets. Equivalently, for every affine V in Y and U in $f^{-1}(V) \subset X$, we have an element $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U))$, that behaves well with respect to localization to distinguished open sets on both affines. By the adjoint property, this corresponds to elements of $\mathrm{Hom}(\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \mathcal{F}(U))$, which behave well with respect to localization. And that's the left side. □

3.7. *Pullback for ringed spaces* $\star$. (This is actually conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of $\mathcal{O}$-modules over ringed spaces. We define pushforward in the usual way, and then define the pullback of an $\mathcal{O}$-module using the adjoint property. Then one must show that (i) it exists, and (ii) the pullback of a quasicoherent sheaf is quasicoherent.

Here is a construction that always works. Suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. We will not show that this definition is equivalent to ours, but the interested reader is welcome to try this as an exercise.

3.D. EXERCISE FOR INTERESTED READERS. Show that π^* and π_* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules. Hint: Justify the following.

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) &= \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

The statements of Theorem 3.6 apply in this more general setting.

In particular, by uniqueness of adjointness, this "sheaf-theoretic" definition of pullback agrees with our scheme-theoretic definition of pullback when π is quasicompact and quasiseparated. The interested reader may wish to show it in general.

3.E. UNIMPORTANT EXERCISE. Show that the scheme-theoretic definition of pullback agrees with the sheaf-theoretic definition in terms of $\mathcal{O}$-modules.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 32

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CONTENTS

1. IMPORTANT EXAMPLE: INVERTIBLE SHEAVES AND MAPS TO PROJECTIVE SCHEMES

Theorem 1.1 will give one reason why line bundles are crucially important: they tell us about maps to projective space, and more generally, to quasiprojective A-schemes. Given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.

Before stating the theorem, we begin with some motivation. Recall that the data of a map to $\mathbb{A}^n$ corresponds to the choice of n functions, which could be called "coordinate functions". (The case $n = 1$ was an earlier exercise, and the general case is no harder.) Our goal is to give a similar characterization of maps to $\mathbb{P}^n$. We have already seen that a choice of $n+1$ functions on X with no common zeros yields a map to $\mathbb{P}^n$. However, this can't give *all* maps to $\mathbb{P}^n$: suppose $n > 0$ and consider the identity map $\mathbb{P}^n_k \to \mathbb{P}^n_k$. This map can't be described in terms of $n+1$ functions on X with no common zeros, as the only functions on $\mathbb{P}^n$ are constants, so they only maps $\mathbb{P}^n_k \to \mathbb{P}^n_k$ that can be described in terms of n functions with no common zeros are constant maps. The resolution of this problem is by considering not just functions — sections of the trivial invertible sheaf — but sections of any invertible sheaf.

1.1. *Important theorem. — Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point (i.e. generated by global sections), modulo global sections of $\mathcal{O}_X^*$.*

This is one of those important theorems in algebraic geometry that is easy to prove, but quite subtle in its effect on how one should think. It takes some time to properly digest.

The theorem describes all morphisms to projective space, and hence by the Yoneda philosophy, this can be taken as the *definition* of projective space: it defines projective space up to unique isomorphism.

Every time you see a map to projective space, you should immediately simultaneously keep in mind the invertible sheaf and sections.

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Maps to projective schemes can be described similarly. For example, if $Y \hookrightarrow \mathbb{P}^2_k$ is the curve $x_2^2 x_0 = x_1^3 - x_1 x_0^2$, then maps from a scheme X to Y are given by an invertible sheaf on X along with three sections s_0, s_1, s_2, with no common zeros, satisfying $s_2^2 s_0 - s_1^3 + s_1 s_0^2 = 0$.

Here more precisely is the correspondence of Theorem 1.1. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \dots, x_{n+1}$. then $f^* x_0, \dots, f^* x_{n+1}$ are sections of $f^* \mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Proof. Given $n+1$ sections $s_0, \dots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each one vanishes. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^* D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*(x_i/x_j)$. Thus starting with the $n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^* x_i$. The map $[s_0; \cdots; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^* x_i) = f^* D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ is precisely by $s_j/s_i = f^* x_j / x_i$. □

Here is some convenient language. A **linear system** on a k-scheme X is a k-vector space V (usually finite-dimensional), an invertible sheaf $\mathcal{L}$, and a linear map $\lambda : V \to \Gamma(X, \mathcal{L})$. Such a linear system is often called "V", with the rest of the data left implicit. If the map λ is an isomorphism, it is called a **complete linear system**, and is often written $|\mathcal{L}|$. Given a linear system, any point $x \in X$ on which all elements of the linear system V vanish, we say that x is a **base-point** of V. If V has no base-points, we say that it is **base-point-free**. The union of base-points is called the **base locus**. The base locus has a scheme-structure — the (scheme-theoretic) intersection of the vanishing loci of the elements of V (or equivalently, of a basis of V). In this incarnation, it is called the **base scheme** of the linear system.

A linear system is sometimes called a **linear series**. I'm not sure of the distinction between these two terms, so I'll not use this second terminology.

1.A. EXERCISE (AUTOMORPHISMS OF PROJECTIVE SPACE). Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^* \mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

Exercise 1.A will be useful later, especially for the case $n = 1$. In this case, these automorphisms are called **fractional linear transformations**.

(A question for experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

Here are some more examples of these ideas in action.

Example 1. Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^n - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We've talked about this morphism before. But now we don't have to worry about gluing.)

Example 2: the Veronese morphism is $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We've checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections x_0^m, x_1^m, $\dots$, x_n^m have no common zeros). Thus the complete linear system is base-point-free, and determines a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is called the **Veronese morphism**. For example, if $n=2$ and $m=2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

We have checked earlier that this is a closed immersion. How can you tell in general if something is a closed immersion, and not just a map? Here is one way.

1.B. EXERCISE. Suppose $\pi : X \to \mathbb{P}^n_A$ corresponds to an invertible sheaf $\mathcal{L}$ on X, and sections $s_0, \dots, s_n$. Show that π is a closed immersion if and only if

(i) each open set X_{s_i} is affine, and
(ii) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_{s_i}, \mathcal{O})$ given by $y_j \mapsto s_j/s_i$ is surjective.

Example 3: The rational normal curve. Recall that the image of the Veronese morphism when $n=1$ is called a **rational normal curve of degree** m. Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x;y] \to [x^m; x^{m-1}y; \cdots; xy^{m-1}; y^m]$.

1.C. EXERCISE. If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is **degenerate** (and otherwise, **non-degenerate**). Show that a base-point-free linear system V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear system is always non-degenerate.

1.D. EXERCISE. Suppose we are given a map $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ where the corresponding invertible sheaf on $\mathbb{P}^1_k$ is $\mathcal{O}(d)$. (We will later call this a *degree* d *map*.) Show that if $d < n$, then the image is degenerate. Show that if $d = n$ and the image is nondegenerate, then the image is isomorphic (via an automorphism of projective space, Exercise 1.A) to a rational normal curve.

Example 4: The Segre morphism in terms of a linear system. The Segre morphism can also be interpreted in this way. This is a useful excuse to define some notation. Suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear system for the invertible sheaf $\mathcal{O}(1, 1)$.

When we first saw the Segre morphism, we saw (in different language) that this complete linear system is base-point-free. We also checked by hand that it is a closed immersion, essentially by Exercise 1.B.

1.E. FUN EXERCISE. Show that any map from projective space to a smaller projective space is constant (over a field). Hint: show that if $m < n$ then m non-empty hypersurfaces in $\mathbb{P}^n$ have non-empty intersection. For this, use the fact that any non-empty hypersurface in $\mathbb{P}^n_k$ has non-empty intersection with any subscheme of dimension at least 1.

1.F. EXERCISE. Show that a base-point-free linear system V on X corresponding to $\mathcal{L}$ induces a morphism to projective space $X \to \mathbb{P}V^* = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written $X \xrightarrow{|V|} \mathbb{P}^n$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 33 AND 34

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CONTENTS

In these two lectures, we will use universal properties to define two more useful constructions, $\underline{\mathrm{Spec}}$ of a sheaf of algebras $\mathcal{A}$, and $\underline{\mathrm{Proj}}$ of a sheaf of graded algebras $\mathcal{A}_\bullet$ on a scheme X. These will both generalize (globalize) our constructions of Spec and Proj of A-algebras and graded A-algebras. We'll see that affine morphisms are precisely those of the form $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$, and so we'll *define* projective morphisms to be those of the form $\underline{\mathrm{Proj}}\,\mathcal{A}_\bullet \to X$.

1. RELATIVE SPEC OF A (QUASICOHERENT) SHEAF OF ALGEBRAS

Given an A-algebra, B, we can take its Spec to get an affine scheme over $\mathrm{Spec}\,A$: $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$. We will now see universal property description of a globalization of that notation. Consider an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{A}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism.

You can think of this in two ways. First, and most concretely, for any affine open set $\mathrm{Spec}\,A \subset X$, $\Gamma(\mathrm{Spec}\,A, \mathcal{A})$ is some A-algebra; call it B. Then above $\mathrm{Spec}\,A$, $\underline{\mathrm{Spec}}\,A$ will be $\mathrm{Spec}\,B$.

Second, it will satisfy a universal property. We could define the A-scheme $\mathrm{Spec}\,B$ by the fact that maps to $\mathrm{Spec}\,B$ (from an A-scheme Y, over $\mathrm{Spec}\,A$) correspond to maps of A-algebras $B \to \Gamma(Y, \mathcal{O}_Y)$. The universal property for $\underline{\mathrm{Spec}}\,\mathcal{A}$ is similar. More precisely, we describe a universal property for the morphism $\beta : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ along with an isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\mathrm{Spec}}\,\mathcal{A}}$: to each morphism $\pi : Y \to X$ along with a morphism of $\mathcal{O}_X$-modules

$$\alpha : \mathcal{A} \to \pi_* \mathcal{O}_Y,$$

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there is a unique map $f : Y \to \underline{\operatorname{Spec}} \mathcal{A}$ factoring π, i.e. so that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\exists! f} & \underline{\operatorname{Spec}} \mathcal{A} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

where α is the composition

$$\mathcal{A} \xrightarrow{\phi} \beta_* \mathcal{O}_{\underline{\operatorname{Spec}} \mathcal{A}} \longrightarrow \beta_* f_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y$$

(For experts: we need to work with $\mathcal{O}_X$-modules, and to leave our category of quasicoherent sheaves on X, because we only showed that the pushforward of quasicoherent sheaves are quasicoherent for quasicompact quasiseparated morphisms, and we don't need such hypotheses here.) This bijection $\operatorname{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y) \leftrightarrow \operatorname{Mor}_X(Y, \underline{\operatorname{Spec}} \mathcal{A})$ is natural in Y, i.e. given $Y' \to Y$ the diagram

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y) & \longleftrightarrow & \operatorname{Mor}_X(Y, \underline{\operatorname{Spec}} \mathcal{A}) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_{Y'}) & \longleftrightarrow & \operatorname{Mor}_X(Y', \underline{\operatorname{Spec}} \mathcal{A}) \end{array}$$

commutes. By universal property nonsense, this determines $\underline{\operatorname{Spec}} \mathcal{A}$ up to unique isomorphism, assuming of course that it exists.

1.A. EXERCISE. Show that if X is affine, say $\operatorname{Spec} A$, and $\mathcal{A} = \tilde{B}$, where B is an A-algebra, then $\operatorname{Spec} B \to \operatorname{Spec} A$ satisfies this universal property. (Hint: recall that maps to an affine scheme correspond to maps of rings of functions in the opposite direction.) Show that this affine construction behaves well with respect to "affine base change": given a map $g : \operatorname{Spec} A' \to \operatorname{Spec} A$, then describe a canonical isomorphism $\underline{\operatorname{Spec}}\, g^* \mathcal{A} \cong \operatorname{Spec} A' \otimes_A B$.

1.1. *Remark.* In particular, if p is a point of $\operatorname{Spec} A$, $k(p)$ is the residue field at p, and $\operatorname{Spec} k(p) \to \operatorname{Spec} A$ is the inclusion, then the fiber of $\operatorname{Spec} B \to \operatorname{Spec} A$ is canonically identified (as a scheme) with $\operatorname{Spec} B \otimes_A k(p)$. This is the motivation for our construction below.

We define $\underline{\operatorname{Spec}} \mathcal{A}$ by describing the points, then the topology, and then the structure sheaf. (Experts: where does the quasicoherence of $\mathcal{A}$ come in?)

First the points: above the point $p \in X$, the points of $\underline{\operatorname{Spec}} \mathcal{A}$ are defined to be the points of $\operatorname{Spec}(\mathcal{A} \otimes k(p))$. (For example, take the stalk, and mod out by the maximal ideal. Or take any affine open neighborhood of p, and apply the construction of Remark 1.1.

We topologize this set as follows. Above the affine open subset $\operatorname{Spec} A \subset X$, the points are identified with the points of $\operatorname{Spec} \Gamma(\operatorname{Spec} A, \mathcal{A})$, by Remark 1.1. We impose that this be an open subset of $\underline{\operatorname{Spec}} A$, and the topology restricted to this open set is required to be the Zariski topology on $\operatorname{Spec} \Gamma(\operatorname{Spec} A, \mathcal{A})$.

1.B. EXERCISE. Show that this topology is well-defined. In other words, show that if $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ are affine open subsets of X, then the topology imposed on $\beta^{-1}(\operatorname{Spec} A \cap \operatorname{Spec} A')$ by the construction using $\operatorname{Spec} A$ agrees with the topology imposed by $\operatorname{Spec} A'$. (Some ideas behind the Affine Communication Lemma may be helpful. For example, this question is much easier if $\operatorname{Spec} A'$ is a distinguished open subset of $\operatorname{Spec} A$.)

Next, we describe the structure sheaf, and the description is precisely what you might expect: on $\beta^{-1}(\operatorname{Spec} A) \subset \underline{\operatorname{Spec}}\,\mathcal{A}$, the sheaf is isomorphic to the structure sheaf on $\operatorname{Spec} \Gamma(\operatorname{Spec} A, \mathcal{A})$.

1.C. EXERCISE. Rigorously define the structure sheaf. How do you glue these sheaves on small open sets together? Once again, the ideas behind the Affine Communication Lemma may help.

1.D. EXERCISE. Describe the isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\,\mathcal{A}}$. Show that given any $\pi : Y \to X$, this construction yields the isomorphism $\operatorname{Mor}_X(Y, \underline{\operatorname{Spec}}\,\mathcal{A}) \to \operatorname{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y)$ via the composition

$$\mathcal{A} \xrightarrow{\phi} \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\,\mathcal{A}} \longrightarrow \beta_* f_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y.$$

1.E. EXERCISE. Show that $\underline{\operatorname{Spec}}\,\mathcal{A}$ satisfies the desired universal property. (Hint: figure out how to reduce to the case X affine, Exercise 1.A.)

We make some quick observations, some verified in exercises. First $\underline{\operatorname{Spec}}\,\mathcal{A}$ can be "computed affine-locally on X".

Second, this gives an important way to understand affine morphisms. Note that $\underline{\operatorname{Spec}}\,\mathcal{A} \to X$ is an affine morphism. The "converse" is also true:

1.F. EXERCISE. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\operatorname{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\underline{\operatorname{Spec}}\,\mathcal{A} \to X$.

1.G. EXERCISE ($\underline{\operatorname{Spec}}$ BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\operatorname{Spec}}\,\mathcal{A} \cong \underline{\operatorname{Spec}}\, f^* \mathcal{A}$.

An important example of this $\underline{\operatorname{Spec}}$ construction is the **total space of a finite rank locally free sheaf** $\mathcal{F}$, which we define to be $\underline{\operatorname{Spec}} \operatorname{Sym}^\bullet \mathcal{F}^\vee$.

1.H. EXERCISE. Show that this is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\mathrm{Spec}}\, \mathrm{Sym}^\bullet \mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of it.

In particular, if $\mathcal{F}$ is a *free* sheaf of rank n, then $\underline{\mathrm{Spec}}\, \mathrm{Sym}^* \mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of a free sheave behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$.

Here is one last fact that can be useful.

1.I. EXERCISE. Suppose $f : \underline{\mathrm{Spec}}\, \mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\mathrm{Spec}}\, A$ is "essentially the same as" (i.e. equivalent to) the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

The reason you could imagine caring is when X is quite simple, and $\underline{\mathrm{Spec}}\, \mathcal{A}$ is complicated. We'll use this before long when $X \cong \mathbb{P}^1$, and $\underline{\mathrm{Spec}}\, \mathcal{A}$ is a more complicated curve.

1.J. IMPORTANT EXERCISE: THE TAUTOLOGICAL BUNDLE ON $\mathbb{P}^n$ IS $\mathcal{O}(-1)$. Define the subset $X \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$ corresponding to "points of $\mathbb{A}^{n+1}_k$ on the corresponding line of $\mathbb{P}^n_k$", so that the fiber of the map $\pi : X \to \mathbb{P}^n$ corresponding to a point $l = [x_0; \cdots ; x_n]$ is the line in $\mathbb{A}^{n+1}$ corresponding to l, i.e. the scalar multiples of $(x_0, \dots, x_n)$. Show that $\pi : X \to \mathbb{P}^n$ is (the line bundle corresponding to) the invertible sheaf $\mathcal{O}(-1)$. (Possible hint: work first over the usual affine open sets of $\mathbb{P}^n$, and figure out transition functions.) (For this reason, $\mathcal{O}(-1)$ is often called the **tautological bundle** of $\mathbb{P}^n$.)

2. RELATIVE PROJ OF A SHEAF OF GRADED ALGEBRAS

In parallel with $\underline{\mathrm{Spec}}$, we will define a relative version of Proj, denoted $\underline{\mathrm{Proj}}$.

Suppose now that $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras of X. We require that $\mathcal{S}_\bullet$ is *locally generated in degree* 1 (i.e. there is a cover by small affine open sets, where for each affine open set, the corresponding algebra is generated in degree 1), and $\mathcal{S}_1$ is finite type. We will define $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ by describing a universal property, and the constructing it.

In order to understand the universal property, let's revisit maps to $\mathrm{Proj}\, S_\bullet$ (over a base ring A), satisfying the analogous assumptions. Suppose S_1 is generated by $x_1, \dots, x_n$. Recall that maps from an A-scheme to projective space

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Proj}\, S_\bullet \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\, A & \end{array}$$

correspond to invertible sheaves $\mathcal{L}$ on Y and sections $s_1, \dots, s_n$,

(i) with no common zeros (they are a *base-point-free linear system*),
(ii) satisfying "the same relations as $x_1, \dots, x_n$".

It is helpful to write this map as

$$Y \xrightarrow{[s_1;\cdots;s_n]} \operatorname{Proj} S_\bullet.$$

The condition that $s_1, \dots, s_n$ satisfy the same conditions as $x_1, \dots, x_n$ can be formalized to say that there is a map of graded A-algebras

$$\Gamma_\bullet(Y, \mathcal{L}) := \oplus_{i=0}^{\infty} \Gamma(Y, \mathcal{L}^{\otimes i}) \longleftarrow S_\bullet$$

given by $x_i \mapsto s_i$. This will yield a "relative" version of (ii).

We now describe a relative version of (i).

2.1. Definition. Given a morphism $\pi : Y \to X$, an invertible sheaf $\mathcal{L}$ on Y is **relatively base-point-free** (with respect to π) if for every point of $y \in Y$, there is an open subset $U \subset X$ and a section s of $\mathcal{L}$ above U ($s \in \Gamma(\pi^{-1}(U), \mathcal{L})$) such that $s(y) \neq 0$.

2.A. EASY EXERCISE. If $X = \operatorname{Spec} A$, and $\mathcal{L}$ is base-point-free, show that $\mathcal{L}$ is relatively base-point-free.

Thus $\mathcal{L}$ is relatively base-point-free if it is "base-point-free over an affine cover X".

2.B. EXERCISE. Suppose π is quasicompact and quasiseparated (so π_* sends quasicoherent sheaves to quasicoherent sheaves). Show that $\mathcal{L}$ is basepoint free if the canonical map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective.

More generally, if $\mathcal{F}$ is a quasicoherent and quasiseparated, we say that a quasicoherent sheaf $\mathcal{F}$ on X is **relatively generated** (with respect to π) if the canonical map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is surjective. We won't be using this notion.

2.C. EXERCISE. Describe why this is the relative version of *generated by global sections*.

Having defined relative versions of (i) and (ii) above, we are now ready to define $\underline{\operatorname{Proj}}$.

2.2. Definition. Suppose $\mathcal{S}_\bullet$ is a graded quasicoherent sheaf of algebras on a scheme X, locally generated in degree 1. In analogy with $\underline{\operatorname{Spec}}$, we define

$$(\beta : \operatorname{Proj} \mathcal{S}_\bullet \to X, \mathcal{O}(1), \phi : \mathcal{S}_\bullet \to \oplus_n \beta_* \mathcal{O}(n))$$

by the following universal property. (Here ϕ is a map of graded sheaves, and is *not* required to be an isomorphism.)

Maps

$$\begin{array}{ccc} Y & \xrightarrow{f} & \underline{\operatorname{Proj}}\, \mathcal{S}_\bullet \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & X & \end{array}$$

correspond to maps $\alpha : \mathcal{S}_\bullet \to \oplus_{n=0}^{\infty} \pi_* \mathcal{L}^{\otimes n}$, where $\mathcal{L}$ is an invertible sheaf on Y, α factors as

$$\mathcal{S}_\bullet \xrightarrow{\phi} \oplus \beta_* \mathcal{O}(n) \longrightarrow \oplus \beta_* f_* \mathcal{L}^{\otimes n} = \oplus \pi_* \mathcal{L}^{\otimes n},$$

and the image of S_1 is relatively base-point free. (You might be worried about what happens if π is not quasicompact and quasiseparated, in which case we don't know that π_* is a quasicoherent sheaf. This isn't a problem: we can work with $\mathcal{O}_X$-modules. This won't cause any complication.)

As usual, if $(\beta : \operatorname{Proj} \mathcal{S}_\bullet \to X, \mathcal{O}(1), \phi : \mathcal{S}_\bullet \to \oplus_n \beta_* \mathcal{O}(n))$ exists, it is unique up to unique isomorphism. We now show that it exists, in analogy with $\underline{\operatorname{Spec}}$.

2.D. EXERCISE. Show the result if X is affine by restating what we know about the Proj construction.

Note that this construction behaves well with respect to affine base change.

Motivated by this, we define the points of $\operatorname{Proj} \mathcal{S}_\bullet$ over a point $p \in X$ as the points of $\operatorname{Proj}(\mathcal{S}_\bullet \otimes k(p))$.

2.E. EXERCISE. Define a topology on this set as follows: above each affine open subset of $\operatorname{Spec} A \subset X$, take the Zariski topology on $\operatorname{Proj} \Gamma(\operatorname{Spec} A, \mathcal{S}_\bullet)$. Be sure to show this is well-defined.

2.F. EXERCISE. Define the structure sheaf on this topological space as follows: above each affine open subset of $\operatorname{Spec} A \subset X$, take the structure sheaf of $\operatorname{Proj} \Gamma(\operatorname{Spec} A, \mathcal{S}_\bullet)$. Be sure to show this is well-defined.

2.G. EXERCISE. Define the map $\phi : \mathcal{S}_\bullet \to \oplus \mathcal{O}(n)$.

2.H. EXERCISE. Show that your construction satisfies the universal property.

2.I. EXERCISE ($\underline{\operatorname{Proj}}$ BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\operatorname{Proj}}\, \mathcal{S}_\bullet$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\operatorname{Proj}}\, f^* \mathcal{S}_\bullet, \mathcal{O}_{\underline{\operatorname{Proj}}\, f^* \mathcal{S}_\bullet}(1)) \cong (Y \times_X \underline{\operatorname{Proj}}\, \mathcal{S}_\bullet, g^* \mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_\bullet}(1))$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet & \xrightarrow{g} & \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

2.3. *Definition.* If $\mathcal{F}$ is a finite rank locally free sheaf on X. Then $\underline{\mathrm{Proj}}\, \mathrm{Sym}^\bullet \mathcal{F}$ is called its *projectivization*. If $\mathcal{F}$ is a free sheaf of rank $n+1$, then we define $\mathbb{P}^n_X := \underline{\mathrm{Proj}}\, \mathrm{Sym}^\bullet \mathcal{F}$. (Then $\mathbb{P}^n_{\mathrm{Spec}\, A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) Clearly this notion behaves well with respect to base change.

This "relative $\mathcal{O}(1)$" we have constructed is a little subtle. Here are couple of exercises to give you practice with the concept.

2.J. EXERCISE. $\underline{\mathrm{Proj}}(\mathcal{S}_\bullet[t]) \cong \underline{\mathrm{Spec}}\, \mathcal{S}_\bullet \coprod \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$, where $\underline{\mathrm{Spec}}\, \mathcal{S}_\bullet$ is an open subscheme, and $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\, \mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\, N}(1)$. (This is the generalization of the projective and affine cone.)

2.K. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0} \mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\, \mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet}(1) \otimes \pi^* \mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

3. PROJECTIVE MORPHISMS

In §1, that we reinterpreted affine morphisms: $X \to Y$ is an affine morphism if there is an isomorphism $X \cong \underline{\mathrm{Spec}}\, \mathcal{A}$ of Y-schemes for some quasicoherent sheaf of algebras $\mathcal{A}$ on Y. We now *define* the notion of a projective morphism similarly.

3.1. *Definition.* A morphism $X \to Y$ is **projective** if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_\bullet$ on Y. X is said to be a **projective Y-scheme**, or **projective over** Y. This generalizes the notion of a projective A-scheme.

3.2. *Warnings.* First, notice that $\mathcal{O}(1)$, an important part of the definition of $\underline{\mathrm{Proj}}$, is not mentioned. As a result, the notion of affine morphism is affine-local on the target, but this notion is not affine-local on the target. (In nice circumstances it is, as we'll see later. We'll also see an example where this is not.)

Second, Hartshorne gives a different definition; we are following the more general definition of Grothendieck. These definitions turn out to be the same in nice circumstances.

We now establish a number of properties of projective morphisms.

Note first that projective morphisms are proper. (Reason: properness is local on the base, and we've seen earlier that projective A-schemes are proper over A.) Equivalently (by definition of properness!) they are separated, finite type, and universally closed.

3.A. IMPORTANT EXERCISE: FINITE MORPHISMS ARE PROJECTIVE. Show that finite morphisms are projective as follows. Suppose $Y \to X$ is finite, and that $Y = \underline{\mathrm{Spec}}\,\mathcal{A}$ where $\mathcal{A}$ is a finite type quasicoherent sheaf on X. Describe a sheaf of graded algebras $\mathcal{S}_\bullet$ where $\mathcal{S}_0 \cong \mathcal{O}_X$ and $\mathcal{S}_n \cong \mathcal{A}$ for $n > 0$. (What is the multiplication in this algebra?) Describe an X-isomorphism $Y \cong \underline{\mathrm{Proj}}\,\mathcal{S}_\bullet$.

In particular, closed immersions are projective. We have the sequence of implications for morphisms

$$\text{closed immersion} \Rightarrow \text{finite} \Rightarrow \text{projective} \Rightarrow \text{proper}.$$

3.B. EXERCISE. Show that a morphism (over $\mathrm{Spec}\,k$) from a projective k-scheme to a separated k-scheme is always projective. (Hint: the Cancellation Theorem for properties of morphisms.)

3.C. EXERCISE. Show that the property of a morphism being projective is preserved by base change.

3.D. HARDER EXERCISE. Show that the property of being projective is preserved by composition. (Ask me for a hint. The main thing is to figure out a candidate $\mathcal{O}(1)$.)

The previous two exercises imply that the property of being projective is preserved by products: if $f : X \to Y$ and $f' : X' \to Y$ are projective, then so is $f \times f' : X \times X' \to Y \times Y'$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 35 AND 36

RAVI VAKIL

CONTENTS

In these two lectures, we will define Cech cohomology and discuss its most important properties, although not in that order.

1. INTRODUCTION

As $\Gamma(X, \cdot)$ is a left-exact functor, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, then

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X)$$

is exact. We dream that this sequence continues off to the right, giving a long exact sequence. More explicitly, there should be some covariant functors H^i ($i \geq 0$) from quasicoherent sheaves on X to groups such that $H^0 = \Gamma$, and so that there is a "long exact sequence in cohomology".

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \tag{1}$$

$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

(In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in good cases our dreams will come true. The machinery behind this is sometimes called *derived functor cohomology*, which we will discuss shortly.)

Before defining cohomology groups of quasicoherent sheaves explicitly, we first describe their important properties. Indeed these fundamental properties are in some ways more important than the formal definition. The boxed properties will be the important ones.

Date: Friday, February 22 and Monday, February 25, 2008.

Suppose X is a separated and quasicompact A-scheme. (The separated and quasicompact hypotheses will be necessary in our construction.) For each quasicoherent sheaf $\mathcal{F}$ on X, we will define A-modules $H^i(X, \mathcal{F})$. In particular, if $A = k$, they are k-vector spaces.

(i) $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})}$.

(ii) Each H^i is a $\boxed{\text{covariant functor in the sheaf } \mathcal{F}}$ extending the usual covariance for $H^0(X, \cdot)$: $\mathcal{F} \to \mathcal{G}$ induces $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$.

(iii) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$ (1). The maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ come from covariance, and similarly for $H^i(X, \mathcal{G}) \to H^i(X, \mathcal{G})$. The *connecting homomorphisms* $H^i(x, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.

(iv) If $f : X \to Y$ is any morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural morphism $\boxed{H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})}$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$. We will later see this as part of a larger story, the *Leray spectral sequence*. If $\mathcal{G}$ is a quasicoherent sheaf on Y, then setting $\mathcal{F} := f^*\mathcal{G}$ and using the adjunction map $\mathcal{G} \to f_*f^*\mathcal{G}$ and covariance of (ii) gives a natural **pullback map** $H^i(Y, \mathcal{G}) \to H^i(X, f^*\mathcal{G})$ (via $H^i(Y, \mathcal{G}) \to H^i(Y, f_*f^*\mathcal{G}) \to H^i(X, f^*\mathcal{G})$) extending $\Gamma(Y, \mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$. In this way, H^i is a "contravariant functor in the space".

(v) If $f : X \hookrightarrow Y$ is an affine morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, the natural map of (iv) is an isomorphism: $\boxed{H^i(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^i(X, \mathcal{F})}$. When f is a closed immersion and $Y = \mathbb{P}^N_A$, this isomorphism will translate calculations on arbitrary projective A-schemes to calculations on $\mathbb{P}^n_A$.

(vi) If X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$. In particular, all higher ($i > 0$) quasicoherent cohomology groups on affine schemes vanish. The vanishing of H^1 in this case, along with the long exact sequence **(iii)** implies that Γ is an exact functor for quasicoherent sheaves on affine schemes, something we already knew. It is also true that if $\dim X = n$, then $H^i(X, \mathcal{F}) = 0$ for all $i > n$ and for all $\mathcal{F}$ (**dimensional vanishing**). We will prove this for quasiprojective A-schemes, but we won't use this fact in general, and hence won't prove it. (A proof is given in Hartshorne (Thm. III.2.7) for derived functors, and we show in a week or two that this agrees with Cech cohomology.)

(vii) The functor H^i behaves well under direct sums, and more generally under colimits: $\boxed{H^i(X, \varinjlim \mathcal{F}_j) = \varinjlim H^i(X, \mathcal{F}_j)}$.

(viii) We will also identify the cohomology of all $\mathcal{O}(m)$ on $\mathbb{P}^n_A$:

1.1. *Theorem.* —

- $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$, *and* 0 *otherwise.*

- $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ *is a free* A*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$, *and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0$ *if* $0 < i < n$.

We already have shown the first statement in an Essential Exercise (class 27 end of section 3).

Theorem 1.1 has a number of features that will be the first appearances of things that we'll prove later.

- The cohomology of these bundles vanish above n (**(vi)** above)
- These cohomology groups are always *finitely-generated* A-modules. This will be true for all coherent sheaves on projective A-schemes (Theorem 1.2(i)).
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of *Kodaira vanishing*.)
- The top cohomology group is one-dimensional for $m = -n-1$ if $A = k$. This is the first appearance of the *dualizing sheaf*.
- There is a natural duality

$$H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1)).$$

  This is the first appearance of *Serre duality*.

Before proving these facts, let's first use them to prove interesting things, as motivation.

By an earlier Theorem from last quarter (class 30 Corollary 3.3), for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_A$ we can find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{2}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

1.2. *Theorem. — (i) For any coherent sheaf* $\mathcal{F}$ *on a projective* A*-scheme where* A *is Noetherian,* $h^i(X, \mathcal{F})$ *is a coherent (finitely generated)* A*-module.*
(ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ *for all* i, *even without Noetherian hypotheses.*

A non-Noetherian generalization of the coherence statement is given in Exercise 1.A.

Proof. Because cohomology of a closed scheme can be computed on the ambient space (see **(v)** above), we may immediately reduce to the case $X = \mathbb{P}^n_A$.

(i) Consider the long exact sequence:

$$0 \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow$$

$$H^1(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^1(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^1(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^{n-1}(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow$$

$$H^n(\mathbb{P}^n_A, \mathcal{G}) \longrightarrow H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^n(\mathbb{P}^n_A, \mathcal{F}) \longrightarrow 0$$

The exact sequence ends here because $\mathbb{P}^n_A$ is covered by $n+1$ affines (**(vi)** above). Then $H^n(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Theorem 1.1, hence $H^n(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_A, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_A, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_A, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (2) by $\mathcal{O}(N)$ for $N \gg 0$. Then

$$H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = \oplus_j H^n(\mathbb{P}^n_A, \mathcal{O}(m+N)) = 0$$

(by **(vii)** above), so $H^n(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_A, \mathcal{F}(N)) = 0$. (The induction proceeds no further, as it is *not* true that $H^0(\mathbb{P}^n_A, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite.) □

1.A. EXERCISE ONLY FOR THOSE WHO LIKE WORKING WITH NON-NOETHERIAN RINGS. Prove part (i) in the above result without the Noetherian hypotheses, assuming only that A is a coherent A-module (A is "coherent over itself"). (Hint: induct downwards as before. Show the following in order: $H^n(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_A, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_A, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_A, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_A, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

1.3. *Corollary. — Any projective* k*-scheme has a finite-dimensional space of global sections. More generally, if* $\mathcal{F}$ *is a coherent sheaf on a projective* A*-scheme, then* $H^0(X, \mathcal{F})$ *is a finitely generated* A*-module.*

This is true more generally for proper k-schemes, not just projective k-schemes, but this requires more work.

Here is three important consequences. They can also be shown directly, without the use of cohomology, but with much more elbow grease.

1.B. EXERCISE. Suppose X is a projective integral scheme over an algebraically closed field. Show that $h^0(X, \mathcal{O}_X) = 1$. Hint: show that $H^0(X, \mathcal{O}_X)$ is a finite-dimensional k-algebra, and a domain. Hence show it is a field. (For experts: the same argument holds with the weaker hypotheses where X is proper, geometrically connected, and reduced over an arbitrary field.)

1.C. CRUCIAL EXERCISE (PUSHFORWARDS OF COHERENTS ARE COHERENT). Suppose $f : X \to Y$ is a projective morphism, and $\mathcal{O}_Y$ is coherent over itself (true in all reasonable circumstances). Show that the pushforward of a coherent sheaf on X is a coherent sheaf on Y.

Finite morphisms are affine (from the definition) and projective (shown earlier, class 33/34 Exercise 3.A). We can now show that this is a characterization of finiteness.

1.4. *Corollary. — If $\pi : X \to Y$ is projective and affine and $\mathcal{O}_Y$ is coherent, then π is finite.*

In fact, more generally, if π is universally closed and affine, then π is finite, by Atiyah-Macdonald Exercise 5.35 (thanks Joe!). We won't use this, so I won't explain why.

Proof. By Exercise 1.C, $\pi_* \mathcal{O}_X$ is coherent and hence finitely generated. □

1.D. EXERCISE. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on projective X with $\mathcal{F}$ coherent. Show that for $n \gg 0$,

$$0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$$

is also exact. (Hint: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

2. DEFINITIONS AND PROOFS OF KEY PROPERTIES

This section could be read much later; the facts we will use are all stated in the previous section. However, the arguments aren't that complicated, so you may feel like reading this right away. As you read this, you should go back and check off all the facts, to assure yourself that I've shown all that I've promised.

2.1. Cech cohomology. Cech cohomology in general settings is often defined using a limit over finer and finer covers of a space. In our algebro-geometric setting, the situation is much cleaner, and we can use a single cover.

Suppose X is quasicompact and separated, e.g. X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set (by separatedness, Class 17 Proposition

1.2). We'll use quasicompactness and separatedness only in order to ensure these two nice properties.

Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* set of affine open sets of X covering U. For $I \subset \{1, \dots, n\}$ define $U_I = \cap_{i \in I} U_i$, which is affine by the separated hypothesis. Consider the **Cech complex**

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots . \tag{3}$$

The maps are defined as follows, in terms of the summands. The map from $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ is 0 unless $I \subset J$, i.e. $J = I \cup \{j\}$. If j is the kth element of J, then the map is $(-1)^{k-1}$ times the restriction map $\operatorname{res}_{U_I, U_J}$.

2.A. EASY EXERCISE (FOR THOSE WHO HAVEN'T SEEN ANYTHING LIKE THE CECH COMPLEX BEFORE). Show that the Cech complex is indeed a complex, i.e. that the composition of two consecutive arrows is 0.

Define $H^i_{\mathcal{U}}(U, \mathcal{F})$ to be the ith cohomology group of the complex (3). Note that if X is an A-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an A-module. We have almost succeeded in defining the Cech cohomology group H^i, except our definition seems to depend on a choice of a cover $\mathcal{U}$.

2.B. EASY EXERCISE. Show that $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. (Hint: use the sheaf axioms for $\mathcal{F}$.)

2.C. EXERCISE. Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection of open subsets in $\mathcal{U}$, the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

2.2. *Theorem/Definition. — Recall that* X *is quasicompact and separated.* $H^i_{\mathcal{U}}(U, \mathcal{F})$ *is independent of the choice of (finite) cover* $\{U_i\}$. *More precisely,*

(*) *for all* k, *for any two covers* $\{U_i\} \subset \{V_i\}$ *of size at most* k, *the maps* $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ *induced by the natural maps of Cech complexes* (3) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ *to be this group.*

The dependence of k in the statement is there because we will prove it by induction on k.

(For experts: maps of complexes inducing isomorphisms are called *quasiisomorphisms*. We are actually getting a finer invariant than cohomology out of this construction; we are getting an element of the *derived category of* A*-modules*.)

Proof. We prove this by induction on k. The base case $k = 1$ is trivial. We need only prove the result for $\{U_i\}_{i=1}^n \subset \{U_i\}_{i=0}^n$, where the case $k = n$ is assumed known. Consider the exact sequence of complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ 0 \in I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ 0 \in I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ 0 \in I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I \subset \{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \bigoplus_{\substack{|I|=i-1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The bottom two rows are Cech complexes with respect to two covers. We get a long exact sequence of cohomology from this short exact sequence of complexes. Thus we wish to show that the top row is exact. But the ith cohomology of the top row is precisely $H^i_{\{U_i \cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off $0 \to \mathcal{F}(U_0) \to \oplus_{j=1}^n \mathcal{F}(U_0 \cap U_j)$). So it suffices to show that higher Cech groups of affine schemes are 0. Hence we are done by the following result. $\square$

2.3. *Theorem. — The higher Cech cohomology $H^i_{\mathcal{U}}(X, \mathcal{F})$ of an affine A-scheme X vanishes (for any affine cover $\mathcal{U}$, $i > 0$, and quasicoherent $\mathcal{F}$).*

Serre describes this as a partition of unity argument.

Proof. We want to show that the "extended" complex

$$0 \to \mathcal{F}(X) \to \oplus_{|I|=1} \mathcal{F}(U_I) \to \oplus_{|I|=2} \mathcal{F}(U_I) \to \cdots \tag{4}$$

(where the global sections are appended to the front) has no cohomology, i.e. is exact. We do this with a trick.

Suppose first that some U_i, say U_0, is X. Then the complex is the middle row of the following short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{|I|=1,0\in I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\in I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \cdots
\end{array}$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (Topological experts will recognize this as a *mapping cone* construction.)

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} R$. We wish to show that the complex of A-modules (4) is exact. It is also a complex of R-modules, so we wish to show that the complex of R-modules (4) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} R$ by distinguished open sets $D(f_i)$ $(1 \leq i \leq r)$ (i.e. $(f_1, \dots, f_r) = 1$ in R) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} A_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} A, \mathcal{F})_f = \Gamma(\operatorname{Spec}(A_j)_f, \mathcal{F})$$

(as this is one of the definitions of a quasicoherent sheaf), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. □

We have now proved properties **(i)–(iii)** of the previous section.

2.D. EXERCISE (PROPERTY **(v)**). Suppose $f : X \to Y$ is an affine morphism, and Y is a quasicompact and separated A-scheme (and hence X is too, as affine morphisms are both quasicompact and separated). If $\mathcal{F}$ is a quasicoherent sheaf on X, describe a natural isomorphism $H^i(Y, f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$. (Hint: if $\mathcal{U}$ is an affine cover of Y, "$f^{-1}(\mathcal{U})$" is an affine cover X. Use these covers to compute the cohomology of $\mathcal{F}$.)

2.E. EXERCISE (PROPERTY **(iv)**). Suppose $f : X \to Y$ is any quasicompact separated morphism, $\mathcal{F}$ is a quasicoherent sheaf on X, and Y is a quasicompact quasiseparated A-scheme. The hypotheses on f ensure that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y. Describe a natural morphism $H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F})$ extending $\Gamma(Y, f_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$.

2.F. UNIMPORTANT EXERCISE. Prove Property **(vii)** of the previous section.

2.4. Useful facts about cohomology for k-schemes.

2.G. EXERCISE (COHOMOLOGY AND CHANGE OF BASE FIELD). Suppose X is a projective k-scheme, and $\mathcal{F}$ is a coherent sheaf on X. Show that

$$h^0(X, \mathcal{F}) = h^0(X \times_{\operatorname{Spec} k} \operatorname{Spec} K, \mathcal{F} \otimes_k K)$$

where K/k is any field extension. Here $\mathcal{F} \otimes_k k$ means the pullback of $\mathcal{F}$ to $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. Note: the two sides of this equality are dimensions of vector spaces over different fields! (This is useful for relating facts about k-schemes to facts about schemes over algebraically closed fields.)

2.5. *Theorem. — Suppose* X *is a projective* k*-scheme, and* $\mathcal{F}$ *is a quasicoherent sheaf on* X*. Then* $H^i(X, \mathcal{F}) = 0$ *for* $i > \dim X$.

In other words, cohomology vanishes above the dimension of X. We will later show that this is true when X is a *quasiprojective* k-scheme.

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. A key Exercise from a couple of months ago shows that there are n effective Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \ldots, U_n$ cover X. Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. $\square$

Remark. We actually *need* n affine open sets to cover X, but I don't see an easy way to prove it. One way of proving it is by showing that the complement of an affine set is always pure codimension 1.

3. COHOMOLOGY OF LINE BUNDLES ON PROJECTIVE SPACE

We will finally prove the last promised basic fact about cohomology, property **(viii)** of §1, Theorem 1.1.

We saw earlier (Essential Exercise in class 27, end of section 3, and the ensuing discussion) that $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \ldots, x_n$ (with A-coefficients). Similarly, $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \ldots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof of Theorem 1.1. As stated above, we showed the H^0 case earlier.

Rather than consider $\mathcal{O}(m)$ for various m, we consider them all at once, by considering $\mathcal{F} = \oplus_m \mathcal{O}(m)$.

We take the standard cover $U_0 = D(x_0), \ldots, U_n = D(x_n)$ of $\mathbb{P}^n_A$. Notice that if $I \subset \{1, \ldots, n\}$, then $\mathcal{F}(U_I)$ corresponds to the Laurent monomials where each x_i for $i \notin I$ appears with non-negative degree.

We first consider the H^n statement. $H^n(\mathbb{P}^n_A, \mathcal{F})$ is the cokernel of the surjection

$$\oplus_{i=0}^n \mathcal{F}(U_{\{1,\ldots,n\}-\{i\}}) \to \mathcal{F}_{U_{\{1,\ldots,n\}}}$$

i.e.

$$\oplus_{i=0}^{n} A[x_0, \ldots, x_n, x_0^{-1}, \ldots, \widehat{x_i^{-1}}, \ldots x_n^{-1}] \to A[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}].$$

This cokernel is precisely as described.

We last consider the H^i statement ($0 < i < n$). (Strangely, the vanishing of these H^i is the hardest part of the Theorem.) We prove this by induction on n. The cases $n = 0$ and 1 are trivial. Consider the exact sequence of quasicoherent sheaves:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

where $\mathcal{F}'$ is analogous sheaf on the hyperplane $x_n = 0$ (isomorphic to $\mathbb{P}_A^{n-1}$). (This exact sequence is just the direct sum over all m of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}_A^n}(m) \longrightarrow \mathcal{O}_{\mathbb{P}_A^{n-1}}(m) \longrightarrow 0\ ,$$

which in turn is obtained by twisting the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}_A^n} \longrightarrow \mathcal{O}_{\mathbb{P}_A^{n-1}} \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}_A^n}(m)$.)

The long exact sequence in cohomology yields

$$0 \longrightarrow H^0(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^0(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}_A^{n-1}, \mathcal{F}') \quad .$$

$$\longrightarrow H^1(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^1(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^1(\mathbb{P}_A^{n-1}, \mathcal{F}')$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^{n-1}(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}')$$

$$\longrightarrow H^n(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow 0$$

We will now show that this gives an isomorphism

$$\times x_n : H^i(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\sim} H^i(\mathbb{P}_A^n, \mathcal{F}) \tag{5}$$

for $0 < i < n$. The inductive hypothesis gives us this except for $i = 1$ and $i = n-1$, where we have to be more careful. For the first, note that $H^0(\mathbb{P}_A^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}_A^{n-1}, \mathcal{F}')$ is surjective: this map corresponds to taking the set of all polynomials in $x_0, \ldots, x_n$, and setting $x_n = 0$. The last is slightly more subtle: $H^{n-1}(\mathbb{P}_A^{n-1}, \mathcal{F}') \to H^n(\mathbb{P}_A^n, \mathcal{F})$ is injective, and corresponds to taking a Laurent polynomial in $x_0, \ldots, x_{n-1}$ (where in each monomial, each x_i appears with degree at most -1) and multiplying by x_n^{-1}, which indeed describes the kernel of $H^n(\mathbb{P}_A^n, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}_A^n, \mathcal{F})$. (This is a worthwhile calculation! See Exercise 3.A below.) We have thus established (5) above.

We will now show that the localization $H^i(\mathbb{P}_A^n, \mathcal{F})_{x_n} = 0$. (Here's what we mean by localization. Notice $H^i(\mathbb{P}_A^n, \mathcal{F})$ is naturally a module over $A[x_0, \ldots, x_n]$ — we know how to multiply by elements of A, and by (5) we know how to multiply by x_i. Then we localize this at x_n to get an $A[x_0, \dots, x_n]_{x_n}$-module.) This means that each element $\alpha \in H^i(\mathbb{P}^n_A, \mathcal{F})$ is killed by some power of x_i. But by (5), this means that $\alpha = 0$, concluding the proof of the theorem.

Consider the Cech complex computing $H^i(\mathbb{P}^n_A, \mathcal{F})$. Localize it at x_n. Localization and cohomology commute (basically because localization commutes with operations of taking quotients, images, etc.), so the cohomology of the new complex is $H^i(\mathbb{P}^n_A, \mathcal{F})_{x_n}$. But this complex computes the cohomology of $\mathcal{F}_{x_n}$ on the affine scheme U_n, and the higher cohomology of *any* quasicoherent sheaf on an affine scheme vanishes (by Theorem 2.3 which we've just proved — in fact we used the same trick there), so $H^i(\mathbb{P}^n_A, \mathcal{F})_{x_n} = 0$ as desired. □

3.A. EXERCISE. Verify that $H^{n-1}(\mathbb{P}^{n-1}_A, \mathcal{F}') \to H^n(\mathbb{P}^n_A, \mathcal{F})$ is injective (likely by verifying that it is the map on Laurent monomials we claimed above).

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 37

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CONTENTS

1. APPLICATION OF COHOMOLOGY: HILBERT POLYNOMIALS AND FUNCTIONS, RIEMANN-ROCH, DEGREES, AND ARITHMETIC GENUS

We have now seen some powerful uses of Cech cohomology, to prove things about spaces of global sections, and to prove Serre vanishing. We will now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Define $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Define the **Euler characteristic**

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, notice also that the leading coefficient is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which is behaving well *always*, and (ii) the other cohomology groups vanish in that range.

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In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it can be easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.) We will see this in action when discussing curves.

The following exercise shows another way in which Euler characteristic behaves well: it is *additive in exact sequences*.

1.A. EXERCISE. Show that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

1.1. The Riemann-Roch Theorem for line bundles on a nonsingular projective curve. Suppose $\mathcal{L}$ is an invertible sheaf on a projective curve C over k. We tentatively define the **degree** of $\mathcal{L}$ as follows. Let s be a non-zero rational section on C. Let D be the divisor of zeros and poles of s:

$$D := \sum_{p \in C} \nu_p(s)[p]$$

Then define $\deg \mathcal{L} := \deg D = \sum \nu_p(s) \deg p$. Here $\deg p$ is the degree of the residue field of $\mathcal{O}_C$ at p, i.e. $\dim k\mathcal{O}_p = \deg p$. It isn't yet clear $\deg \mathcal{L}$ is well-defined: a priori it depends on the choice of s. Nonetheless you should prove the following.

1.B. EXERCISE: THE RIEMANN-ROCH THEOREM FOR LINE BUNDLES ON A NONSINGULAR PROJECTIVE CURVE. Show that

$$\chi(C, \mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C).$$

Here is a possible hint. Suppose $p \in C$ is a closed point of C, of degree d. Then twisting the closed exact sequence

$$0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$$

by $\mathcal{L}$ (as $\otimes \mathcal{L}$ is an exact functor) we obtain

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{O}_p \to 0$$

(where we are using a non-canonical isomorphism $\mathcal{L}|_p \cong \mathcal{O}_C|_p = \mathcal{O}_p$). Use the additivity of χ in exact sequences to show that the result is true for $\mathcal{L}$ if and only if it is true for $\mathcal{L}(-p)$. The result is also clearly true for $\mathcal{L} = \mathcal{O}$. Then argue by "induction" that it is true for all $\mathcal{L}$.

In particular, $\deg \mathcal{L}$ is well-defined!

1.C. EXERCISE. If $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on a nonsingular projective curve C, show that $\deg \mathcal{L} \otimes \mathcal{M} = \deg \mathcal{L} + \deg \mathcal{M}$. (Hint: choose rational sections of $\mathcal{L}$ and $\mathcal{M}$.)

In fact we could have *defined* the degree of a line bundle $\mathcal{L}$ on a nonsingular projective curve C to be $\chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$. Then Riemann-Roch would be true by definition; but we would still want to relate this notion of degree to the classical notion of zeros and poles, which we would do by the argument in the previous paragraph. Otherwise, for example, Exercise 1.C isn't obvious from the cohomological definition.

Definition. Suppose C is a reduced projective curve (pure dimension 1, over a field k). If $\mathcal{L}$ is a line bundle on C, define $\deg \mathcal{L} = \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)$. If C is irreducible, and $\mathcal{F}$ is a coherent sheaf on C, define the **rank** of $\mathcal{F}$, denoted $\operatorname{rank} \mathcal{F}$, to be its rank at the generic point of C.

1.D. EASY EXERCISE. Show that the rank is additive in exact sequences: if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves, show that $\operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G} + \operatorname{rank} \mathcal{H} = 0$.

Definition. Define

$$\deg \mathcal{F} = \chi(C, \mathcal{F}) - (\operatorname{rank} \mathcal{F}) \chi(C, \mathcal{O}_C). \tag{1}$$

If $\mathcal{F}$ is a line bundle, we can drop the hypothesis of irreducibility in the definition.

This generalizes the notion of the degree of a line bundle on a nonsingular curve.

1.E. EASY EXERCISE. Show that degree is additive in exact sequences.

The statement (1) is often called Riemann-Roch for coherent sheaves (or vector bundles) on a projective curve.

If $\mathcal{F}$ is a coherent sheaf on X, define the **Hilbert function of** $\mathcal{F}$:

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The **Hilbert function of** X is the Hilbert function of the structure sheaf. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the **Hilbert polynomial** (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). This polynomial contains lots of interesting geometric information, as we will soon see. In modern language, we expect that this "eventual polynomiality" arises because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

1.2. *Theorem. — If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme $X \hookrightarrow \mathbb{P}^n_k$, $\chi(X, \mathcal{F}(m))$ is a polynomial of degree equal to $\dim \operatorname{Supp} \mathcal{F}$. Hence by Serre vanishing (Theorem 1.2(ii) in the class*

35/36 notes), for $m \gg 0$, $h^0(X, \mathcal{F}(m))$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$. In particular, for $m \gg 0$, $h^0(X, \mathcal{O}_X(m))$ is polynomial with degree = $\dim X$.

Here $\mathcal{O}_X(m)$ is the restriction or pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Both the degree of the 0 polynomial and the dimension of the empty set is defined to be -1. In particular, the only coherent sheaf Hilbert polynomial 0 is the zero-sheaf.

Proof. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$. We will show that $p_{\mathcal{F}}(m)$ is a polynomial of the desired degree.

Step 1. Assume first that k is infinite. (This is one of those cases where even if you are concerned with potentially arithmetic questions over some non-algebraically closed field like $\mathbb{F}_p$, you are forced to consider the "geometric" situation where the base field is algebraically closed.)

$\mathcal{F}$ has a finite number of associated points. Then there is a hyperplane $x = 0$ ($x \in \Gamma(X, \mathcal{O}(1))$) missing this finite number of points. (This is where we use the algebraic closure, or more precisely, the infinitude of k.)

Then the map $\mathcal{F}(-1) \xrightarrow{\times x} \mathcal{F}$ is injective (on any affine open subset, $\mathcal{F}$ corresponds to a module, and x is not a zero-divisor on that module, as it doesn't vanish at any associated point of that module). Thus we have a short exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0 \qquad (2)$$

where $\mathcal{G}$ is a coherent sheaf.

1.F. EXERCISE. Show that $\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap V(x)$.

Hence $\dim \operatorname{Supp} \mathcal{G} = \dim \operatorname{Supp} \mathcal{F} - 1$ by Krull's Principal Ideal Theorem unless $\mathcal{F} = 0$ (in which case we already know the result, so assume this is not the case).

Twisting (2) by $\mathcal{O}(m)$ yields

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{G}(m) \to 0$$

Euler characteristics are additive in exact sequences, from which $p_{\mathcal{F}}(m) - p_{\mathcal{F}}(m-1) = p_{\mathcal{G}}(m)$. Now $p_{\mathcal{G}}(m)$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F} - 1$.

The result follows from a basic fact about polynomials.

1.G. EXERCISE. Suppose f and g are functions on the integers, $f(m+1) - f(m) = g(m)$ for all m, and $g(m)$ is a polynomial of degree $d \geq 0$. Show that f is a polynomial of degree $d+1$.

Step 2: k finite.

1.H. EXERCISE. Complete the proof using Exercise 2.G from the notes from class 35/36 (on cohomology and change of base field), using $K = \overline{k}$.

$\square$

Definition. $p_{\mathcal{F}}(m)$ was defined in the above proof. If $X \subset \mathbb{P}^n$ is a projective k-scheme, define $p_X(m) := p_{\mathcal{O}_X}(m)$.

Example 1. $p_{\mathbb{P}^n}(m) = \binom{m+n}{n}$, where we interpret this as the polynomial $(m+1)\cdots(m+n)/n!$.

Example 2. Suppose H is a degree d hypersurface in $\mathbb{P}^n$. Then from the closed subscheme exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0,$$

we have

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

1.I. EXERCISE. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m+1$.

1.J. EXERCISE. Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$).

From the Hilbert polynomial, we can extract many invariants, of which two are particularly important. The first is the *degree*. Classically, the degree of a complex projective variety of dimension n was defined as follows. We slice the variety with n generally chosen hyperplane. Then the intersection will be a finite number of points. The degree is this number of points. Of course, this requires showing all sorts of things. Instead, we will define the *degree of a projective k-scheme of dimension* n to be leading coefficient of the Hilbert polynomial (the coefficient of m^n) times $n!$.

Using the examples above, we see that the degree of $\mathbb{P}^n$ in itself is 1. The degree of the twisted cubic is 3.

1.K. EXERCISE. Show that the degree is always an integer. Hint: by induction, show that any polynomial in m of degree k taking on only integral values must have coefficient of m^k an integral multiple of $1/k!$. Hint for this: if $f(x)$ takes on only integral values and is of degree k, then $f(x+1) - f(x)$ takes on only integral values and is of degree $k-1$.

1.L. EXERCISE. Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

1.M. EXERCISE. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. In other words, this line bundle determines a closed immersion. Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch, Exercise 1.B.)

1.N. EXERCISE. Show that the degree of the dth Veronese embedding of $\mathbb{P}^n$ is d^n.

1.O. EXERCISE (BEZOUT'S THEOREM). Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

This is a very handy theorem! For example: if two projective plane curves of degree m and degree n share no irreducible components, then they intersect in mn points, counted with appropriate multiplicity. The notion of multiplicity of intersection is just the degree of the intersection as a k-scheme.

We trot out a useful example we have used before: let $k = \mathbb{Q}$, and consider the parabola $x = y^2$. We intersect it with the four lines, $x = 1$, $x = 0$, $x = -1$, and $x = 2$, and see that we get 2 each time (counted with the same convention as with the last time we saw this example).

If we intersect it with $y = 2$, we only get one point — but that's of course because this isn't a projective curve, and we really should be doing this intersection on $\mathbb{P}^2_k$ — and in this case, the conic meets the line in two points, one of which is "at ∞".

]

1.P. EXERCISE. Show that the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ is d^n in a different way (from Exercise 1.N) as follows. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image, we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem (Exercise 1.O).

There is another nice important of information residing in the Hilbert polynomial. Notice that $p_X(0) = \chi(X, \mathcal{O}_X)$, which is an *intrinsic* invariant of the scheme X, which does not depend on the projective embedding.

Imagine how amazing this must have seemed to the ancients: they defined the Hilbert function by counting how many "functions of various degrees" there are; then they noticed that when the degree gets large, it agrees with a polynomial; and then when they plugged 0 into the polynomial — extrapolating backwards, to where the Hilbert function and Hilbert polynomials didn't agree — they found a magic invariant!

Now we can finally see a nonsingular curve over an algebraically closed field that is provably not $\mathbb{P}^1$! Note that the Hilbert polynomial of $\mathbb{P}^1$ is $(m+1)/1 = m+1$, so $\chi(\mathcal{O}_{\mathbb{P}^1}) =$

1. Suppose C is a degree d curve in $\mathbb{P}^2$. Then the Hilbert polynomial of C is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m-d) = (m+1)(m+2)/2 - (m-d+1)(m-d+2)/2.$$

Plugging in $m = 0$ gives us $-(d^2 - 3d)/2$. Thus when $d > 2$, we have a curve that cannot be isomorphic to $\mathbb{P}^1$! (I think I gave you an earlier exercise that there is a *nonsingular* degree d curve.)

Now from $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$, using $h^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$, we have that $h^0(C, \mathcal{O}_C) = 1$. As $h^0 - h^1 = \chi$, we have

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2.$$

Motivated by geometry, we define the **arithmetic genus** of a scheme X as $1-\chi(X, \mathcal{O}_X)$. This is sometimes denoted $p_a(X)$. In the case of nonsingular complex curves, this corresponds to the topological genus. For irreducible reduced curves (or more generally, curves with $h^0(X, \mathcal{O}_X) \cong k$), $p_a(X) = h^1(X, \mathcal{O}_X)$. (In higher dimension, this is a less natural notion.)

We thus now have examples of curves of genus $0, 1, 3, 6, 10, \dots$ (corresponding to degree 1 or $2, 3, 4, 5, \dots$).

This begs some questions, such as: are there curves of other genera? (We'll see soon that the answer is yes.) Are there other genus 1 curves? (Not if k is algebraically closed, but yes otherwise.) Do we have all the curves of genus 3? (Almost all, but not quite.) Do we have all the curves of genus 6? (We're missing most of them.)

Caution: The Euler characteristic of the structure sheaf doesn't distinguish between isomorphism classes of nonsingular projective schemes over algebraically closed fields — for example, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ both have Euler characteristic 1, but are not isomorphic — $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ while $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$.

Important Remark. We can restate the Riemann-Roch formula for curves (Exercise 1.B) as:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - p_a + 1.$$

This is the most common formulation of the Riemann-Roch formula.

If C is a nonsingular irreducible projective complex curve, then the corresponding complex-analytic object, a compact *Riemann surface*, has a notion called the *genus* g, which is the number of holes. It turns out that $g = p_a$ in this case, and for this reason, we will often write g for p_a when discussing nonsingular (projective irreducible) curves, over any field.

1.3. *Complete intersections.* We define a **complete intersection** in $\mathbb{P}^n$ as follows. $\mathbb{P}^n$ is a complete intersection in itself. A closed subscheme $X_r \hookrightarrow \mathbb{P}^n$ of dimension r (with $r < n$) is a complete intersection if there is a complete intersection X_{r+1}, and X_r is an effective Cartier divisor in class $\mathcal{O}_{X_{r+1}}(d)$.

1.Q. EXERCISE. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective. (Hint: long exact sequences.)

Now X_r is the divisor of a section of $\mathcal{O}_{X_{r+1}}(m)$ for some m. But this section is the restriction of a section of $\mathcal{O}(m)$ on $\mathbb{P}^n$. Hence X_r is the scheme-theoretic intersection of X_{r+1} with a hypersurface. Thus inductively X_r is the scheme-theoretic intersection of $n-r$ hypersurfaces. (By Bezout's theorem, Exercise 1.O, $\deg X_r$ is the product of the degree of the defining hypersurfaces.)

1.R. EXERCISE (COMPLETE INTERSECTIONS ARE CONNECTED). Show that complete intersections of *positive* dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

1.S. EXERCISE. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this! At this point we need to worry about whether there are any nonsingular curves of this form. We can check this by hand, but later Bertini's Theorem will save us this trouble.)

1.T. EXERCISE. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$. (Hint: If it *were* the complete intersection of $d-1$ hypersurfaces, what would the degree of the hypersurfaces be? Why could none of the degrees be 1?)

1.U. EXERCISE. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. Hint: it is connected, but you can slice with another plane and get something not connected (see Exercise 1.R).

This is another important scheme in algebraic geometry that is an example of many sorts of behavior. We will see more of it later!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 38

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CONTENTS

1. A useful very general fact from homological algebra

Here is a fact that is very useful, because it applies in so many situations.

1.A. IMPORTANT EXERCISE IN ABSTRACT NONSENSE. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant additive functor from one abelian category to another. Suppose $C^\bullet$ is a complex in $\mathcal{A}$.

(a) Describe a natural morphism $FH^\bullet \to H^\bullet F$.
(b) If F is exact, show that the morphism of (a) is an isomorphism.

If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathcal{A}$ and $\mathcal{B}$ are the category of A-modules, and F is $\cdot \otimes N$ for some fixed N-module. Your argument in this case will translate without change to yield a solution to Exercise 1.A. If $\otimes N$ is exact, then N is called a **flat** A-module.

For example, localization is exact, so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets S. Thus taking cohomology of a complex of A-modules commutes with localization — something you could verify directly.

2. Higher direct image sheaves

Cohomology groups were defined for $X \to \operatorname{Spec} A$ where the structure morphism is quasicompact and separated; for any quasicoherent $\mathcal{F}$ on X, we defined $H^i(X, \mathcal{F})$. We'll now define a "relative" version of this notion, for quasicompact and separated morphisms $\pi : X \to Y$: for any quasicoherent $\mathcal{F}$ on X, we'll define $R^i\pi_*\mathcal{F}$, a quasicoherent sheaf on Y.

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We have many motivations for doing this. In no particular order:

(1) It "globalizes" what we did before.
(2) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we know that $0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \pi_*\mathcal{H}$ is exact, and higher pushforwards will extend this to a long exact sequence.
(3) We'll later see that this will show how cohomology groups vary in families, especially in "nice" situations. Intuitively, if we have a nice family of varieties, and a family of sheaves on them, we could hope that the cohomology varies nicely in families, and in fact in "nice" situations, this is true. (As always, "nice" usually means "flat", whatever that means.)

All of the important properties of cohomology described earlier will carry over to this more general situation. Best of all, there will be no extra work required.

In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and corresponds to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). Later this year, we will see that in good circumstances, if we have a left-exact functor, there is be a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Suppose $\pi : X \to Y$, and $\mathcal{F}$ is a quasicoherent sheaf on X. For each $\operatorname{Spec} A \subset Y$, we have A-modules $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$. We will show that these patch together to form a quasicoherent sheaf. We need check only one fact: that this behaves well with respect to taking distinguished open sets. In other words, we must check that for each $f \in A$, the natural map $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F}) \to H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})_f$ (induced by the map of spaces in the opposite direction — H^i is contravariant in the space) is precisely the localization $\otimes_A A_f$. But this can be verified easily: let $\{U_i\}$ be an affine cover of $\pi^{-1}(\operatorname{Spec} A)$. We can compute $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$ using the Cech complex. But this induces a cover $\operatorname{Spec} A_f$ in a natural way: If $U_i = \operatorname{Spec} A_i$ is an affine open for $\operatorname{Spec} A$, we define $U_i' = \operatorname{Spec}(A_i)_f$. The resulting Cech complex for $\operatorname{Spec} A_f$ is the localization of the Cech complex for $\operatorname{Spec} A$. As taking cohomology of a complex commutes with localization (as discussed in Exercise 1.A), we have defined a quasicoherent sheaf on Y by one of our definitions of quasicoherent sheaves by Definition 2′ of a quasicoherent sheaf.

Define the **ith higher direct image sheaf** or the **ith (higher) pushforward sheaf** to be this quasicoherent sheaf.

2.1. *Theorem. —*

(a) $R^0\pi_*\mathcal{F}$ *is canonically isomorphic to* $\pi_*\mathcal{F}$.
(b) $R^i\pi_*$ *is a covariant functor from the category of quasicoherent sheaves on* X *to the category of quasicoherent sheaves on* Y, *and a contravariant functor in* Y-*schemes* X.

(c) *(the* **long exact sequence of higher pushforward sheaves***) A short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves on* X *induces a long exact sequence*

$$0 \longrightarrow R^0\pi_*\mathcal{F} \longrightarrow R^0\pi_*\mathcal{G} \longrightarrow R^0\pi_*\mathcal{H} \longrightarrow$$

$$R^1\pi_*\mathcal{F} \longrightarrow R^1\pi_*\mathcal{G} \longrightarrow R^1\pi_*\mathcal{H} \longrightarrow \cdots$$

of sheaves on Y.

(d) *(projective pushforwards of coherent are coherent) If* π *is a projective morphism and* $\mathcal{O}_Y$ *is coherent on* Y *(this hypothesis is automatic for* Y *locally Noetherian), and* $\mathcal{F}$ *is a coherent sheaf on* X*, then for all* i, $R^i\pi_*\mathcal{F}$ *is a coherent sheaf on* Y.

Proof. Because it suffices to check each of these results on affine open sets, they all follow from the analogous statements in Cech cohomology. □

The following result is handy, and essentially immediate from our definition.

2.A. EXERCISE. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$.

Remark. This is in fact a characterization of affineness. Serre's criterion for affineness states that if f is quasicompact and separated, then f is affine if and only if f_* is an exact functor from the category of quasicoherent sheaves on X to the category of quasicoherent sheaves on Y. exact on the category of quasicoherent sheaves (EGA II.5.2). We won't use this fact.

2.B. EXERCISE (HIGHER PUSHFORWARDS AND COMMUTATIVE DIAGRAMS). (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$ of sheaves on Z. (Hint: Exercise 1.A.)

(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, and the diagram in (a) is a fiber square, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

A useful special case of (a) if the following.

2.C. EXERCISE. Show that if $y \in Y$, there is a natural morphism $H^i(Y, f_*\mathcal{F})_y \to H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$. (Hint: if you take a complex, and tensor it with a module, and take cohomology, there is a map from that to what you would get if you take cohomology and tensor it with a module.)

We'll later see that in good situations this is an isomorphism, and thus the higher direct image sheaf indeed "patches together" the cohomology on fibers.

2.D. EXERCISE (PROJECTION FORMULA). Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(Hint: Exercise 1.A.) (b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

3. FUN APPLICATIONS OF THE HIGHER PUSHFORWARD

Here are a series of useful geometric facts shown using similar tricks.

3.1. *Theorem (relative dimensional vanishing). — If $f : X \to Y$ is a projective morphism and $\mathcal{O}_Y$ is coherent, then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.*

This is false without the projective hypothesis, as shown by the following exercise.

3.A. EXERCISE. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n-0} \neq 0$.

Proof of Theorem 3.1. Let m be the maximum dimension of all the fibers.

The question is local on Y, so we'll show that the result holds near a point p of Y. We may assume that Y is affine, and hence that $X \hookrightarrow \mathbb{P}^n_Y$.

Let k be the residue field at p. Then $f^{-1}(p)$ is a projective k-scheme of dimension at most m. Thus we can find affine open sets $D(f_1), \dots, D(f_{m+1})$ that cover $f^{-1}(p)$. In other words, the intersection of $V(f_i)$ does not intersect $f^{-1}(p)$.

If $Y = \operatorname{Spec} A$ and $p = [\mathfrak{p}]$ (so $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$), then arbitrarily lift each f_i from an element of $k[x_0, \dots, x_n]$ to an element f_i' of $A_\mathfrak{p}[x_0, \dots, x_n]$. Let F be the product of the denominators of the f_i'; note that $F \notin \mathfrak{p}$, i.e. $p = [\mathfrak{p}] \in D(F)$. Then $f_i' \in A_F[x_0, \dots, x_n]$. The intersection of their zero loci $\cap V(f_i') \subset \mathbb{P}^n_{A_F}$ is a closed subscheme of $\mathbb{P}^n_{A_F}$. Intersect it with X to get another closed subscheme of $\mathbb{P}^n_{A_F}$. Take its image under f; as projective morphisms are closed, we get a closed subset of $D(F) = \operatorname{Spec} A_F$. But this closed subset does not include p; hence we can find an affine neighborhood $\operatorname{Spec} B$ of p in Y missing the image. But if f_i'' are the restrictions of f_i' to $B[x_0, \dots, x_n]$, then $D(f_i'')$ cover $f^{-1}(\operatorname{Spec} B)$; in other words, over $f^{-1}(\operatorname{Spec} B)$ is covered by $m + 1$ affine open sets, so by the affine-cover vanishing theorem, its cohomology vanishes in degree at least $m + 1$. But the higher-direct image sheet is computed using these cohomology groups, hence the higher direct image sheaf $R^i f_* \mathcal{F}$ vanishes on $\operatorname{Spec} B$ too. $\square$

3.B. IMPORTANT EXERCISE. Use a similar argument to prove *semicontinuity of fiber dimension of projective morphisms*: suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave!

Here is another handy theorem, that is proved by a similar argument. We know that finite morphisms are projective, and have finite fibers. Here is the converse.

3.2. *Theorem (projective + finite fibers = finite). — Suppose $\pi : X \to Y$ is such that $\mathcal{O}_Y$ is coherent. Then π is projective and finite fibers if and only if it is finite. Equivalently, π is projective and quasifinite if and only it is finite.*

(Recall that quasifinite = finite fibers + finite type. But projective includes finite type.)

It is true more generally that proper + quasifinite = finite.

Proof. We show it is finite near a point $y \in Y$. Fix an affine open neighborhood $\operatorname{Spec} A$ of y in Y. Pick a hypersurface H in $\mathbb{P}^n_A$ missing the preimage of y, so $H \cap X$ is closed. (You can take this as a hint for Exercise 3.B!) Let $H' = \pi_*(H \cap X)$, which is closed, and doesn't contain y. Let $U = \operatorname{Spec} R - H'$, which is an open set containing y. Then above U, π is projective and affine, so we are done by the Corollary from last day (that projective + affine = finite). $\square$

Here is one last potentially useful fact.

3.C. EXERCISE. Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ the invertible sheaf on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) [Should I relate this to fact 1.A?] Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^i f_* \mathcal{F}(n) = 0$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 39

RAVI VAKIL

CONTENTS

We'll conclude this quarter by discussing derived functor cohomology, which was introduced by Grothendieck in his celebrated Tôhoku article. For quasicoherent sheaves on quasicompact separated schemes, derived functor will agree with Cech cohomology. Furthermore, Cech cohomology will suffice for most of our purposes, and is quite down to earth and computable. But derived functor cohomology is worth seeing for a number of reasons. First of all, it generalizes readily to a wide number of situations. Second, it will easily provide us with some useful notions, such as Ext-groups and the Leray spectral sequence.

But to be honest, we won't use it much for the rest of the course, so feel free to just skim these notes, and come back to them later.

1. The Tor functors

We begin with a warm-up: the case of Tor. This is a hands-on example. But if you understand it well, you will understand derived functors in general. Tor will be useful to prove facts about flatness, which we'll discuss later. Tor is short for "torsion". The reason for this name is that the 0th and/or 1st Tor-group measures common torsion in abelian groups (aka $\mathbb{Z}$-modules).

If you have never seen this notion before, you may want to just remember its properties, which are natural. But I'd like to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence,

Date: Monday, March 10, 2008.

we are hoping to see a long exact sequence

$$(1)\qquad \cdots \longrightarrow \mathrm{Tor}_i^A(M,N') \longrightarrow \mathrm{Tor}_i^A(M,N) \longrightarrow \mathrm{Tor}_i^A(M,N'') \longrightarrow \cdots$$

$$\longrightarrow \mathrm{Tor}_1^A(M,N') \longrightarrow \mathrm{Tor}_1^A(M,N) \longrightarrow \mathrm{Tor}_1^A(M,N'')$$

$$\longrightarrow M\otimes_A N' \longrightarrow M\otimes_A N \longrightarrow M\otimes_A N'' \longrightarrow 0.$$

More precisely, we are hoping for *covariant functors* $\mathrm{Tor}_i^A(\cdot,N)$ from A-modules to A-modules (giving 2/3 of the morphisms in that long exact sequence), with $\mathrm{Tor}_0^A(M,N) \equiv M\otimes_A N$, and natural δ morphisms $\mathrm{Tor}_{i+1}^A(M,N'') \to \mathrm{Tor}_i^A(M,N')$ for every short exact sequence giving the long exact sequence. (In case you care, "natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute. I'm not going to state this explicitly.)

It turns out to be not too hard to make this work, and this will also motivate derived functors. Let's now define $\mathrm{Tor}_i^A(M,N)$.

Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \longrightarrow 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite, although if N is a finitely-generated module and A is Noetherian (or more generally if N is coherent and A is coherent over itself), we can choose the n_i to be finite. Truncate the resolution, by stripping off the last term. Then tensor with M (which may lose exactness!). Let $\mathrm{Tor}^i_A(M,N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of the resolution.

We make some quick observations.

• $\mathrm{Tor}_0^A(M,N)_{\mathcal{R}} \cong M\otimes_A N$, and this isomorphism is canonical. Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M\otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M\otimes_A N$.

• If $M\otimes\cdot$ is exact (i.e. M is *flat*), then $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} = 0$ for all i.

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{\oplus n_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n_1} & \longrightarrow & A^{\oplus n_0} & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n'_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n'_1} & \longrightarrow & A^{\oplus n'_0} & \longrightarrow & N' & \longrightarrow & 0
\end{array}$$

Here we are using the freeness of $A^{\otimes n_i}$: if $a_1, \ldots, a_{n_i}$ are generators of $A^{\otimes n_i}$, to lift the map $b : A^{\otimes n_i} \to A^{\otimes n'_{i-1}}$ to $c : A^{\otimes n_i} \to A^{\otimes n'_i}$, we arbitrarily lift $b(a_i)$ from $A^{\otimes n'_{i-1}}$ to $A^{\otimes n'_i}$, and declare this to be $c(a_i)$.

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes (remove column $N \to N'$) and tensor with M. Maps of complexes induce maps of homology, so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_* \to C'_*$ are **homotopic** if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$. Two homotopic maps give the same map on homology. (Exercise: verify this if you haven't seen this before.)

1.A. CRUCIAL EXERCISE. Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

(1) We get a covariant functor from $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}$, independent of the lift $\mathcal{R} \to \mathcal{R}'$.
(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ we get a canonical isomorphism $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \cong \mathrm{Tor}_i^1(M, N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R} \to \mathcal{R}'} : \mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M, N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}' \to \mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}' \to \mathcal{R}} \circ f_{\mathcal{R} \to \mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R} \to \mathcal{R}'} \circ f_{\mathcal{R}' \to \mathcal{R}}$ is the identity.
(3) Hence the covariant functor doesn't depend on the resolutions!

Finally:
(4) For any short exact sequence we get a long exact sequence of Tor's (1). Here's why: given a short exact sequence, choose resolutions of N' and N''. Then use these to get a resolution for N in the obvious way (see below; the map $A^{\oplus(n'_0 \to n''_0)} \to N$ is the composition $A^{\oplus n'_0} \to N' \to N$ along with any lift of $A^{n''_0} \to N''$ to N) so that we have a short exact

sequence of resolutions

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_1'} & \longrightarrow & A^{\oplus n_0'} & \longrightarrow & N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus (n_1'+n_1'')} & \longrightarrow & A^{\oplus (n_0'+n_0'')} & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_1''} & \longrightarrow & A^{\oplus n_0''} & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then truncate (removing the right column $0 \to N' \to N \to N'' \to 0$), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding a long exact sequence.

We have thus established the foundations of Tor!

Note that if N is a free module, then $\mathrm{Tor}_i^A(M, N) = 0$ for all M and all $i > 0$, as N has the trivial resolution $0 \to N \to N \to 0$ (it is "its own resolution").

1.B. EXERCISE. Show that the following are equivalent conditions on an A-module M.

(i) M is flat
(ii) $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and all A-modules N,
(iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

2. FROM TOR TO DERIVED FUNCTORS IN GENERAL

2.1. *Projective resolutions.* We used very little about free modules in the above construction of Tor; in fact we used only that free modules are **projective**, i.e. those modules M such that for any surjection $M' \to M''$, it is possible to lift any morphism $M \to M''$ to $M \to M'$. This is summarized in the following diagram.

$$\begin{array}{ccc}
M & & \\
{\scriptstyle \text{exists}}\downarrow & \searrow & \\
M' & \twoheadrightarrow & M''
\end{array}$$

Equivalently, $\mathrm{Hom}(M, \cdot)$ is an *exact functor* ($\mathrm{Hom}(M, \cdot)$ is always left-exact for any M). More generally, the same idea yields the definition of a **projective object in any abelian category**. Hence (i) we can compute $\mathrm{Tor}_i^A(M, N)$ by taking any projective resolution of N, and (ii) $\mathrm{Tor}_i^A(M, N) = 0$ for any projective A-module N.

2.A. INTERESTING EXERCISE: DERIVED FUNCTORS CAN BE COMPUTED USING ACYCLIC RESOLUTIONS. Show that you can also compute derived functor cohomology using *flat resolutions*, i.e. by a resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$$

by flat A-modules. Hint: show that you can construct a double complex

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_{2,1}} & \longrightarrow & A^{\oplus n_{1,1}} & \longrightarrow & A^{\oplus n_{0,1}} & \longrightarrow & A^{\oplus n_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_{2,0}} & \longrightarrow & A^{\oplus n_{1,0}} & \longrightarrow & A^{\oplus n_{0,0}} & \longrightarrow & A^{\oplus n_0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

where the rows and columns are exact. Do this by constructing the $A^{\oplus ??}$ inductively from the bottom left. Tensor the double complex with M, to obtain a new double complex. Remove the bottom row, and the right-most nonzero column. Use a spectral sequence argument to show that (i) the double complex has homology equal to Tor, and (ii) the homology of the double complex agrees with the homology of the free resolution (truncated) tensored with M.

You will notice in the solution to the above exercise that what mattered was that flat modules had no higher Tor's (Exercise 1.B). This will later directly generalize to the statements that *derived functors can be computed with acyclic resolutions* ("acyclic" means "no higher (co)homology").

2.2. Derived functors of right-exact functors.

The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A$ is a right-exact functor. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ (called left-derived functors of F) such that $L_0 F = F$ and the L_i's give a long-exact sequence. We can make this more general still. We say that an abelian category **has enough projectives** if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact functor from an abelian category with enough projectives to any abelian category, then F has left-derived functors.

2.B. UNIMPORTANT EXERCISE. Show that an object P is projective if and only if every short exact sequence $0 \to A \to B \to P \to 0$ splits.

2.C. EXERCISE. The notion of an **injective object** in an abelian category is dual to the notion of a projective object. (a) State precisely the definition of an injective object. (b) Define derived functors for (i) covariant left-exact functors (these are called **right-derived**

functors), (ii) contravariant left-exact functors (also called **right-derived functors**), and (iii) contravariant right-exact functors (these are called **left-derived functors**), making explicit the necessary assumptions of the category having enough injectives or projectives.

2.3. *Notation.* If F is a right-exact functor, its (left-)derived functors are denoted L_iF ($i \geq 0$, with $L_0F = F$). If F is a left-exact functor, its (right-) derived functors are denoted R^iF.

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SPECTRAL SEQUENCES: FRIEND OR FOE?

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Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams. They were introduced by Leray in the 1940's at the same time as he introduced sheaves. They have a reputation for being abstruse and difficult. It has been suggested that the name 'spectral' was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.

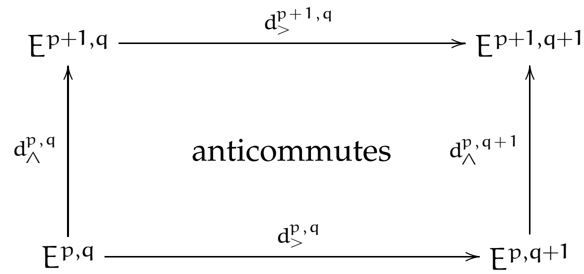
Nonetheless, the goal of this note is to tell you enough that you can use spectral sequences without hesitation or fear, and why you shouldn't be frightened when they come up in a seminar. What is different in this presentation is that we will use spectral sequence to prove things that you may have already seen, and that you can prove easily in other ways. This will allow you to get some hands-on experience for how to use them. We will also see them only in a "special case" of double complexes (which is the version by far the most often used in algebraic geometry), and not in the general form usually presented (filtered complexes, exact couples, etc.). See chapter 5 of Weibel's marvelous book for more detailed information if you wish. If you want to become comfortable with spectral sequences, you *must* try the exercises.

For concreteness, we work in the category vector spaces over a given field. However, everything we say will apply in any abelian category, such as the category $\mathbf{Mod}_A$ of A-modules.

0.1. Double complexes.

A **first-quadrant double complex** is a collection of vector spaces $E^{p,q}$ ($p, q \in \mathbb{Z}$), which are zero unless $p, q \geq 0$, and "rightward" morphisms $d_>^{p,q} : E^{p,q} \to E^{p,q+1}$ and "upward" morphisms $d_\wedge^{p,q} : E^{p,q} \to E^{p+1,q}$. In the superscript, the first entry denotes the row number, and the second entry denotes the column number, in keeping with the convention for matrices, but opposite to how the (x, y)-plane is labeled. The subscript is meant to suggest the direction of the arrows. We will always write these as $d_>$ and $d_\wedge$ and ignore the superscripts. We require that $d_>$ and $d_\wedge$ satisfying (a) $d_>^2 = 0$, (b) $d_\wedge^2 = 0$, and one more condition: (c) either $d_> d_\wedge = d_\wedge d_>$ (all the squares commute) or $d_> d_\wedge + d_\wedge d_> = 0$ (they all anticommute). Both come up in nature, and you can switch from one to the other by replacing $d_\wedge^{p,q}$ with $d_\wedge^p (-1)^q$. So I'll assume that all the squares anticommute, but that you know how to turn the commuting case into this one. (You will see that there is no difference in the recipe, basically because the image and kernel of a homomorphism f equal the image and kernel respectively of $-f$.)

Date: Tuesday, March 12, 2008. Updated version later Tuesday afternoon.



There are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $E^{p,q}$ are required to be zero, but I'll leave these straightforward variations to you.

From the double complex we construct a corresponding (single) complex $E^\bullet$ with $E^k = \oplus_i E^{i,k-i}$, with $d = d_> + d_\wedge$. In other words, when there is a *single* superscript k, we mean a sum of the kth antidiagonal of the double complex. The single complex is somtimes called the **total complex**. Note that $d^2 = (d_> + d_\wedge)^2 = d_>^2 + (d_> d_\wedge + d_\wedge d_>) + d_\wedge^2 = 0$, so $E^\bullet$ is indeed a complex.

The **cohomology** of the single complex is sometimes called the **hypercohomology** of the double complex. We will instead use the phrase "cohomology of the double complex".

Our initial goal will be to find the cohomology of the double complex. You will see later that we secretly also have other goals.

A spectral sequence is a recipe for computing some information about the cohomology of the double complex. I won't yet give the full recipe. Surprisingly, this fragmentary bit of information is sufficent to prove lots of things.

0.2. Approximate Definition. A **spectral sequence** with **rightward orientation** is a sequence of tables or **pages** ${}_>E_0^{p,q}$, ${}_>E_1^{p,q}$, ${}_>E_2^{p,q}$, ... $(p, q \in \mathbb{Z})$, where ${}_>E_0^{p,q} = E^{p,q}$, along with a differential

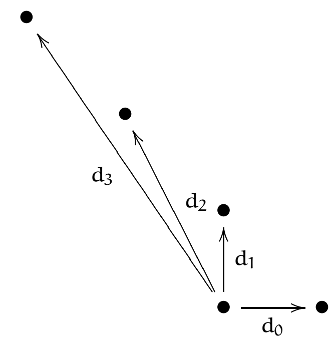
$$ {}_>d_r^{p,q} : {}_>E_r^{p,q} \to {}_>E^{p+r,q-r+1} $$

with ${}_>d_r^{p,q} \circ {}_>d_r^{p,q} = 0$, along with an isomorphism of the cohomology of ${}_>d_r$ at ${}_>E^{p,q}$ (i.e. $\ker {}_>d_r^{p,q} / \operatorname{im} {}_>d_r^{p-r,q+r-1}$) with ${}_>E_{r+1}^{p,q}$.

The orientation indicates that our 0th differential is the rightward one: $d_0 = d_>$. The left subscript ">" is usually omitted.

The order of the morphisms is best understood visually:

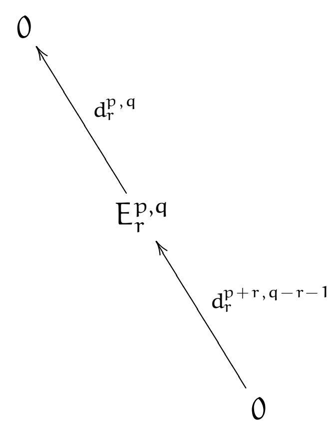
(1)



(the morphisms each apply to different pages). Notice that the map always is "degree 1" in the grading of the single complex $E^\bullet$.

The actual definition describes what $E_r^{\bullet,\bullet}$ and $d_r^{\bullet,\bullet}$ actually are, in terms of $E^{\bullet,\bullet}$. We will describe d_0, d_1, and d_2 below, and you should for now take on faith that this sequence continues in some natural way.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the 0th page $E_0^{p,q} = E^{p,q}$. In particular, if $E^{p,q} = 0$, then $E_r^{p,q} = 0$ for all r, so $E_r^{p,q} = 0$ unless $p, q \in \mathbb{Z}^{\geq 0}$. Notice also that for any fixed p, q, once r is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex



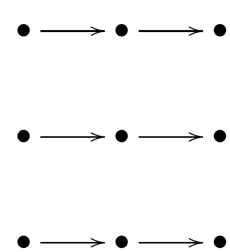
and thus we have canonical isomorphisms

$$E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$$

We denote this module $E_\infty^{p,q}$.

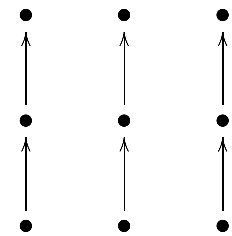
We now describe the first few pages of the spectral sequence explicitly. As stated above, the differential d_0 on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_>$. The rows are complexes:

The 0th page E_0:



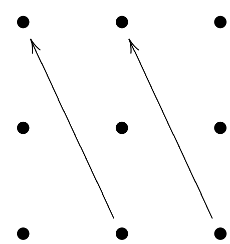
and so E_1 is just the table of cohomologies of the rows. You should check that there are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ of the row cohomology groups, induced by $d_\wedge$, and that these make the columns into complexes. (We have "used up the horizontal morphisms", but "the vertical differentials live on".)

The 1st page E_1:



We take cohomology of d_1 on E_1, giving us a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is 0. (It is a very worthwhile exercise to work out how this natural morphism d_2 should be defined. Your argument may be reminiscent of the connecting homomorphism in the Snake Lemma 0.5 or in the long exact sequence in cohomology arising from a short exact sequence of complexes, Exercise 0.D. This is no coincidence.)

The 2nd page E_2:



This is the beginning of a pattern.

Then it is a theorem that there is a filtration of $H^k(E^\bullet)$ by $E_\infty^{p,q}$ where $p+q=k$. (We can't yet state it as an official **Theorem** because we haven't precisely defined the pages and differentials in the spectral sequence.) More precisely, there is a filtration

$$E_\infty^{0,k} \xhookrightarrow{E_\infty^{1,k-1}} \;?\; \xhookrightarrow{E_\infty^{2,k-2}} \cdots \xhookrightarrow{E^{k,0}} H^k(E^\bullet) \tag{2}$$

where the quotients are displayed above each inclusion. (I always forget which way the quotients are supposed to go, i.e. whether $E^{k,0}$ or $E^{0,k}$ is the subobject. One way of remembering it is by having some idea of how the result is proved.)

We say that the spectral sequence ${}_{>}E_\bullet^{\bullet,\bullet}$ **converges** to $H^\bullet(E^\bullet)$. We often say that ${}_{>}E_2^{\bullet,\bullet}$ (or any other page) abuts to $H^\bullet(E^\bullet)$.

Although the filtration gives only partial information about $H^\bullet(E^\bullet)$, sometimes one can find $H^\bullet(E^\bullet)$ precisely. One example is if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero (e.g. if $E_r^{i,k-i}$ has precisely one non-zero row or column, in which case one says that the spectral sequence *collapses* at the rth step, although we will not use this term). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^\bullet)$. Also, in lucky circumstances, E_2 (or some other small page) already equals E_∞.

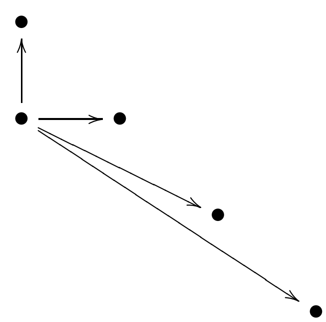
0.A. EXERCISE: INFORMATION FROM THE SECOND PAGE. Show that $H^0(E^\bullet) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \longrightarrow E_2^{0,1} \longrightarrow H^1(E^\bullet) \longrightarrow E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \longrightarrow H^2(E^\bullet).$$

0.3. *The other orientation.*

You may have observed that we could as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this (compare to (1)).

(3)



This spectral sequence is denoted ${}_\wedge E_\bullet^{\bullet,\bullet}$ ("with the upwards orientation"). Then we would again get pieces of a filtration of $H^\bullet(E^\bullet)$ (where we have to be a bit careful with the order with which ${}_\wedge E_\infty^{p,q}$ corresponds to the subquotients — it in the opposite order to that of (2) for ${}_> E_\infty^{p,q}$). Warning: in general there is no isomorphism between ${}_> E_\infty^{p,q}$ and ${}_\wedge E_\infty^{p,q}$.

In fact, this observation that we can start with either the horizontal or vertical maps was our secret goal all along. Both algorithms compute information about the same thing ($H^\bullet(E^\bullet)$), and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

0.4. Examples.

We're now ready to see how this is useful. The moral of these examples is the following. In the past, you may have proved various facts involving various sorts of diagrams, which involved chasing elements around. Now, you'll just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

0.5. *Example: Proving the Snake Lemma.* Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the rows are exact and the squares commute. (Normally the Snake Lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0. \tag{4}$$

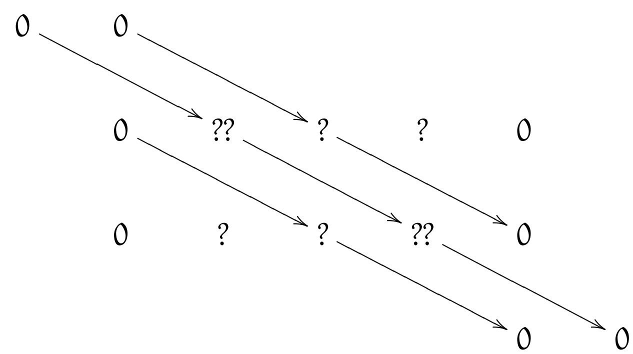
We plug this into our spectral sequence machinery. We first compute the cohomology using the rightwards orientation, i.e. using the order (1). Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

We next compute this "0" in another way, by computing the spectral sequence using the upwards orientation. Then ${}_\wedge E_1^{\bullet,\bullet}$ (with its differentials) is:

$$0 \longrightarrow \operatorname{im}\alpha \longrightarrow \operatorname{im}\beta \longrightarrow \operatorname{im}\gamma \longrightarrow 0$$

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow 0.$$

Then ${}_\wedge E_2^{\bullet,\bullet}$ is of the form:



We see that after ${}_\wedge E_2$, all the terms will stabilize except for the double-question-marks — all maps to and from the single question marks are to and from 0-entries. And after ${}_\wedge E_3$, even these two double-quesion-mark terms will stabilize. But in the end our complex must be the 0 complex. This means that in ${}_\wedge E_2$, all the entries must be zero, except for the two double-question-marks, and these two must be the isormorphic. This means that $0 \to \ker\alpha \to \ker\beta \to \ker\gamma$ and $\operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker\beta \to \ker\gamma) \cong \ker(\operatorname{im}\alpha \to \operatorname{im}\beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the exactness of (4), and hence the Snake Lemma!

Spectral sequences make it easy to see how to generalize results further. For example, if $A \to B$ is no longer assumed to be injective, how would the conclusion change?

0.6. *Example: the Five Lemma.* Suppose

(5)
$$\begin{CD} F @>>> G @>>> H @>>> I @>>> J \\ @A{\alpha}AA @A{\beta}AA @A{\gamma}AA @A{\delta}AA @A{\epsilon}AA \\ A @>>> B @>>> C @>>> D @>>> E \end{CD}$$

where the rows are exact and the squares commute.

Suppose α, β, δ, ϵ are isomorphisms. We'll show that γ is an isomorphism.

We first compute the cohomology of the total complex using the rightwards orientation (1). We choose this because we see that we will get lots of zeros. Then ${}_{>}E_1^{\bullet,\bullet}$ looks like this:

$$\begin{CD} ? @. 0 @. 0 @. 0 @. ? \\ @AAA @AAA @AAA @AAA @AAA \\ ? @. 0 @. 0 @. 0 @. ? \end{CD}$$

Then ${}_{>}E_2$ looks similar, and the sequence will converge by E_2, as we will never get any arrows between two non-zero entries in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, it vanishes in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this using the upwards orientation (3). Then ${}_{\wedge}E_1$ looks like this:

$$\begin{CD} 0 @>>> 0 @>>> ? @>>> 0 @>>> 0 \\ @. @. @. @. \\ 0 @>>> 0 @>>> ? @>>> 0 @>>> 0 \end{CD}$$

and the spectral sequence converges at this step. We wish to show that those two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we're done!

The best way to become comfortable with this sort of argument is to try it out yourself several times, and realize that it really is easy. So you should do the following exercises!

0.B. EXERCISE: THE SUBTLE FIVE LEMMA. By looking at the spectral sequence proof of the Five Lemma above, prove a subtler version of the Five Lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I am deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.)

0.C. EXERCISE. If β and δ (in (5)) are injective, and α is surjective, show that γ is injective. State the dual statement (whose proof is of course essentially the same).

0.D. EXERCISE. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

0.E. EXERCISE (THE MAPPING CONE). Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes. Suppose $C^\bullet$ is the single complex associated to the double complex $A^\bullet \to B^\bullet$. ($C^\bullet$ is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, we will use the fact that μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

You are now ready to go out into the world and use spectral sequences to your heart's content!

0.7. $\star\star$ **Complete definition of the spectral sequence, and proof.**

You should most definitely not read this section any time soon after reading the introduction to spectral sequences above. Instead, flip quickly through it to convince yourself that nothing fancy is involved.

We consider the rightwards orientation. The upwards orientation is of course a trivial variation of this.

0.8. *Goals.* We wish to describe the pages and differentials of the spectral sequence explicitly, and prove that they behave the way we said they did. More precisely, we wish to:

(a) describe $E_r^{p,q}$,
(b) verify that $H^k(E^\bullet)$ is filtered by $E_\infty^{p,k-p}$ as in (2),
(c) describe d_r and verify that $d_r^2 = 0$, and
(d) verify that $E_{r+1}^{p,q}$ is given by cohomology using d_r.

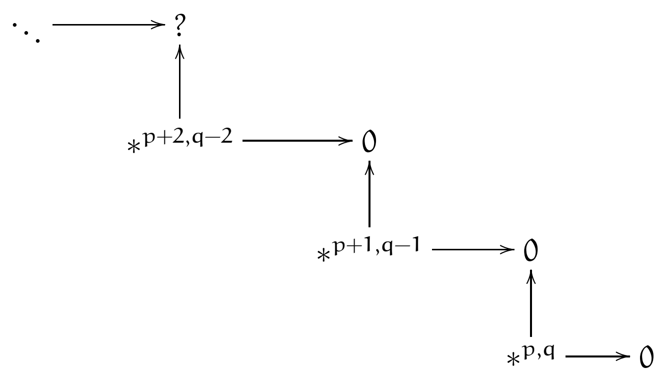
Before tacking these goals, you can impress your friends by giving this short description of the pages and differentials of the spectral sequence. We say that an element of $E^{\bullet,\bullet}$ is a (p,q)-*strip* if it is an element of $\oplus_{l \geq 0} E^{p+l,q-l}$ (see Fig. 1). Its non-zero entries lie on a semi-infinite antidiagonal starting with position (p,q). We say that the (p,q)-entry (the projection to $E^{p,q}$) is the *leading term* of the (p,q)-strip. Let $\boxed{S^{p,q}} \subset E^{\bullet,\bullet}$ be the submodule of all the (p,q)-strips. Clearly $S^{p,q} \subset E^{p+q}$, and $S^{0,k} = E^k$.

Note that the differential $d = d_\wedge + d_>$ sends a (p,q)-strip x to a $(p,q+1)$-strip dx. If dx is furthermore a $(p+r, q+r+1)$-strip ($r \in \mathbb{Z}^{\geq 0}$), we say that x is an r-*closed* (p,q)-*strip*. We denote the set of such $\boxed{S_r^{p,q}}$ (so for example $S_0^{p,q} = S^{p,q}$, and $S_0^{0,k} = E^k$). An element of

$$\begin{array}{ccccc}
\ddots & 0 & 0 & 0 & 0 \\
0 & *^{p+2,q-2} & 0 & 0 & 0 \\
0 & 0 & *^{p+1,q-1} & 0 & 0 \\
0 & 0 & 0 & *^{p,q} & 0 \\
0 & 0 & 0 & 0 & 0^{p-1,q+1}
\end{array}$$

FIGURE 1. A (p,q)-strip (in $S^{p,q} \subset E^{p+q}$). Clearly $S^{0,k} = E^k$.

$S_r^{p,q}$ may be depicted as:



0.9. *Preliminary definition of* $E_r^{p,q}$. We are now ready to give a first definition of $E_r^{p,q}$, which by construction should be a subquotient of $E^{p,q} = E_0^{p,q}$. We describe it as such by describing two submodules $Y_r^{p,q} \subset X_r^{p,q} \subset E^{p,q}$, and defining $E_r^{p,q} = X_r^{p,q}/Y_r^{p,q}$. Let $X_r^{p,q}$ be those elements of $E^{p,q}$ that are the leading terms of r-closed (p,q)-strips. Note that by definition, d sends $(r-1)$-closed $S^{p-(r-1),q+(r-1)-1}$-strips to (p,q)-strips. Let $Y_r^{p,q}$ be the leading $((p,q))$-terms of the differential d of $(r-1)$-closed $(p-(r-1), q+(r-1)-1)$-strips (where the differential is considered as a (p,q)-strip).

We next give the definition of the differential d_r of such an element $x \in X_r^{p,q}$. We take *any* r-closed (p,q)-strip with leading term x. Its differential d is a $(p+r, q-r+1)$-strip, and we take its leading term. The choice of the r-closed (p,q)-strip means that this is not a well-defined element of $E^{p,q}$. But it is well-defined modulo the $(r-1)$-closed $(p+1, r+1)$-strips, and hence gives a map $E_r^{p,q} \to E_r^{p+r,q-r+1}$.

This definition is fairly short, but not much fun to work with, so we will forget it, and instead dive into a snakes' nest of subscripts and superscripts.

We begin with making some quick but important observations about (p, q)-strips.

0.F. EXERCISE. Verify the following.

(a) $S^{p,q} = S^{p+1,q-1} \oplus E^{p,q}$.
(b) (Any closed (p, q)-strip is r-closed for all r.) Any element x of $S^{p,q} = S_0^{p,q}$ that is a cycle (i.e. $dx = 0$) is automatically in $S_r^{p,q}$ for all r. For example, this holds when x is a boundary (i.e. of the form dy).
(c) Show that for fixed p, q,

$$S_0^{p,q} \supset S_1^{p,q} \supset \cdots \supset S_r^{p,q} \supset \cdots$$

stabilizes for $r \gg 0$ (i.e. $S_r^{p,q} = S_{r+1}^{p,q} = \cdots$). Denote the stabilized module $S_\infty^{p,q}$. Show $S_\infty^{p,q}$ is the set of closed (p, q)-strips (those (p, q)-strips annihilated by d, i.e. the cycles). In particular, $S_r^{0,k}$ is the set of cycles in E^k.

0.10. *Defining* $E_r^{p,q}$.

Define $X_r^{p,q} := S_r^{p,q}/S_{r-1}^{p+1,q-1}$ and $Y := dS_{r-1}^{p-(r-1),q+(r-1)-1}/S_{r-1}^{p+1,q-1}$.

Then $Y_r^{p,q} \subset X_r^{p,q}$ by Exercise 0.F(b). We define

$$E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}} \tag{6}$$

We have completed Goal 0.8(a).

You are welcome to verify that these definitions of $X_r^{p,q}$ and $Y_r^{p,q}$ and hence $E_r^{p,q}$ agree with the earlier ones of §0.9 (and in particular $X_r^{p,q}$ and $Y_r^{p,q}$ are both submodules of $E^{p,q}$), but we won't need this fact.

0.G. EXERCISE: $E_\infty^{p,k-p}$ GIVES SUBQUOTIENTS OF $H^k(E^\bullet)$. By Exercise 0.F(c), $E_r^{p,q}$ stabilizes as $r \to \infty$. For $r \gg 0$, interpret $S_r^{p,q}/dS_{r-1}^{p-(r-1),q+(r-1)-1}$ as the cycles in $S_\infty^{p,q} \subset E^{p+q}$ modulo those boundary elements of dE^{p+q-1} contained in $S_\infty^{p,q}$. Finally, show that $H^k(E^\bullet)$ is indeed filtered as described in (2).

We have completed Goal 0.8(b).

0.11. *Definition of* d_r.

We shall see that the map $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$ is just induced by our differential d. Notice that d sends r-closed (p, q)-strips $S_r^{p,q}$ to $(p + r, q - r + 1)$-strips $S^{p+r,q-r+1}$, by the definition "r-closed". By Exercise 0.F(b), the image lies in $S_r^{p+r,q-r+1}$.

0.H. EXERCISE. Verify that d sends

$$dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1} \to dS_{r-1}^{(p+r)-(r-1),(q-r+1)+(r-1)-1} + S_{r-1}^{(p+r)+1,(q-r+1)-1}.$$

(The first term on the left goes to 0 from $d^2 = 0$, and the second term on the left goes to the first term on the right.)

Thus we may define

$$\boxed{d_r : E_r^{p,q} = \frac{S_r^{p,q}}{dS_{r-1}^{p-(r-1),q+(r-1)-1} + S_{r-1}^{p+1,q-1}} \to}$$

$$\boxed{\frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}} = E_r^{p+r,q-r+1}}$$

and clearly $d_r^2 = 0$ (as we may interpret it as taking an element of $S_r^{p,q}$ and applying d twice).

We have accomplished Goal 0.8(c).

0.12. *Verifying that the cohomology of* d_r *at* $E_r^{p,q}$ *is* $E_{r+1}^{p,q}$. We are left with the unpleasant job of verifying that the cohomology of

$$(7)\qquad \frac{S_r^{p-r,q+r-1}}{dS_{r-1}^{p-2r+1,q-3}+S_{r-1}^{p-r+1,q+r-2}} \xrightarrow{d_r} \frac{S_r^{p,q}}{dS_{r-1}^{p-r+1,q+r-2}+S_{r-1}^{p+1,q-1}} \xrightarrow{d_r} \frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1}+S_{r-1}^{p+r+1,q-r}}$$

is naturally identified with

$$\frac{S_{r+1}^{p,q}}{dS_r^{p-r,q+r-1} + S_r^{p+1,q-1}}$$

and this will conclude our final Goal 0.8(d).

Let's begin by understanding the kernel of the right map of (7). Suppose $a \in S_r^{p,q}$ is mapped to 0. This means that $da = db + c$, where $b \in S_{r-1}^{p+1,q-1}$. If $u = a - b$, then $u \in S^{p,q}$, while $du = c \in S_{r-1}^{p+r+1,q-r} \subset S^{p+r+1,q-r}$, from which u is r-closed, i.e. $u \in S_{r+1}^{p,q}$. Hence $a = b + u + x$ where $dx = 0$, from which $a - x = b + c \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$. However, $x \in S^{p,q}$, so $x \in S_{r+1}^{p,q}$ by Exercise 0.F(b). Thus $a \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$. Conversely, any $a \in S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}$ satisfies

$$da \in dS_{r-1}^{p+r,q-r+1} + dS_{r+1}^{p,q} \subset dS_{r-1}^{p+r,q-r+1} + S_{r-1}^{p+r+1,q-r}$$

(using $dS_{r+1}^{p,q} \subset S_0^{p+r+1,q-r}$ and Exercise 0.F(b)) so any such a is indeed in the kernel of

$$S_r^{p,q} \to \frac{S_r^{p+r,q-r+1}}{dS_{r-1}^{p+1,q-1} + S_{r-1}^{p+r+1,q-r}}.$$

Hence the kernel of the right map of (7) is

$$\ker = \frac{S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}.$$

Next, the image of the left map of (7) is immediately

$$\mathrm{im} = \frac{dS_r^{p-r,q+r-1} + dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}} = \frac{dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}}{dS_{r-1}^{p-r+1,q+r-2} + S_{r-1}^{p+1,q-1}}$$

(as $S_r^{p-r,q-r+1}$ contains $S_{r-1}^{p-r+1,q+r-1}$).

Thus the cohomology of (7) is

$$\ker / \operatorname{im} = \frac{S_{r-1}^{p+1,q-1} + S_{r+1}^{p,q}}{dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}} = \frac{S_{r+1}^{p,q}}{S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1})}$$

where the equality on the right uses the fact that $dS_r^{p-r,q+r+1} \subset S_{r+1}^{p,q}$ and an isomorphism theorem. We thus must show

$$S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}) = dS_r^{p-r,q+r-1} + S_r^{p+1,q-1}.$$

However,

$$S_{r+1}^{p,q} \cap (dS_r^{p-r,q+r-1} + S_{r-1}^{p+1,q-1}) = dS_r^{p-r,q+r-1} + S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$$

and $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$ consists of (p,q)-strips whose differential vanishes up to row $p+r$, from which $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1} = S_r^{p,q}$ as desired.

This completes the explanation of how spectral sequences work for a first-quadrant double complex. The argument applies without significant change to more general situations, including filtered complexes.

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