FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 1

RAVI VAKIL

This set is due at noon on Friday October 5. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 1 and 2.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in ten solutions, where each "-" problem is worth half a solution. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 1.

1-. A category in which each morphism is an isomorphism is called a *groupoid*.
(a) A perverse definition of a group is: a groupoid with one element. Make sense of this.
(b) Describe a groupoid that is not a group.
(For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of x, and the morphisms from $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and the $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

2-. If A is an object in a category $\mathcal{C}$, show that the isomorphisms of A with itself $\mathrm{Isom}(A, A)$ form a group (called the *automorphism group of* A, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in the **Sets** and $\mathbf{Vec}_k$ (k-vector spaces)? Show that two isomorphic objects have isomorphic automorphism groups.

3. *(if you haven't seen tensor products before)* Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (This exercise is intended to give some hands-on practice with tensor products.)

4. *(right-exactness of* $\cdot \otimes_A N$*)* Show that $\cdot \otimes_A N$ gives a covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. Show that $\cdot \otimes_A N$ is a *right-exact functor*, i.e. if

$$M' \to M \to M'' \to 0$$

is an exact sequence of A-modules, then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. (For experts: is there a universal property proof?)

Date: Thursday, September 27, 2007. Problem 2 fixed Oct. 13, 2007.

5. In the universal property definition of tensor product, show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product. (This exercise will prime you for Yoneda's Lemma.)

6. Show that the construction of tensor product given in class satisfies the universal property of tensor product.

7-. Show that any two initial objects are canonically isomorphic. Show that any two final objects are canonically isomorphic.

Class 2.

8. *Important Exercise that everyone should do once in their life.* Prove the form of Yoneda's lemma stated in class. (See the class notes for a hint.)

9. Show that in **Sets**, show that

$$X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}.$$

More precisely, describe a natural isomorphism between the left and right sides. (This will help you build intuition for fibered products.)

10-. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is. (Hint: it has a one-word description.)

11-. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$": "the" fibered product over Z is canonically isomorphic to "the" product. (This is an exercise about unwinding the definition.)

12-. Show that in the category **Ab** of abelian groups, the kernel K of $f : A \to B$ can be interpreted as a fibered product:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

13-. Prove a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (What is this natural morphism?!) We may then take this as the definition of monomorphism. (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

14-. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. We will use this later when talking about fibered products. (Hint: for any object V, give a natural bijection between maps from V to the first and maps from V to the second.)

15-. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

16-. Define *coproduct* in a category by reversing all the arrows in the definition of product. Show that coproduct for **Sets** is disjoint union.

17. Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Define a ring structure $A \otimes_C B$ with multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. There is a natural morphism $A \to A \otimes_C B$ given by $a \mapsto (a, 1)$. (Warning: This is not necessarily an inclusion.) Similarly, there is a natural morphism $B \to A \otimes_C B$. Show that this gives a coproduct on rings, i.e. that

$$\begin{tikzcd} A \otimes_C B & B \arrow[l] \\ A \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd}$$

satisfies the universal property of coproduct.

18. *Important Exercise for Later.* We continue the notation of the previous exercise. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right exactness of $\otimes_C B$.)

19. Show that in the category **Sets**,

$$\{(a_i)_{i \in I} \in \prod_i A_i : F(m)(a_i) = a_j \text{ for all } [m : i \to j] \in \operatorname{Mor}(\mathcal{I})\},$$

along with the projection maps to each A_i, is the limit $\varprojlim_{\mathcal{I}} A_i$.

20-. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.)

21. Consider the set $\{(i \in \mathcal{I}, a_i \in A_i)\}$ modulo the equivalence generated by: if $m : i \to j$ is an arrow in $\mathcal{I}$, then $(i, a_i) \sim (j, F(m)(a_j))$. Show that this set, along with the obvious maps from each A_i, is the colimit.

22. Verify that the construction of colimits of A-modules given in class are indeed colimits.

23-. Write down what the condition not mentioned in class in the definition of adjoint should be. (See the class notes.)

24. Suppose M, N, and P are A-modules. Describe a natural bijection $\operatorname{Mor}_A(M \otimes_A N, P) = \operatorname{Mor}_A(M, \operatorname{Mor}_A(N, P))$. (Hint: try to use the universal property.) If you want, you could check that $\cdot \otimes_A N$ and $\operatorname{Mor}_A(N, \cdot)$ are adjoint functors. (Checking adjointness is never any fun!) We may later see why problem 24 implies problem 4.

25. Define groupification H from the category of abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are (a, b), which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d = b + c$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let F be the forgetful morphism from the category of abelian groups **Ab** to the category of abelian semigroups. Show that H is left-adjoint to F.

26-. Show that if a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

27. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory of the category of A-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 2

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This set is due at noon on Friday October 12. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 3 and 4.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in ten solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 3.

1. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of k-vector spaces (often called $A^\bullet$ for short). Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice.)

2. *(important)* Suppose $\mathcal{C}$ is an abelian category. Define the category $\mathbf{Com}_{\mathcal{C}}$ as follows. The objects are infinite complexes

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$\begin{tikzcd}
A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\
B^\bullet : & \cdots \arrow[r] & B^{i-1} \arrow[r, "f^{i-1}"] & B^i \arrow[r, "f^i"] & B^{i+1} \arrow[r, "f^{i+1}"] & \cdots
\end{tikzcd}$$

Date: Wednesday, October 3, 2007.

Show that $\mathbf{Com}_{\mathcal{C}}$ is an abelian category. Show that a short exact sequence of complexes

$$\begin{tikzcd}
0 : \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : \arrow[d] & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d] & A^i \arrow[r, "f^i"] \arrow[d] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d] & \cdots \\
B^\bullet : \arrow[d] & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] \arrow[d] & B^i \arrow[r, "g^i"] \arrow[d] & B^{i+1} \arrow[r, "g^{i+1}"] \arrow[d] & \cdots \\
C^\bullet : \arrow[d] & \cdots \arrow[r] & C^{i-1} \arrow[r, "h^{i-1}"] \arrow[d] & C^i \arrow[r, "h^i"] \arrow[d] & C^{i+1} \arrow[r, "h^{i+1}"] \arrow[d] & \cdots \\
0 : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

induces a *long exact sequence in cohomology*

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H^{i-1}(C^\bullet) \arrow[r] & {} \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[r] & {} & \\
H^{i+1}(A^\bullet) \arrow[r] & \cdots & & &
\end{tikzcd}$$

3. $\mathrm{Hom}(X, \cdot)$ *commutes with limits.* Suppose A_i ($i \in \mathcal{I}$) is a diagram in $\mathcal{D}$ indexed by $\mathcal{I}$, and $\varprojlim A_i \to A_i$ is its limit. Then for any $X \in \mathcal{D}$, $\mathrm{Hom}(X, \varprojlim A_i) \to \mathrm{Hom}(X, A_i)$ is the limit $\varprojlim \mathrm{Hom}(X, A_i)$.

4. *(for those familiar with differentiable functions)* In the "motivating example" of the sheaf of differentiable functions, show that $\mathfrak{m}_x$ is the only maximal ideal of $\mathcal{O}_x$.

5-. *"A presheaf is the same as a contravariant functor"* Given any topological space X, we can get a category, called the "category of open sets" (discussed last week), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets. (This interpretation is suprisingly useful.)

6-. *(unimportant exercise for category-lovers)* The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain limit. What is that limit?

7. *(important Exercise: constant presheaf and locally constant sheaf*
(a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. We denote this presheaf $\underline{S}^{\mathrm{pre}}$. Show that this needn't form a sheaf. This is called the *constant presheaf with values in* S.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. Show that this is a *sheaf*. (A better

description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$. Using this description, this follows immediately from Exercise 9 below.) We will call this the *locally constant sheaf*. This is usually called the *constant sheaf*.

8-. *(more examples of presheaves that are not sheaves)* Show that the following are presheaves on $\mathbb{C}$ (with the usual topology), but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

9. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.

10. This is a fancier example of the previous exercise.
(a) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.)
(b) (This exercise is for those who know topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that Y is a topological group. Show that maps to Y form a sheaf of *groups*. (A special case turned up in class.)

11. *(important exercise: the direct image sheaf or pushforward sheaf)* Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *direct image sheaf* of *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by* f.

12. *(pushforward induces maps of stalks)* Suppose $\mathcal{F}$ is a sheaf of sets (or rings or A-modules). If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$. (You can use the explicit definition of stalk using representatives, or the universal property. If you prefer one way, you should try the other.)

Class 4.

13. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor from $\{$ sheaves of sets on X $\}$ to $\{$ sheaves of sets on Y $\}$. Here "sets" can be replaced by any category.

14. *(important exercise and definition: "Sheaf* Hom*")* Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set U, see last day's notes.) Show that this is a sheaf. This is called the "sheaf $\underline{\mathrm{Hom}}$". Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups.

15. Show that $\ker_{\text{pre}} f$ is a presheaf. (Hint: if $U \hookrightarrow V$, there is a natural map $\text{res}_{V,U} : \mathcal{G}(V)/f_V(\mathcal{F}(V)) \to \mathcal{G}(U)/f_U(\mathcal{F}(U))$ by chasing the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker_{\text{pre}} f_V \arrow[r] \arrow[d, dotted, "\exists!"] & \mathcal{F}(V) \arrow[r] \arrow[d, "\text{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\text{res}_{V,U}"] \\ 0 \arrow[r] & \ker_{\text{pre}} f_U \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \end{tikzcd}$$

You should check that the restriction maps compose as desired.)

16. *(the cokernel deserves its name)* Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

17. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves of abelian groups, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

18. *(important)* Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} f$ is in fact a sheaf. Show that it satisfies the universal property of kernels. (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} f$ satisfies the universal property in the category of *presheaves*.)

19. *(important exercise)* Let X be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the locally constant sheaf on X with group $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. (Why is $\mathcal{F}$ not a sheaf?) Show that

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion. Show that this is an exact sequence *of presheaves*. Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm that locally have a logarithm.)

20+. *(important exercise: sections are determined by stalks)* Prove that a section of a sheaf is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \tag{1}$$

is injective. (Hint # 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint # 2: it is false for presheaves in general, see Exercise , so you *will* use the identity axiom.)

21+. *(important)* Prove that any choice of compatible germs for $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U. (Hint: you will use gluability.)

22. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on X, and $x \in X$, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

23. *(morphisms are determined by stalks)* Show that morphisms of sheaves are determined by morphisms of stalks. Hint: consider the following diagram.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x\in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd} \tag{2}$$

24. *(tricky: isomorphisms are determined by stalks)* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (2). Injectivity uses the previous exercise . Surjectivity will use gluability, and is more subtle.)

25. Problems 20, 21, 23, and 24 are all false for general presheaves. Give counterexamples to three of them. (General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

26-. Show that sheafification (as defined by universal property) is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$.

27. Show that $\mathcal{F}^{\mathrm{sh}}$ (using the tautological restriction maps) forms a sheaf.

28-. Describe a natural map $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$.

29+. Show that the map sh satisfies the universal property of sheafification.

30. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$. Show that sheafification is a functor from presheaves to sheaves.

31+. *(useful exercise for category-lovers)* Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

32. Show $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility: judicious use of adjoints.)

33+. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of sets) on at topological space X. Show that the following are equivalent.

(a) ϕ is a monomorphism in the category of sheaves.
(b) ϕ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective for all $x \in X$.
(c) ϕ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise . For (a) implies (b), judiciously choose a skyscraper sheaf. For (a) implies (c), judiciously the "indicator sheaf" with one section over every open set contained in U, and no section over any other open set.)

34. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) ϕ is a epimorphism in the category of sheaves.
(b) ϕ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ surjective for all $x \in X$.

35. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

36. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

37. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

38. *(Left-exactness of the global section functor)* Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give an example to show that the global section functor is not exact. (Hint: the exponential exact sequence.)

39+. *(Left-exactness of pushforward)* Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.)

40. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.)

41. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There isn't much more to check!)

42. *(important exercise: tensor products of $\mathcal{O}_X$-modules)* (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context.
(b) Show that the tensor product of stalks is the stalk of tensor product.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 3

RAVI VAKIL

This set is due at noon on Friday October 19. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 5 and 6.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in eight solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 5.

1. If $f : X \to Y$ is a continuous map, and $\mathcal{G}$ is a sheaf on Y, show that $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) := \varinjlim_{V \supset f(U)} \mathcal{G}(V)$ defines a presheaf on X. (Possible hint: Recall the explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

2. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Possible hint: use the concrete description of the stalk, as a direct limit. Recall that stalks are preserved by sheafification.)

3-. *(easy but useful)* If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}\mathcal{G}$ is naturally isomorphic to $\mathcal{G}|_U$.

4-. *(easy but useful)* If $y \in Y$, $i : \{y\} \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}(\mathcal{G})$ is naturally isomorphic to the stalk $\mathcal{G}_y$.

5. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks.) The identical argument will show that f^{-1} is an exact functor from $\mathcal{O}_Y$-modules (on Y) to $f^{-1}\mathcal{O}_Y$-modules (on X), but don't bother writing that down. (Remark for experts: f^{-1} is a left-adjoint, hence right-exact by abstract nonsense. The left-exactness is true for "less categorical" reasons.)

6+. *(The construction of f^{-1} satisfies the adjoint property* If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, describe a bijection

$$\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Observe that your bijection is "natural" in the sense of the definition of adjoints.

Date: Sunday, October 13, 2007.

7. (a) Suppose $Z \subset Y$ is a closed subset, and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathcal{F}$ is a sheaf on Z, then show that the stalk $(i_*\mathcal{F})_y$ is 0 if $y \in Z$, and $\mathcal{F}_y$ if $y \in Z$.
(b) *Important definition:* Define the *support* of a sheaf $\mathcal{F}$ of sets, denoted $\operatorname{Supp} \mathcal{F}$, as the locus where the stalks are non-empty:

$$\operatorname{Supp} \mathcal{F} := \{x \in X : \mathcal{F}_x \neq \emptyset\}.$$

(More generally, if the sheaf has value in some category, the support consists of points where the stalk is not the initial object. For sheaves of abelian groups, the support consists of points with non-zero stalks.) Suppose $\operatorname{Supp} \mathcal{F} \subset Z$ where Z is closed. Show that the natural map $\mathcal{F} \to f_* f^{-1} \mathcal{F}$ is an isomorphism. Thus a sheaf supported in a closed subset can be considered a sheaf on that closed subset.

8. Suppose $\mathcal{F}$ is a sheaf. Show that you can recover $\mathcal{F}$ from just knowing its behavior on a base.

9+. In class, we mostly prove the following theorem: *Suppose* $\{B_i\}$ *is a base on* X*, and* F *is a sheaf of sets on this base. Then there is a unique sheaf* $\mathcal{F}$ *extending* F *(with isomorphisms* $\mathcal{F}(B_i) \cong F(B_i)$ *agreeing with the restriction maps).* In the proof, I did not describe a certain inverse map $\mathcal{F}(B) \to F(B)$. Do so, and verify that it is inverse to the obvious map $F(B) \to \mathcal{F}(B)$.

10+. *(morphisms of sheaves correspond to morphisms of sheaf on a base)* Suppose $\{B_i\}$ is a base for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{tikzcd}
F(B_i) \arrow[r] \arrow[d] & G(B_i) \arrow[d] \\
F(B_j) \arrow[r] & G(B_j)
\end{tikzcd}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

11+. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.) (You can use the ideas of this section to solve this problem, but you don't necessarily need to. Hint: As the base, take those open sets contained in *some* U_i.)

12. *(for those with a little experience with manifolds)* Prove that a continuous function of differentiable manifolds $f : X \to Y$ is differentiable if differentiable functions pull back to differentiable functions. (Hint: check this on small patches. Once you figure out what you are trying to show, you'll realize that the result is immediate.)

13. Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$.

14-. *A small exercise about small schemes.* (a) Describe the set $\operatorname{Spec} k[\epsilon]/\epsilon^2$. This is called the ring of dual numbers, and will turn out to be quite useful. You should think of ϵ as a very small number, so small that its square is 0 (although it itself is not 0).
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$. (We will see this scheme again later.)

15-. Show that for primes of the form $\mathfrak{p} = (x^2 + ax + b)$ in $\mathbb{R}[x]$, the quotient $\mathbb{R}[x]/\mathfrak{p}$ is *always* isomorphic to $\mathbb{C}$.

16-. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$. (This is harder to picture in a way analogous to $\mathbb{A}^1_{\mathbb{R}}$; but the rough cartoon of points on a line remains a reasonable sketch.)

Class 6.

17. Show that all the prime ideals of $\mathbb{C}[x, y]$ are of the form (0), $(f(x, y))$, or $(x - a, y - b)$.

18. Ring elements that have a power that is 0 are called *nilpotents*. If I is an ideal of nilpotents, show that $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection. Thus nilpotents don't affect the underlying set.

19. *(only if you haven't already seen this fact)* Prove that the nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of A.

20-. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

21. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence show that the intersection of any finite number of open sets is open.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $\cap_i V(I_i) = V(\sum_i I_i)$.

22. If $I \subset R$ is an ideal, then define its *radical* by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

For example, the nilradical $\mathfrak{N}$ is $\sqrt{(0)}$. Show that $V(\sqrt{I}) = V(I)$. We say an *ideal is radical* if it equals its own radical.

23. *(practice with the concept)* If $I_1, \ldots, I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. (We will use this property without referring back to this exercise.)

24. *(for future use)* Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint: Use Problem 19 on an appropriate ring.)

25+. Suppose $A \to B$ is a ring homomorphism, and $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ is the induced map of sets. By showing that closed sets pull back to closed sets, show that π is a *continuous map*.

26+. Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. Show that the Zariski topology on $\operatorname{Spec} B/I$

(resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

27. *(useful for later)* Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n.

28-. Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

29-. Show that on an irreducible topological space, any nonempty open set is dense. (The moral of this is: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "very big".)

30. Show that $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime. (Minimality is with respect to inclusion.) In particular, if A is an integral domain, then $\operatorname{Spec} A$ is irreducible.

31-. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

32-. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that we have made our intuition of "containment of points" precise: it means that the one point is contained in the *closure* of another.

33. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

34. (a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]$ is irreducible, by showing that I is prime. (One possible approach: Show that quotient ring is a domain, by showing that it is isomorphic to the subring of $k[a, b]$ including only monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve*.
(b) Note that the ideal of part (a) may be rewritten as

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} = 1,$$

i.e. that all determinants of 2×2 submatrices vanish. Generalize this to the ideal of rank 1 $2 \times n$ matrices. This will correspond to the cone over the *degree n rational normal curve*.

35. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in class.)

36. Suppose $0 \to M' \to M \to M'' \to 0$, and M' and M'' satisfy the ascending chain condition for modules. Show that M does too. (The converse also holds; we won't use this, but you can show it if you wish.)

37. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Show that the converse is not true. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

38. If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 4

RAVI VAKIL

This set is due at noon on Friday October 26. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 7 and 8.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 1), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

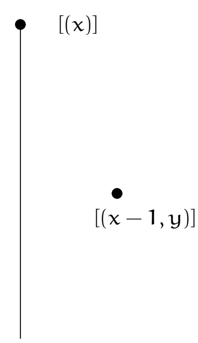

FIGURE 1. The set $S = \{[(x)], (1, 0)\}$, pictured as a subset of $\mathbb{A}^2$

2. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are deined analogously.) Give generators for the ideal $I(X)$. Be sure to prove it! Hint: We will see later that this ideal is not generated by less than three elements.

3. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

4+. *(important)* Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between

Date: Thursday, October 18, 2007.

points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

The next six problems on distinguished open sets will be very useful. Please think about them!

5. Show that the distinguished open sets form a base for the Zariski topology. (Hint: Given an ideal I, show that the complement of $V(I)$ is $\cup_{f \in I} D(f)$.)

6+. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$. (One of the directions will use the fact that any proper ideal of A is contained in some maximal ideal.)

7. Show that if $\operatorname{Spec} A$ is an infinite union $\cup_{i \in J} D(f_i)$, then in fact it is a union of a finite number of these. (Hint: use the previous exercise.) Show that $\operatorname{Spec} A$ is quasicompact.

8-. Show that $D(f) \cap D(g) = D(fg)$.

9. Show that if $D(f) \subset D(g)$, if and only if $f^n \in (g)$ for some n if and only if g is a unit in A_f. (Hint for the first equivalence: $f \in I(V((g)))$. We will use this shortly.

10. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

11+. Prove base identity for the structure sheaf for any distinguished open $D(f)$. (Possible strategy: show that the argument is the same as the argument in class for $\operatorname{Spec} A$.)

12+. Prove base gluability for any distinguished open $D(f)$.

13+. Suppose M is an A-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_A A_f$; the restriction map is the "obvious" one. This is an $\mathcal{O}_{\operatorname{Spec} A}$-module! This sort of sheaf $\tilde{M}$ will be very important soon; it is an example of a *quasicoherent sheaf*.

14. *(important)* Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

15. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

16. If X is a scheme, and U is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

17. *(important)* Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the ring $A_{\mathfrak{p}}$.

18. Show that the affine line with doubled origin is not an affine scheme. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0, 0)$.

19-. Define the *affine plane with doubled origin*. Use this example to show that the intersection of two affine open sets need not be an affine open set.

20+. Figure out how to define projective n-space $\mathbb{P}_k^n$. Glue together $n + 1$ opens each isomorphic to $\mathbb{A}_k^n$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}_k^n$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" (to be stated rigorously later) says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

21. Show that if k is algebraically closed, the closed points of $\mathbb{P}_k^n$ may be interpreted in the same way as we interpreted the points of $\mathbb{P}_k^1$. (The points are of the form $[a_0; \dots; a_n]$, where the a_i are not all zero, and $[a_0; \dots; a_n]$ is identified with $[ca_0; \dots; ca_n]$ where $c \in k^*$.)

22. (a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)
(b) Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 5

RAVI VAKIL

This set is due at noon on Friday November 2. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 9 and 10.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1-. Show that $\mathbb{P}^n_k$ is irreducible.

2. An earlier exercise showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

3. Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian (i.e. if X is a Noetherian scheme), then X is a Noetherian topological space.

4-. Show that an irreducible topological space is connected.

5-. Give (with proof!) an example of a scheme that is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

6-. Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine schemes (Hence $\mathbb{P}^n_k$ is quasicompact.)

7. *(quasicompact schemes have closed points)* Show that if X is a nonempty quasicompact scheme, then it has a closed point. (Warning: there exist non-empty schemes with no closed points, so your argument had better use the quasicompactness hypothesis!)

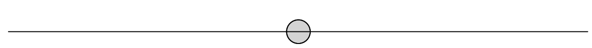

FIGURE 1. A picture of the scheme $\operatorname{Spec} k[x,y]/(xy, y^2)$

8. Show that $\left(k[x,y]/(y^2, xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture.)

Date: Friday, October 26, 2007.

9. *(reducedness is stalk-local)* Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from an earlier Exercise, and the nilradical is intersection of all prime ideals.)

10-. Suppose X is quasicompact, and f is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of x. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

11+. Show that a scheme X is integral if and only if it is irreducible and reduced.

12. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

13. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $\mathrm{FF}(A)$, the fraction field of A. This is called the **function field** $\mathrm{FF}(X)$ of X. It can be computed on any non-empty open set of X, as any such open set contains the generic point.

14. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X (so A is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \mathrm{FF}(A)$ (where U is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient that sections over different open sets can be considered subsets of the same thing. This makes restriction maps and gluing easy to consider; this is one reason why varieties are usually introduced before schemes.

15. Show that all open subsets of a Noetherian topological space (e.g. a Noetherian scheme) are quasicompact.

16. Show that a Noetherian scheme has a finite number of irreducible components.

17. If X is a Noetherian scheme, show that every point p has a closed point in its closure. (In particular, every non-empty Noetherian scheme has closed points; this is not true for every scheme.)

18. If X is an affine scheme or Noetherian scheme, show that it suffices to check reducedness at *closed points*.

19. Show that a locally Noetherian scheme X is integral if and only if X is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains (informally: "the scheme is locally integral"). Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are domains) and global (connected) conditions.

20. Show that X is reduced if and only if X can be covered by affine opens $\operatorname{Spec} A$ where A is reduced (nilpotent-free).

21. Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. (Recall the following form of Hilbert's Nullstellensatz, richer than the version stated before: the maximal ideals of $k[x_1, \dots, x_n]$ are precisely those with residue of the form a finite extension of k.) Show that the closed points are dense on such a scheme.

22. Finish the proof that Noetherianness is an affine-local property: show that if A is a ring, and $(f_1, \dots, f_n) = A$, and A_{f_i} is Noetherian, then A is Noetherian.

23. Prove that reducedness is an affine-local property.

24. Show that finite-generatedness over k is an affine-local property (see the notes for an outline).

25. Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (The domain portion is easy. Hint for integral closure: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in $A[x]$ that also has a root in the fraction field.)

26. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes.

27. If A is an integral domain, show that $A = \cap A_m$, where the intersection runs over all maximal ideals of A. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

28. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[a, b, c, d]/(ad - bc)$ (which we will later recognize as the cone over the quadric surface), and $a/c = b/d \in FF(A)$. Show that $I = (c, d)$. (If you can, show that this is not principal.)

29. Show that any localization of a Unique Factorization Domain is a Unique Factorization Domain.

30+. Show that unique factorization domains are integrally closed. Hence factorial schemes are are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we'll see in Exercise . An example without proof: $\mathbb{Z}[\sqrt{17}]$ again.)

31-. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$.

32+. *(this will give us a number of enlightening examples later)* Suppose A is a Unique Factorization Domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if f is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is a domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) = 0$ with $F(T) \in B[T]$ which has a solution α in $FF(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume

$F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in FF(A)$. Now α is the solution of monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f)T \in FF(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - rs^2 f)/4t^2$. Then $t = 1$, and r is even.)

33+. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$;
(c) $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. (This is our cone over a quadric surface example.)

34+. Suppose A is a k-algebra where $\operatorname{char} k = 0$, and l/k is a finite field extension. Show that A is normal if and only if $A \otimes_k l$ is normal. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal if k is characteristic 0. (In fact the hypothesis on the characteristic is unnecessary.) Possible hint: reduced to the case where l/k is Galois.

35-. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. (Caution: $\sqrt{\mathfrak{q}}$ can be prime without $\mathfrak{q}$ being primary — consider our example (y^2, xy) in $k[x, y]$.)

36-. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

37-. *(reality check)* Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

38+. *(existence of primary decomposition for Noetherian rings)* Suppose A is a Noetherian ring. Show that every proper ideal $I \subset A$ has a primary decomposition. (Hint: mimic the Noetherian induction argument we saw last week.)

39+. (a) Find a minimal primary decomposition of (y^2, xy). (b) Find another one. (Possible hint: see Figure 1. You might be able to draw sketches of your different primary decompositions.)

40+. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup_{i=1}^n \mathfrak{p}_i$. (The right side is not an ideal!) Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n. Don't look in the literature — you might find a much longer argument!)
(Parts (a) and (b) are "geometric facts"; try to draw pictures of what they mean.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 6

RAVI VAKIL

This set is due at noon on Friday November 9. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 11 and 12.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. *(associated primes behave well with respect to localization)* Show that if A is a Noetherian ring, and S is a multiplicative subset (so there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

2. (a) Show that the minimal primes of A are associated primes. We have now proved important fact **(1)**. (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by a previous exercise, $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \subset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.)
(b) Show that there can be other associated primes that are not minimal. (Hint: we've seen an example...) Your argument will show more generally that the minimal primes of I are associated primes of I.

3. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

4. Show that

$$Z = \cup_{x \neq 0} (0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq Z.$$

5-. *(Rabinoff's Theorem)* Here is an interesting variation on **(4)**: show that $a \in A$ is nilpotent if and only if it vanishes at the associated points of $\operatorname{Spec} A$. Algebraically: we know that the nilpotents are the intersection of *all* prime ideals; now show that in the Noetherian case, the nilpotents are in fact the intersection of the (finite number of) associated prime ideals.

6-. Prove fact **(3)**.

Date: Thursday, November 1, 2007. Updated November 9.

7-. Let $\vee\vee$: **f.d. Vec**$_k$ $\to$ **f.d. Vec**$_k$ be the double dual functor from the category of vector spaces over k to itself. Show that $\vee\vee$ is naturally isomorphic to the identity. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from id to $\vee\vee$.)

8-. Show that $\mathcal{V} \to$ **f.d.Vec**$_k$ gives an equivalence of categories, by describing an "inverse" functor. (You'll need the axiom of choice, as you'll simultaneously choose bases for each vector space in **f.d.Vec**$_k$!)

9. Assuming that morphisms of schemes are defined so that Motivation (a) holds, show that the category of rings and the opposite category of affine schemes are equivalent.

10. *(morphisms of ringed spaces glue)* Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (An earlier exercise essentially showed this for topological spaces.)

11+. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

12++. Suppose $f^\# : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given earlier. To describe a morphism of sheaves $\mathcal{O}_B \to f_* \mathcal{O}_A$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subset \operatorname{Spec} B$, we define

$$\mathcal{O}_B(D(g)) \to \mathcal{O}_A(f^{-1} D(g)) = \mathcal{O}_A(D(f^\# g))$$

by $B_g \to A_{f^\# g}$. Verify that this makes sense (e.g. is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We showed that a morphism of rings induces a map of sets first, a map of topological spaces later, and now a map of ringed spaces here.)

13-. Recall that $\operatorname{Spec} k[x]_{(x)}$ has two points, corresponding to (0) and (x), where the second point is closed, and the first is not. Consider the map *of ringed spaces* $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ sending the point of $\operatorname{Spec} k(x)$ to $[(x)]$, and the pullback map $f^\# \mathcal{O}_{\operatorname{Spec} k(x)} \to \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}$ is induced by $k \hookrightarrow k(x)$. Show that this map of ringed spaces is not of the form described in Key Exercise .

14. Show that morphisms of locally ringed spaces glue

15+. (a) Show that $\operatorname{Spec} A$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by a ring morphism $f^\# : B \to A$ is a morphism of locally ringed spaces.

16+. Show that a morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

17+. (This exercise will give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

18+. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

19-. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

20-. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

21-. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

22. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

23-. Show that open immersions are monomorphisms.

24. Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

25-. Show that the composition of two quasicompact morphisms is quasicompact.

26. *(the notions "locally of finite type" and "finite type" are affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

27. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

28+. *(not hard, but important —)*

(a) Show that a closed immersion is a morphism of finite type.
(b) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.

(c) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)

(d) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.

(e) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

29. *(the property of finiteness is affine-local on the target)* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

30-. Show that the composition of two finite morphisms is also finite.

31+. Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed. (Hint: going-up theorem.)

32. (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

33. Show that the property of being a closed immersion is affine-local on the target.

34. In analogy with closed subsets, define the notion of a finite union of closed subschemes of X, and an arbitrary intersection of closed subschemes. Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.

35-. Show that closed immersions are finite morphisms.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 7

RAVI VAKIL

This set is due at noon on Friday November 16. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers class 13.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to two problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1+. *(a useful criterion for when ideals in affine open sets define a closed subscheme)* It will be convenient to define certain closed subschemes of Y by defining on any affine subset $\operatorname{Spec} B$ of Y an ideal $I_B \subset B$. Show that these $\operatorname{Spec} B/I_B \hookrightarrow \operatorname{Spec} B$ glue together to form a closed subscheme precisely if for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$ and each $f \in B$, $I_{(B_f)} = (I_B)_f$.

You might hope that closed subschemes correspond to ideal sheaves of $\mathcal{O}_Y$. Sadly not every ideal sheaf arises in this way. Here is an example.

2. Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that this sheaf of ideals does not correspond to a closed subscheme.

3. (a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$. This curve is called the *twisted cubic*. The twisted cubic is a good non-trivial example of many things, so it you should make friends with it as soon as possible. (b) Show that the twisted cubic is isomorphic to $\mathbb{P}^1_k$.

4. The usual definition of a closed immersion is a morphism $f : X \to Y$ such that f induces a homeomorphism of the underlying topological space of Y onto a closed subset of the topological space of X, and the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. Show that this definition agrees with the one given above. (To show that our definition involving surjectivity on the level of affine open sets implies this definition, you can use the fact that surjectivity of a morphism of sheaves can be checked on a base, which you can verify yourself.)

Date: Friday, November 9, 2007.

5. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} A$ then $Y = \operatorname{Spec} A_f$. However, Y is not always of this form.)

6-. If X is reduced, show that the scheme-theoretic image of $f : X \to Y$ is also reduced.

7. If $f : X \to Y$ is a morphism of locally Noetherian schemes, show that the associated points of the image subscheme are a subset of the image of the associated points of X.

8. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

9. If $X \to Y$ is quasicompact and quasiseparated (e.g. if X is Noetherian) or if X is reduced, show that the following three notions are the same.

(a) V is an open subscheme of X intersect a closed subscheme of X
(b) V is an open subscheme of a closed subscheme of X
(c) V is a closed subscheme of an open subscheme of X.

(Hint: it will be helpful to note that the scheme-theoretic image may be computed on each open subset of the base.)

10. If $f : X \to Y$ is a locally closed immersion into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic image are (naturally in bijection with) the associated points of X. (Hint: Exercise .) Informally, we get no non-reduced structure on the scheme-theoretic closure not "forced by" that on X.

11. *(the notions "locally of finite type" and "finite type" are affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

12. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

13-. Show that finite morphisms are of finite type. Hence closed immersions are of finite type.

14+. *(not hard, but important)*

(a) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
(b) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)

(c) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.

(d) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

The following are double-plus problems because I'd like to see people try them.

15++. Show that the notion of "locally finite presentation" is affine-local.

16++. A scheme is *quasiseparated* if the intersection of two affine open sets is the finite union of affine schemes. Show that this notion is affine-local.

17++. A morphism is *quasiseparated* if the preimage of every affine scheme is a quasiseparated scheme. Show that this notion is affine-local on the target.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 8

RAVI VAKIL

This set is due at noon on Friday November 30. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 14 through 16.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1-. If k is algebraically closed, describe a natural map of *sets* $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k$. Show that this map is not surjective. On the other hand, show that it is a bijection on closed points.

2. The reason for the phrase or "base change" or "pullback" is the following. If X is a point of Z (i.e. f is the natural map of Spec of the residue field of a point of Z into Z), then W is interpreted as the fiber of the family. Show that in the category of topological spaces, this is true, i.e., if $Y \to Z$ is a continuous map, and X is a point p of Z, then the fiber of Y over p is naturally identified with $X \times_Z Y$.

3++. *(only for experts)* Suppose X and Z are affine, and Y_i is an affine open cover of Y. Suppose the covariant functor $F_Y : (\mathbf{Sch}_Y)^{opp} \to \mathbf{Sets}$ is a sheaf on the category of Y-schemes $\mathbf{Sch}_Y$, and F_{Y_i} is the "restriction of the sheaf to Y_i" (where we include only those Y-schemes that are in fact Y_i-schemes, i.e. those $T \to Y$ whose structure morphisms factor through Y_i, $T \to Y_i \to Y$). Show that if F_{Y_i} is representable, then so is F_Y.

4++. *(only for experts)* Suppose F_Y is given by

$$\left(T \xrightarrow{f} Y \right) \mapsto \left(\begin{tikzcd} \\ T \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right).$$

(The diagram on the right isn't intended to have a blank line on top!) Check that this F_Y is a sheaf.

Date: Friday, November 16, 2007. Last updated Nov. 27.

5+. Show that if X and Y are schemes, then there is a natural bijection between morphisms of schemes $X \to Y$ and morphisms of functor spaces $h^X \to h^Y$. (Hint: this has nothing to do with schemes; your argument will work in any category.)

6++. *(only for experts)* If a functor-space h is a sheaf that has an open cover by representable functor-spaces ("is covered by schemes"), then h is representable.

7++. *(only for experts)* Suppose $(Z_i)_i$ is an affine cover of Z, $(X_{ij})_j$ is an affine cover of the preimage of Z_i in X, and $(Y_{ik})_k$ is an affine cover of the preimage of Z_i in Y. Show that $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is an open cover of the functor $h_{X \times_Z Y}$. (Hint: use the definition of open covers!)

8. Show that $B \otimes_A A[t] \cong B[t]$.

9. *(repeat of older exercise; do this only if you haven't done it before)* Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right-exactness of $\otimes_C B$.)

10. Suppose $C \to B, A$ are two morphisms of rings. Suppose S is a multiplicative set of A. Then $(S \otimes 1)$ is a multiplicative set of $A \otimes_C B$. Show that there is a natural morphism $(S^{-1}A) \otimes_C B \cong (S \otimes 1)^{-1}(A \otimes_C B)$.

11. *(the three important types of monomorphisms of schemes)* Show that the following are monomorphisms: open immersions, closed immersions, and localization of affine schemes. As monomorphisms are closed under composition, compositions of the above are also monomorphisms (e.g. locally closed immersions, or maps from Spec of stalks at points of X to X).

12-. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

13. Show that the underlying topological space of the (scheme-theoretic) fiber $X \to Y$ above a point p is naturally identified with the topological fiber of $X \to Y$ above p.

14. Show that for finite-type schemes over $\mathbb{C}$, the closed points (=complex-valued points by the Nullstellensatz) of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

15. More generally, describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion.

16. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$, where $\operatorname{char} k \neq 2$. Show that $X \times_Y X$ has 2 irreducible components. (What happens if $\operatorname{char} k = 2$? See problem 25...)

17+. *(exercise generalizing $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$)* Suppose L/K is a finite Galois field extension. What is $L \otimes_K L$?

18++. *(hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$)* Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

19. *(weird but fun)* Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

20-. Show that locally principal closed subschemes pull back to locally principal closed subschemes.

21. (Each one of these counts for half a problem.) Show that the following properties of morphisms are preserved by base change.

(a) quasicompact
(b) quasiseparated
(c) affine morphism
(d) finite
(e) locally of finite type
(f) finite type
(g) locally of finite presentation
(h) finite presentation

22+. Show that the notion of "quasifinite morphism" (finite type + finite fibers) is preserved by base change. (Warning: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 18.)

23. Show that surjectivity is preserved by base change. (*Surjectivity* has its usual meaning: surjective as a map of sets.) (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

24. If P is a property of morphisms preserved by base change, and $X \to Y$ and $X' \times Y'$ are two morphisms of S-schemes with property P, show that $X \times_S X' \to Y \times_S Y'$ has property P as well.

25-. Suppose k is a field of characteristic p, so $k(u^p)/k(u)$ is an inseparable extension. By considering $k(u^p) \otimes_{k(u)} k(u^p)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (The fact that I'm giving you this example should show that this happens only in characteristic p, in the presence of something as strange as inseparability.)

26. Show that the notion of "connected (resp. irreducible, integral, reduced)" geometric fibers behaves well under base change.

27. *(for the arithmetically-minded)* Show that for the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, all geometric fibers consist of two reduced points.

28. Recall the example of the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fibers over $0 = [(x)]$.

29++. Suppose X is a k-scheme.

(a) Show that X is geometrically irreducible if and only if $X \times_k k^s$ is irreducible if and only if $X \times_k K$ is irreducible for all field extensions K/k. (Here k^s is the separable closure of k.)
(b) Show that X is geometrically connected if and only if $X \times_k k^s$ is connected if and only if $X \times_k K$ is connected for all field extensions K/k.
(c) Show that X is geometrically reduced if and only if $X \times_k k^p$ is reduced if and only if $X \times_k K$ is reduced for all field extensions K/k. (Here k^p is the perfect closure of k.) Thus if $\operatorname{char} k = 0$, then X is geometrically reduced if and only if it is reduced.
(d) Combining (a) and (c), show that X is geometrically integral if and only if $X \times_k K$ is geometrically integral for all field extensions K/k.

30. Check that the maps defined in class glue to give a well-defined morphism $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$.

31+. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank}\begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

32. *(A co-ordinate-free description of the Segre embedding)* Show that the Segre embedding can be interpreted as $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$ via the surjective map of graded rings

$$\operatorname{Sym}^\bullet(V^\vee \otimes W^\vee) \twoheadrightarrow \sum_{i=0}^\infty \left(\operatorname{Sym}^i V^\vee\right) \otimes \left(\operatorname{Sym}^i W^\vee\right)$$

"in the opposite direction".

33. *(important but easy)* Show that open immersions and closed immersions are separated.

34. *(also important but easy)* Show that every morphism of affine schemes is separated.

35. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 9

RAVI VAKIL

This set covers classes 17 and 18.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} A$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} A$, $U \cap V$ is a *finite* union of affine open sets. (Hint: compare this to the proposition showing that the intersection of two affine open sets on a separated scheme over an affine scheme is affine.)

2. *(a nonquasiseparated scheme)* Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated. (This open immersion $U \hookrightarrow X$ came up earlier, as an example of a nonquasicompact open subset of an affine scheme.)

3. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target; use a similar argument.)

4. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion.

5. Show that a A-scheme is separated (over A) if and only if it is separated over $\mathbb{Z}$. (In particular, a complex scheme is separated over $\mathbb{C}$ if and only if it is separated over $\mathbb{Z}$, so complex geometers and arithmetic geometers can communicate about separated schemes without confusion.)

6+. *(useful exercise: The locus where two morphisms agree)* Suppose f and g are two morphisms $X \to Y$, over some scheme Z. We can now give meaning to the phrase 'the locus where f and g agree', and that in particular there is a smallest locally closed subscheme where they agree. Suppose $h : W \to X$ is some morphism (perhaps a locally closed immersion). We say that f and g agree on h if $f \circ h = g \circ h$. Show that there is a locally closed subscheme $i : V \hookrightarrow X$ such that any morphism $h : W \to X$ on which f and g agree factors

Date: Friday, November 30, 2007.

uniquely through i, i.e. there is a unique $j : W \to V$ such that $h = i \circ j$. (You may recognize this as a universal property statement.) Show further that if $V \to Z$ is separated, then $i : V \hookrightarrow X$ is a closed immersion. Hint: define V to be the following fibered product:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\ X \arrow[r, "(f,g)"] & Y \times_Z Y. \end{tikzcd}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V.

7. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given in an exercise, I believe last day.)

8. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $f : X \to Y$ is in P then $f^{\text{red}} : X^{\text{red}} \to Y^{\text{red}}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) Hint:

$$\begin{tikzcd} X^{\text{red}} \arrow[r] \arrow[dr] & X \times_Y Y^{\text{red}} \arrow[r] \arrow[d] & Y^{\text{red}} \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

9-. Interpret rational functions on a separated integral scheme as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$. (This is analogous to functions corresponding to morphisms to $\mathbb{A}^1_{\mathbb{Z}}$, an earlier exercise.)

10. In class, we prove that two S-morphisms $f_1, f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same. Give examples to show how this breaks down when we give up reducedness of the base or separatedness of the target. Here are some possibilities. For the first, consider the two maps $\operatorname{Spec} k[x, y]/(y^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto x$ and f_2 given by $t \mapsto x + y$; f_1 and f_2 agree on the distinguished open set $D(x)$. (See Figure 1.) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$. (See Figure 2.)

11. Show that the graph of a rational map is independent of the choice of representative of the rational map.

12. *(important)* Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant. In particular, integral separated schemes and dominant rational maps between them form a category which is geometrically interesting.

13-. Show that dominant rational maps give morphisms of function fields in the opposite direction.

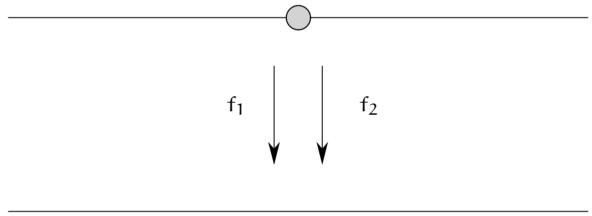

FIGURE 1. Two different maps from a nonreduced scheme agreeing on an open set

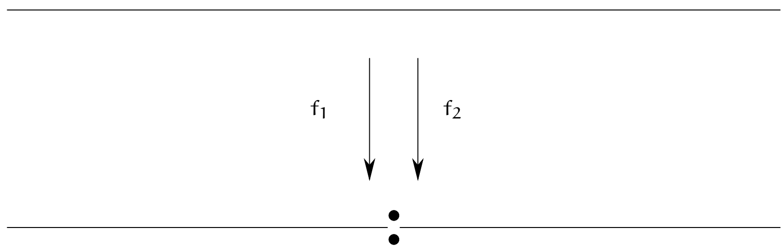

FIGURE 2. Two different maps to a nonseparated scheme agreeing on an open set

14. Let K be a finitely generated field extension of k. Show there exists an irreducible k-variety with function field K. (Hint: let $x_1, \dots, x_n$ be generators for K over k. Consider the map $k[t_1, \dots, t_n] \to K$ given by $t_i \mapsto x_i$, and show that the kernel is a prime ideal $\mathfrak{p}$, and that $k[t_1, \dots, t_n]/\mathfrak{p}$ has fraction field K. This can be interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \dots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the image.)

15. Use our discussion in class to find a "formula" yielding all Pythagorean triples.

16. Show that the conic $x^2+y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (We've done this earlier in the case where k is algebraically closed, by diagonalizing quadrics.)

17. Find all rational solutions to $y^2 = x^3+x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

18. Find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

19++. *(a first view of a blow-up)* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Describe the fiber of the morphism $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\operatorname{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor (a closed subscheme that is locally principal and not a zero-divisor). It is called the *exceptional divisor*.

20. *(the Cremona transformation, a useful classical construction)* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

21. Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, by finding a base change that turns this into a non-closed map. (Hint: Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 10

RAVI VAKIL

This set covers classes 19 and 20.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Show that if Y is an irreducible subset of a scheme X with generic point y, show that the codimension of Y is the dimension of the local ring $\mathcal{O}_{X,y}$.

2-. Show that

$$\operatorname{codim}_X Y + \dim Y \leq \dim X. \tag{1}$$

3++. Show that if $f : B \to A$ is a ring homomorphism, and $(b_1, \dots, b_n) = 1$ in B, and $B_{b_i} \to A_{f(b_i)}$ is integral, then f is integral. Thus we can define the notion of **integral morphism** of schemes.

4+. Show that the notion of integral homomorphism is well behaved with respect to localization and quotient of B, and quotient of A, but not localization of A. Show that the notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A. If possible, draw pictures of your examples.

5. Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.

6-. *(finite = integral + finite type)* Show that a morphism is finite if and only if it is integral and finite type.

7-. *(reality check)* The morphism $k[t] \to k[t]_{(t)}$ is not integral, as $1/t$ satisfies no monic polynomial with coefficients in $k[t]$. Show that the conclusion of the Going-up theorem fails.

8. Show that the special case of the Going-Up Theorem where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

Date: Friday, December 7, 2007.

9+. *(important but straightforward exercise, sometimes also called the going-up theorem)* Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

10++. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$. (Hint: show that a chain of prime ideals downstairs gives a chain upstairs, by the previous exercise, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element $[\mathfrak{q}] \in \operatorname{Spec} B$ of the chain downstairs. As integral extensions are well-behaved by localization and quotients of $\operatorname{Spec} B$ (Exercise), we can replace B by $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (and A by $A \otimes_B (B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}})$). Thus we can assume B is a field. Hence we must show that if $\phi : k \to A$ is an integral extension, then $\dim A = 0$. Outline of proof: Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible: Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible — recall the coefficients are in a field.)

11. *(Nakayama's lemma version 3)* Suppose A is a ring, and I is an ideal of A contained in all maximal ideals. Suppose M is a *finitely generated* A-module, and $N \subset M$ is a submodule. If $N/IN \to M/IM$ an isomorphism, then $M = N$. (This can be useful, although it won't come up again for us.)

12+. *(Nakayama's lemma version 4)* Suppose $(A, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated A-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

13. *(Nakayama's lemma version 5)* Prove Nakayama version 1 without the hypothesis that M is finitely generated, but with the hypothesis that $I^n = 0$ for some n. (This argument does *not* use the trick.) This result is quite useful, although we won't use it.

14+. *(used in the proof of Algebraic Hartogs' Lemma)* Suppose S is a subring of a ring A, and $r \in A$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Show that r is integral over S. (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

15+. *(Nullstellensatz from dimension theory)*
(a) Suppose $A = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$. This version (the "weak Nullstellensatz") was stated earlier.
(b) Suppose $A = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of A has a residue field that is a finite extension of k. This version was stated in earlier. (Hint for both parts: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.)

16. *(important)* Suppose X is an irreducible variety. Show that $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X. (This is not true in general, see the pathology in the notes.)

17. Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions. (This isn't essential for what follows. But it is a basic fact, and very believable.)

18. Suppose $X \subset Y$ is an inclusion of irreducible k-varieties, and η is the generic point of X. Show that $\dim X + \dim \mathcal{O}_{Y,\eta} = \dim Y$. Hence show that $\dim X + \operatorname{codim}_Y X = \dim Y$. Thus for varieties, the inequality $\dim X + \operatorname{codim}_Y X \leq \dim Y$ is always an equality.

19. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (You may recognize this as the affine cone over the twisted cubic.) It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem, see later). The second equation cuts out a surface: our surface, along with another surface. The third equation cuts out our surface, and removes the "extraneous component". One last aside: notice once again that the cone over the quadric surface $k[w, x, y, z]/(wz - xy)$ makes an appearance.)

20++. Reduce the proof of Chevalley's theorem to the following statement: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Then the image of f contains a dense open subset of $\operatorname{Spec} B$. (Hint: Make a series of reductions. The notion of constructable is local, so reduce to the case where Y is affine. Then X can be expressed as a finite union of affines; reduce to the case where X is affine. X can be expressed as the finite union of irreducible components; reduce to the case where X is irreducible. Reduce to the case where X is reduced. By considering the closure of the image of the generic point of X, reduce to the case where Y also is integral (irreducible and reduced), and $X \to Y$ is dominant. Use Noetherian induction in some way on Y.)

21. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wz - xy, y^{17} + z^{17})$? (Be careful to check they hypotheses before invoking Krull!)

22. *(important for later)* (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called **linear subspaces**. Dimension $1, 2$ linear subspaces are called **lines** and **planes** respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. Hint: Refine your argument in (a). (In fact any two things in projective space that you might expect to meet for dimensional reasons do in fact meet. We won't prove that here.)
(c) Show further that there is an intersection of $r + 1$ hypersurfaces missing X. (The key

step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X. Show this by induction on the number of generic points. To get from n to $n+1$: take a hypersurface not vanishing on $p_1, \dots, p_n$. If it doesn't vanish on p_{n+1}, we're done. Otherwise, call this hypersurface f_{n+1}. Do something similar with $n+1$ replaced by i $(1 \leq i \leq n)$. Then consider $\sum_i f_1 \cdots \widehat{f_i} \cdots f_{n+1}$.)

23-. Show that it is false that if X is an integral scheme, and U is a non-empty open set, then $\dim U = \dim X$.

24. Suppose f is an element of a normal domain A, and f is contained in no codimension 1 primes. Show that f is a unit.

25. Suppose f and g are two global sections of a Noetherian normal scheme, not vanishing at any associated point, with the same poles and zeros. Show that each is a unit times the other.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 11

RAVI VAKIL

This set covers classes 21 through 27.

Please *read all of the problems in these classes*, and ask me about any statements that you are unsure of, even of the many problems you won't try.

Hand in seven solutions. Also, pick one problem (which need not be one of the seven) that you find the most interesting, and explain why (in a couple of sentences).

If you are ambitious (and have the time), hand in more problems. Try to solve problems on a range of topics, perhaps even one from each class. Try some hard problems as well as some easy problems. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Please hand in the problem sets in my mailbox by the afternoon of **Friday, February 15.**

E-mail address: `vakil@math.stanford.edu`

Date: Tuesday, February 5, 2008.

FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 12

RAVI VAKIL

This set covers classes 17 through 29.

Please *read all of the problems in these classes (if you haven't before)*, and ask me about any statements that you are unsure of, even of the many problems you won't try.

Hand in six solutions, including at least three from classes 28 and 29. Also, pick one problem (which need not be one of the seven) that you find the most interesting, and explain why (in a couple of sentences).

If you are ambitious (and have the time), hand in more problems. Try to solve problems on a range of topics, perhaps even one from each class. Try some hard problems as well as some easy problems. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Please hand in the problem sets in my mailbox by the afternoon of **Wednesday, February 20.**

E-mail address: vakil@math.stanford.edu

Date: Friday, February 8, 2008.

FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 13

RAVI VAKIL

This set covers all classes this quarter.

Please *read all of the problems in these classes for which there are notes (if you haven't before)*, and ask me about any statements that you are unsure of, even of the many problems you won't try.

Hand in six solutions, including at least three from class 30 onward. Also, pick one problem (which need not be one of the six) that you find the most interesting, and explain why (in a couple of sentences).

If you are ambitious (and have the time), hand in more problems. Try to solve problems on a range of topics, perhaps even one from each class. Try some hard problems as well as some easy problems. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Please hand in the problem sets in my mailbox by the afternoon of **Sunday, March 23.**

E-mail address: `vakil@math.stanford.edu`

Date: Friday, March 14, 2008.