FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 45

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1. HYPERELLIPTIC CURVES

A curve C of genus at least 2 is **hyperelliptic** if it admits a degree 2 cover of $\mathbb{P}^1$. This map is often called the **hyperelliptic map**.

1.A. EXERCISE. Verify that a curve C of genus at least 1 admits a degree 2 cover of $\mathbb{P}^1$ if and only if it admits a degree 2 invertible sheaf $\mathcal{L}$ with $h^0(C, \mathcal{L}) = 2$. Possibly in the course of doing this, verify that if C is a curve, and $\mathcal{L}$ has a degree 2 invertible sheaf with at least 2 (linearly independent) sections, then $\mathcal{L}$ has precisely two sections, and that this $\mathcal{L}$ is base-point free and gives a hyperelliptic map.

The degree 2 map $C \to \mathbb{P}^1$ gives a degree 2 extension of function fields $FF(C)$ over $FF(\mathbb{P}^1) \cong k(t)$. If the characteristic is not 2, this extension is necessarily Galois, and the involution on C induces (via the equivalence of various categories of curves, Class 42 Theorem 1.1) an involution on C is called the **hyperelliptic involution**.

1.1. *Proposition. — If $\mathcal{L}$ corresponds to a hyperelliptic cover $C \to \mathbb{P}^1$, then $\mathcal{L}^{\otimes(g-1)} \cong \mathcal{K}_C$.*

Proof. Compose the hyperelliptic map with the $(g-1)$th Veronese map:

$$C \xrightarrow{\mathcal{L}} \mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(g-1)} \mathbb{P}^{g-1}.$$

The composition corresponds to $\mathcal{L}^{\otimes(g-1)}$. This invertible sheaf has degree $2g-2$, and the image is nondegenerate in $\mathbb{P}^{g-1}$, and hence has at least g sections. But by Exercise 1.C of Class 44, the only invertible sheaf of degree $2g-2$ with (at least) g sections is the canonical sheaf. □

Date: Tuesday, April 15, 2008.

1.2. *Proposition.* — *Any curve C of genus at least 2 admits at most one double cover of* $\mathbb{P}^1$. *In other words, a curve can be in "only one way".*

Proof. If C is hyperelliptic, then we can recover the hyperelliptic map by considering the canonical linear system given by $\mathcal{K}$ (the *canonical map*, which we'll use again soon): it is a double cover of a degree $g-1$ rational normal curve (by the previous Proposition), and this double cover is the hyperelliptic cover (also by the proof of the previous Proposition). □

Next, we invoke the Riemann-Hurwitz formula. In order to do so, we need to assume $\operatorname{char} k = 0$, and $k = \overline{k}$. However, when we actually discuss differentials, and prove the Riemann-Hurwitz formula, we will see that we can just require $\operatorname{char} k \neq 2$ (and $k = \overline{k}$).

The Riemann-Hurwitz formula implies that hyperelliptic covers have precisely $2g+2$ (distinct) branch points. These branch points determine the cover:

1.3. *Claim.* — *Assume* $\operatorname{char} k \neq 2$ *and* $k = \overline{k}$. *Given* n *distinct points* $r_1, \dots, r_n \in \mathbb{P}^1$, *there is precisely one cover branched at precisely these points if* n *is even, and none if* n *is odd.*

Proof. The result when n is odd is immediate from the Riemann-Hurwitz formula, so assume n is even.

Pick a point of $\mathbb{P}^1$ distinct from the n branch points, so all n branch points are in the "complementary" $\mathbb{A}^1$. Suppose we have a double cover of $\mathbb{A}^1$, $C \to \mathbb{A}^1$, where x is the coordinate on $\mathbb{A}^1$. This induces a quadratic field extension K over $k(x)$. As $\operatorname{char} k \neq 2$, this extension is Galois. Let σ be the hyperelliptic involution. Let y be an element of K such that $\sigma(y) = -y$, so 1 and y form a basis for K over the field $k(x)$, and are eigenvectors of σ. Now $\sigma(y^2) = y^2$, so $y^2 \in k(x)$. We can replace y by an appropriate $k(x)$-multiple so that y^2 is a polynomial, with no repeated factors, and monic. (This is where we use the hypothesis that k is algebraically closed, to get leading coefficient 1.)

Thus $y^2 = x^N + a_{N-1}x^{N-1} + \cdots + a_0$, where the polynomial on the right (call it $f(x)$) has no repeated roots. The Jacobian criterion implies that this curve C' in $\mathbb{A}^2$ (with co-ordinates x and y) is nonsingular. then C' is normal and has the same function field as C — but so is C. Thus C' and C are both normalizations of $\mathbb{A}^1$ in the finite field extension generated by y, and hence are isomorphic. Thus we have identified C in terms of an explicit equation!

The branch points correspond to those values of x for which there is exactly one value of y, i.e. the roots of $f(x)$. In particular, $N = n$, and $f(x) = (x - r_1)\cdots(x - r_n)$ (where the r_i are interpreted as elements of $\overline{k}$).

Having mastered the situation over $\mathbb{A}^1$, we return to the situation over $\mathbb{P}^1$. We have identified the function field extension K of $\mathrm{FF}(\mathbb{P}^1) = k(x)$ corresponding to any C double-covering $\mathbb{P}^1$ over the n points — there is only one up to isomorphism, given by adjoining y with $y^2 = (x - r_1)\cdots(x - r_n)$. There is a unique curve branched over $r_1, \dots, r_n$: the normalization of $\mathbb{P}^1$ in the field extension $K/k(x)$. (You might fear that we haven't accidentally acquired a branch point at the missing point $\infty = \mathbb{P}^1 - \mathbb{A}^1$. But the total number of branch points is even, and we already have an even number of points, so there is no branching at ∞.) $\square$

We can now extract a lot of useful information.

1.4. *Curves of every genus.* For the first time, we see that there are curves of every genus $g \geq 0$ over an algebraically closed field of characteristic 0: to get a curve of genus g, consider the branched cover branched over $2g+2$ distinct points. The unique genus 0 curve is of this form, and we have seen above that every genus 2 curve is of this form. We'll soon see that every genus 1 curve is too. But it is too much to hope that all curves are of this form, and in Exercise 2.A we'll see that there are genus 3 curves that are *not* hyperelliptic, and we'll get heuristic evidence that "most" genus 3 curves are not hyperelliptic. We'll later get heuristic evidence that "most" genus g curves are not hyperelliptic if $g > 2$.

We can also classify hyperelliptic curves. Hyperelliptic curves of genus g correspond to precisely $2g+2$ points on $\mathbb{P}^1$ modulo S_{2g+2}, and modulo automorphisms of $\mathbb{P}^1$. Thus "the space of hyperelliptic curves" has dimension

$$2g + 2 - \dim \operatorname{Aut} \mathbb{P}^1 = 2g - 1.$$

This is not a well-defined statement, because we haven't rigorously defined "the space of hyperelliptic curves" — and example of a *moduli space*. For now, take it as a plausibility statement. It is also plausible that this space is irreducible and reduced — it is the image of something irreducible and reduced.

1.5. *Genus* 2 *in particular.* In particular, if $g = 2$, we see that we have a "three-dimensional space of genus 2 curves". This isn't rigorous, but we can certainly show that there are an infinite number of non-isomorphic genus 2 curves.

1.B. EXERCISE. Fix an algebraically closed field k of characteristic 0. Show that there are an infinite number of (pairwise) non-isomorphic genus 2 curves k.

1.6. *If* k *is not algebraically closed.* If k is not algebraically closed (but of characteristic not 2), the above argument shows that if we have a double cover of $\mathbb{A}^1$, then it is of the form $y^2 = af(x)$, where f is monic, and $a \in k^*/(k^*)^2$. You may be able to use this to show that (assuming the $k^* \neq (k^*)^2$) a double cover is *not* determined by its branch points. Moreover, see that this failure is classified by $k^*/(k^*)^2$. Thus we have lots of curves that are not isomorphic over k, but become isomorphic over $\overline{k}$. These are often called *twists* of each other.

(In particular, even though haven't talked about elliptic curves yet, we definitely have two elliptic curves over $\mathbb{Q}$ with the same j-invariant, that are not isomorphic.)

2. CURVES OF GENUS 3

Suppose C is a curve of genus 3. Then $\mathcal{K}$ has degree $2g-2=4$, and has $g=3$ sections.

2.1. *Claim. — $\mathcal{K}$ is base-point-free, and hence gives a map to $\mathbb{P}^2$.*

Proof. We check base-point-freeness by working over the algebraic closure $\overline{k}$. For any point p, by Riemann-Roch,

$$h^0(C,\mathcal{K}(-p)) - h^0(C,\mathcal{O}(p)) = \deg(\mathcal{K}(-p)) - g + 1 = 3 - 3 + 1 = 1.$$

But $h^0(C,\mathcal{O}(p)) = 1$ by Claim 2.3 of Class 44, so

$$h^0(C,\mathcal{K}(-p)) = 2 = h^0(C,\mathcal{K}) - 1.$$

Thus p is not a base-point of $\mathcal{K}$ for any p, so by Criterion 1.4 of Class 44 for base-point-freeness, $\mathcal{K}$ is base-point-free. □

The next natural question is: Is this a closed immersion? Again, we can check over algebraic closure. We use our "closed immersion test" (again, see our useful facts). If it *isn't* a closed immersion, then we can find two points p and q (possibly identical) such that

$$h^0(C,\mathcal{K}) - h^0(C,\mathcal{K}(-p-q)) = 2,$$

i.e. $h^0(C,\mathcal{K}(-p-q)) = 2$. But by Serre duality, this means that $h^0(C,\mathcal{O}(p+q)) = 2$. We have found a degree 2 divisor with 2 sections, so C is hyperelliptic. (Indeed, I could have skipped that sentence, and made this observation about $\mathcal{K}(-p-q)$, but I've done it this way in order to generalize to higher genus.) Conversely, if C is hyperelliptic, then we already know that $\mathcal{K}$ gives a double cover of a nonsingular conic in $\mathbb{P}^2$ (also known as a rational normal curve of degree 2), and hence $\mathcal{K}$ does not give a closed immersion.

Thus we conclude that if (and only if) C is not hyperelliptic, then the canonical map describes C as a degree 4 curve in $\mathbb{P}^2$.

Conversely, any quartic plane curve is canonically embedded. Reason: the curve has genus 3 (we can compute this — see our discussion of Hilbert functions), and is mapped by an invertible sheaf of degree 4 with 3 sections. But by Exercise 1.C of Class 44, the only invertible sheaf of degree $2g-2$ with g sections is $\mathcal{K}$.

In particular, each non-hyperelliptic genus 3 curve can be described as a quartic plane curve in only one way (up to automorphisms of $\mathbb{P}^2$).

In conclusion, there is a bijection between non-hyperelliptic genus 3 curves, and plane quartics up to projective linear transformations.

2.A. EXERCISE. Show that there are non-hyperelliptic genus 3 curves.

2.B. EXERCISE. Give a heuristic (non-rigorous) argument that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = \mathsf{PGL}(3)$.)

The genus 3 curves thus seem to come in two families: the hyperelliptic curves (a family of dimension 5), and the nonhyperelliptic curves (a family of dimension 6). This is misleading — they actually come in a single family of dimension 6.

In fact, hyperelliptic curves are naturally limits of nonhyperelliptic curves. We can write down an explicit family. (This explanation necessarily requires some hand-waving, as it involves topics we haven't seen yet.) Suppose we have a hyperelliptic curve branched over $2g + 2 = 8$ points of $\mathbb{P}^1$. Choose an isomorphism of $\mathbb{P}^1$ with a conic in $\mathbb{P}^2$. There is a nonsingular quartic meeting the conic at precisely those 8 points. (This requires Bertini's theorem, so I'll skip that argument.) Then if f is the equation of the conic, and g is the equation of the quartic, then $f^2 + t^2 g$ is a family of quartics that are nonsingular for most t (nonsingular is an open condition as we will see). The $t = 0$ case is a double conic. Then it is a fact that if you normalize the family, the central fiber (above $t = 0$) turns into our hyperelliptic curve. Thus we have expressed our hyperelliptic curve as a limit of nonhyperelliptic curves.

I then discussed the 28 bitangents to any smooth quartic curve, and their relationship to other interesting geometry, for example the 6 branch points of a genus 2 hyperelliptic cover (both are examples of theta characteristics) and the 27 lines on any smooth cubic surface (and the link to the Weyl groups of E_7 and E_6). I likely won't type those up in these notes.

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