FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 1

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This set is due at noon on Friday October 5. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 1 and 2.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in ten solutions, where each "-" problem is worth half a solution. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 1.

1-. A category in which each morphism is an isomorphism is called a *groupoid*.
(a) A perverse definition of a group is: a groupoid with one element. Make sense of this.
(b) Describe a groupoid that is not a group.
(For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of x, and the morphisms from $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and the $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

2-. If A is an object in a category $\mathcal{C}$, show that the isomorphisms of A with itself $\mathrm{Isom}(A, A)$ form a group (called the *automorphism group of* A, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in the **Sets** and $\mathbf{Vec}_k$ (k-vector spaces)? Show that two isomorphic objects have isomorphic automorphism groups.

3. *(if you haven't seen tensor products before)* Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (This exercise is intended to give some hands-on practice with tensor products.)

4. *(right-exactness of $\cdot \otimes_A N$)* Show that $\cdot \otimes_A N$ gives a covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. Show that $\cdot \otimes_A N$ is a *right-exact functor*, i.e. if

$$M' \to M \to M'' \to 0$$

is an exact sequence of A-modules, then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. (For experts: is there a universal property proof?)

Date: Thursday, September 27, 2007. Problem 2 fixed Oct. 13, 2007.

5. In the universal property definition of tensor product, show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product. (This exercise will prime you for Yoneda's Lemma.)

6. Show that the construction of tensor product given in class satisfies the universal property of tensor product.

7-. Show that any two initial objects are canonically isomorphic. Show that any two final objects are canonically isomorphic.

Class 2.

8. *Important Exercise that everyone should do once in their life.* Prove the form of Yoneda's lemma stated in class. (See the class notes for a hint.)

9. Show that in **Sets**, show that

$$X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}.$$

More precisely, describe a natural isomorphism between the left and right sides. (This will help you build intuition for fibered products.)

10-. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is. (Hint: it has a one-word description.)

11-. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$": "the" fibered product over Z is canonically isomorphic to "the" product. (This is an exercise about unwinding the definition.)

12-. Show that in the category **Ab** of abelian groups, the kernel K of $f : A \to B$ can be interpreted as a fibered product:

$$\begin{array}{ccc} K & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array}$$

13-. Prove a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (What is this natural morphism?!) We may then take this as the definition of monomorphism. (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

14-. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. We will use this later when talking about fibered products. (Hint: for any object V, give a natural bijection between maps from V to the first and maps from V to the second.)

15-. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

16-. Define *coproduct* in a category by reversing all the arrows in the definition of product. Show that coproduct for **Sets** is disjoint union.

17. Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Define a ring structure $A \otimes_C B$ with multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. There is a natural morphism $A \to A \otimes_C B$ given by $a \mapsto (a, 1)$. (Warning: This is not necessarily an inclusion.) Similarly, there is a natural morphism $B \to A \otimes_C B$. Show that this gives a coproduct on rings, i.e. that

$$\begin{array}{ccc} A \otimes_C B & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & C \end{array}$$

satisfies the universal property of coproduct.

18. *Important Exercise for Later.* We continue the notation of the previous exercise. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right exactness of $\otimes_C B$.)

19. Show that in the category **Sets**,

$$\{(a_i)_{i \in I} \in \prod_i A_i : F(m)(a_i) = a_j \text{ for all } [m : i \to j] \in \operatorname{Mor}(\mathcal{I})\},$$

along with the projection maps to each A_i, is the limit $\varprojlim_{\mathcal{I}} A_i$.

20-. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.)

21. Consider the set $\{(i \in \mathcal{I}, a_i \in A_i)\}$ modulo the equivalence generated by: if $m : i \to j$ is an arrow in $\mathcal{I}$, then $(i, a_i) \sim (j, F(m)(a_j))$. Show that this set, along with the obvious maps from each A_i, is the colimit.

22. Verify that the construction of colimits of A-modules given in class are indeed colimits.

23-. Write down what the condition not mentioned in class in the definition of adjoint should be. (See the class notes.)

24. Suppose M, N, and P are A-modules. Describe a natural bijection $\operatorname{Mor}_A(M \otimes_A N, P) = \operatorname{Mor}_A(M, \operatorname{Mor}_A(N, P))$. (Hint: try to use the universal property.) If you want, you could check that $\cdot \otimes_A N$ and $\operatorname{Mor}_A(N, \cdot)$ are adjoint functors. (Checking adjointness is never any fun!) We may later see why problem 24 implies problem 4.

25. Define groupification H from the category of abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are (a, b), which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d = b + c$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let F be the forgetful morphism from the category of abelian groups **Ab** to the category of abelian semigroups. Show that H is left-adjoint to F.

26-. Show that if a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

27. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory of the category of A-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

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