FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 5

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This set is due at noon on Friday November 2. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 9 and 10.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1-. Show that $\mathbb{P}^n_k$ is irreducible.

2. An earlier exercise showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

3. Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian (i.e. if X is a Noetherian scheme), then X is a Noetherian topological space.

4-. Show that an irreducible topological space is connected.

5-. Give (with proof!) an example of a scheme that is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

6-. Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine schemes (Hence $\mathbb{P}^n_k$ is quasicompact.)

7. *(quasicompact schemes have closed points)* Show that if X is a nonempty quasicompact scheme, then it has a closed point. (Warning: there exist non-empty schemes with no closed points, so your argument had better use the quasicompactness hypothesis!)

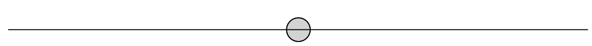

FIGURE 1. A picture of the scheme $\operatorname{Spec} k[x,y]/(xy, y^2)$

8. Show that $\left(k[x,y]/(y^2, xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture.)

Date: Friday, October 26, 2007.

9. *(reducedness is stalk-local)* Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from an earlier Exercise, and the nilradical is intersection of all prime ideals.)

10-. Suppose X is quasicompact, and f is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of x. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

11+. Show that a scheme X is integral if and only if it is irreducible and reduced.

12. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

13. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $\mathrm{FF}(A)$, the fraction field of A. This is called the **function field** $\mathrm{FF}(X)$ of X. It can be computed on any non-empty open set of X, as any such open set contains the generic point.

14. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X (so A is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \mathrm{FF}(A)$ (where U is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient that sections over different open sets can be considered subsets of the same thing. This makes restriction maps and gluing easy to consider; this is one reason why varieties are usually introduced before schemes.

15. Show that all open subsets of a Noetherian topological space (e.g. a Noetherian scheme) are quasicompact.

16. Show that a Noetherian scheme has a finite number of irreducible components.

17. If X is a Noetherian scheme, show that every point p has a closed point in its closure. (In particular, every non-empty Noetherian scheme has closed points; this is not true for every scheme.)

18. If X is an affine scheme or Noetherian scheme, show that it suffices to check reducedness at *closed points*.

19. Show that a locally Noetherian scheme X is integral if and only if X is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains (informally: "the scheme is locally integral"). Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are domains) and global (connected) conditions.

20. Show that X is reduced if and only if X can be covered by affine opens $\operatorname{Spec} A$ where A is reduced (nilpotent-free).

21. Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. (Recall the following form of Hilbert's Nullstellensatz, richer than the version stated before: the maximal ideals of $k[x_1, \dots, x_n]$ are precisely those with residue of the form a finite extension of k.) Show that the closed points are dense on such a scheme.

22. Finish the proof that Noetherianness is an affine-local property: show that if A is a ring, and $(f_1, \dots, f_n) = A$, and A_{f_i} is Noetherian, then A is Noetherian.

23. Prove that reducedness is an affine-local property.

24. Show that finite-generatedness over k is an affine-local property (see the notes for an outline).

25. Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (The domain portion is easy. Hint for integral closure: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in $A[x]$ that also has a root in the fraction field.)

26. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes.

27. If A is an integral domain, show that $A = \cap A_m$, where the intersection runs over all maximal ideals of A. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

28. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[a, b, c, d]/(ad - bc)$ (which we will later recognize as the cone over the quadric surface), and $a/c = b/d \in FF(A)$. Show that $I = (c, d)$. (If you can, show that this is not principal.)

29. Show that any localization of a Unique Factorization Domain is a Unique Factorization Domain.

30+. Show that unique factorization domains are integrally closed. Hence factorial schemes are are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we'll see in Exercise . An example without proof: $\mathbb{Z}[\sqrt{17}]$ again.)

31-. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$.

32+. *(this will give us a number of enlightening examples later)* Suppose A is a Unique Factorization Domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if f is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is a domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) = 0$ with $F(T) \in B[T]$ which has a solution α in $FF(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume

$F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in FF(A)$. Now α is the solution of monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f)T \in FF(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - rs^2 f)/4t^2$. Then $t = 1$, and r is even.)

33+. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$;
(c) $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. (This is our cone over a quadric surface example.)

34+. Suppose A is a k-algebra where $\operatorname{char} k = 0$, and l/k is a finite field extension. Show that A is normal if and only if $A \otimes_k l$ is normal. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal if k is characteristic 0. (In fact the hypothesis on the characteristic is unnecessary.) Possible hint: reduced to the case where l/k is Galois.

35-. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. (Caution: $\sqrt{\mathfrak{q}}$ can be prime without $\mathfrak{q}$ being primary — consider our example (y^2, xy) in $k[x, y]$.)

36-. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

37-. *(reality check)* Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

38+. *(existence of primary decomposition for Noetherian rings)* Suppose A is a Noetherian ring. Show that every proper ideal $I \subset A$ has a primary decomposition. (Hint: mimic the Noetherian induction argument we saw last week.)

39+. (a) Find a minimal primary decomposition of (y^2, xy). (b) Find another one. (Possible hint: see Figure 1. You might be able to draw sketches of your different primary decompositions.)

40+. (a) If $\mathfrak{p}$, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup_{i=1}^n \mathfrak{p}_i$. (The right side is not an ideal!) Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n. Don't look in the literature — you might find a much longer argument!)
(Parts (a) and (b) are "geometric facts"; try to draw pictures of what they mean.)

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