FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 4

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This set is due at noon on Friday October 26. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 7 and 8.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 1), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

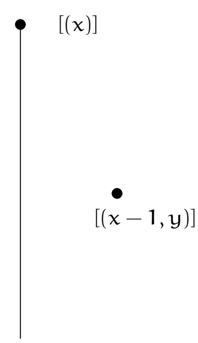

FIGURE 1. The set $S = \{[(x)], (1, 0)\}$, pictured as a subset of $\mathbb{A}^2$

2. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are deined analogously.) Give generators for the ideal $I(X)$. Be sure to prove it! Hint: We will see later that this ideal is not generated by less than three elements.

3. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

4+. *(important)* Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between

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points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

The next six problems on distinguished open sets will be very useful. Please think about them!

5. Show that the distinguished open sets form a base for the Zariski topology. (Hint: Given an ideal I, show that the complement of $V(I)$ is $\cup_{f \in I} D(f)$.)

6+. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$. (One of the directions will use the fact that any proper ideal of A is contained in some maximal ideal.)

7. Show that if $\operatorname{Spec} A$ is an infinite union $\cup_{i \in J} D(f_i)$, then in fact it is a union of a finite number of these. (Hint: use the previous exercise.) Show that $\operatorname{Spec} A$ is quasicompact.

8-. Show that $D(f) \cap D(g) = D(fg)$.

9. Show that if $D(f) \subset D(g)$, if and only if $f^n \in (g)$ for some n if and only if g is a unit in A_f. (Hint for the first equivalence: $f \in I(V((g)))$. We will use this shortly.

10. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

11+. Prove base identity for the structure sheaf for any distinguished open $D(f)$. (Possible strategy: show that the argument is the same as the argument in class for $\operatorname{Spec} A$.)

12+. Prove base gluability for any distinguished open $D(f)$.

13+. Suppose M is an A-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_A A_f$; the restriction map is the "obvious" one. This is an $\mathcal{O}_{\operatorname{Spec} A}$-module! This sort of sheaf $\tilde{M}$ will be very important soon; it is an example of a *quasicoherent sheaf*.

14. *(important)* Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

15. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

16. If X is a scheme, and U is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

17. *(important)* Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the ring $A_{\mathfrak{p}}$.

18. Show that the affine line with doubled origin is not an affine scheme. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

19-. Define the *affine plane with doubled origin*. Use this example to show that the intersection of two affine open sets need not be an affine open set.

20+. Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" (to be stated rigorously later) says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

21. Show that if k is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. (The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.)

22. (a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)
(b) Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme.

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