FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 8

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This set is due at noon on Friday November 30. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 14 through 16.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1-. If k is algebraically closed, describe a natural map of *sets* $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k$. Show that this map is not surjective. On the other hand, show that it is a bijection on closed points.

2. The reason for the phrase or "base change" or "pullback" is the following. If X is a point of Z (i.e. f is the natural map of Spec of the residue field of a point of Z into Z), then W is interpreted as the fiber of the family. Show that in the category of topological spaces, this is true, i.e., if $Y \to Z$ is a continuous map, and X is a point p of Z, then the fiber of Y over p is naturally identified with $X \times_Z Y$.

3++. *(only for experts)* Suppose X and Z are affine, and Y_i is an affine open cover of Y. Suppose the covariant functor $F_Y : (\mathbf{Sch}_Y)^{opp} \to \mathbf{Sets}$ is a sheaf on the category of Y-schemes $\mathbf{Sch}_Y$, and F_{Y_i} is the "restriction of the sheaf to Y_i" (where we include only those Y-schemes that are in fact Y_i-schemes, i.e. those $T \to Y$ whose structure morphisms factor through Y_i, $T \to Y_i \to Y$). Show that if F_{Y_i} is representable, then so is F_Y.

4++. *(only for experts)* Suppose F_Y is given by

$$\left(T \xrightarrow{f} Y \right) \mapsto \left(\begin{array}{ccc} & & \\ T & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \right).$$

(The diagram on the right isn't intended to have a blank line on top!) Check that this F_Y is a sheaf.

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5+. Show that if X and Y are schemes, then there is a natural bijection between morphisms of schemes $X \to Y$ and morphisms of functor spaces $h^X \to h^Y$. (Hint: this has nothing to do with schemes; your argument will work in any category.)

6++. *(only for experts)* If a functor-space h is a sheaf that has an open cover by representable functor-spaces ("is covered by schemes"), then h is representable.

7++. *(only for experts)* Suppose $(Z_i)_i$ is an affine cover of Z, $(X_{ij})_j$ is an affine cover of the preimage of Z_i in X, and $(Y_{ik})_k$ is an affine cover of the preimage of Z_i in Y. Show that $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is an open cover of the functor $h_{X \times_Z Y}$. (Hint: use the definition of open covers!)

8. Show that $B \otimes_A A[t] \cong B[t]$.

9. *(repeat of older exercise; do this only if you haven't done it before)* Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right-exactness of $\otimes_C B$.)

10. Suppose $C \to B, A$ are two morphisms of rings. Suppose S is a multiplicative set of A. Then $(S \otimes 1)$ is a multiplicative set of $A \otimes_C B$. Show that there is a natural morphism $(S^{-1}A) \otimes_C B \cong (S \otimes 1)^{-1}(A \otimes_C B)$.

11. *(the three important types of monomorphisms of schemes)* Show that the following are monomorphisms: open immersions, closed immersions, and localization of affine schemes. As monomorphisms are closed under composition, compositions of the above are also monomorphisms (e.g. locally closed immersions, or maps from Spec of stalks at points of X to X).

12-. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

13. Show that the underlying topological space of the (scheme-theoretic) fiber $X \to Y$ above a point p is naturally identified with the topological fiber of $X \to Y$ above p.

14. Show that for finite-type schemes over $\mathbb{C}$, the closed points (=complex-valued points by the Nullstellensatz) of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

15. More generally, describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion.

16. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$, where $\operatorname{char} k \neq 2$. Show that $X \times_Y X$ has 2 irreducible components. (What happens if $\operatorname{char} k = 2$? See problem 25...)

17+. *(exercise generalizing $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$)* Suppose L/K is a finite Galois field extension. What is $L \otimes_K L$?

18++. *(hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$)* Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

19. *(weird but fun)* Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

20-. Show that locally principal closed subschemes pull back to locally principal closed subschemes.

21. (Each one of these counts for half a problem.) Show that the following properties of morphisms are preserved by base change.

(a) quasicompact
(b) quasiseparated
(c) affine morphism
(d) finite
(e) locally of finite type
(f) finite type
(g) locally of finite presentation
(h) finite presentation

22+. Show that the notion of "quasifinite morphism" (finite type + finite fibers) is preserved by base change. (Warning: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 18.)

23. Show that surjectivity is preserved by base change. (*Surjectivity* has its usual meaning: surjective as a map of sets.) (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

24. If P is a property of morphisms preserved by base change, and $X \to Y$ and $X' \times Y'$ are two morphisms of S-schemes with property P, show that $X \times_S X' \to Y \times_S Y'$ has property P as well.

25-. Suppose k is a field of characteristic p, so $k(u^p)/k(u)$ is an inseparable extension. By considering $k(u^p) \otimes_{k(u)} k(u^p)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (The fact that I'm giving you this example should show that this happens only in characteristic p, in the presence of something as strange as inseparability.)

26. Show that the notion of "connected (resp. irreducible, integral, reduced)" geometric fibers behaves well under base change.

27. *(for the arithmetically-minded)* Show that for the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, all geometric fibers consist of two reduced points.

28. Recall the example of the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fibers over $0 = [(x)]$.

29++. Suppose X is a k-scheme.

(a) Show that X is geometrically irreducible if and only if $X \times_k k^s$ is irreducible if and only if $X \times_k K$ is irreducible for all field extensions K/k. (Here k^s is the separable closure of k.)
(b) Show that X is geometrically connected if and only if $X \times_k k^s$ is connected if and only if $X \times_k K$ is connected for all field extensions K/k.
(c) Show that X is geometrically reduced if and only if $X \times_k k^p$ is reduced if and only if $X \times_k K$ is reduced for all field extensions K/k. (Here k^p is the perfect closure of k.) Thus if $\operatorname{char} k = 0$, then X is geometrically reduced if and only if it is reduced.
(d) Combining (a) and (c), show that X is geometrically integral if and only if $X \times_k K$ is geometrically integral for all field extensions K/k.

30. Check that the maps defined in class glue to give a well-defined morphism $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$.

31+. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank}\begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

32. *(A co-ordinate-free description of the Segre embedding)* Show that the Segre embedding can be interpreted as $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$ via the surjective map of graded rings

$$\operatorname{Sym}^\bullet(V^\vee \otimes W^\vee) \longrightarrow\!\!\!\!\!\rightarrow \textstyle\sum_{i=0}^{\infty} \left(\operatorname{Sym}^i V^\vee\right) \otimes \left(\operatorname{Sym}^i W^\vee\right)$$

"in the opposite direction".

33. *(important but easy)* Show that open immersions and closed immersions are separated.

34. *(also important but easy)* Show that every morphism of affine schemes is separated.

35. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed.

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