FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 44

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CONTENTS

1. A SERIES OF CRUCIAL OBSERVATIONS

We are now ready to start understanding curves in a hands on way. We will repeatedly make use of the following series of crucial remarks, and it will be important to have them at the tip of your tongue.

In what follows, C will be a projective, nonsingular, geometrically integral, over a field k. (Often, what matters is integrality rather than geometric integrality, but most readers aren't worrying about this distinction, and those that are can weaken hypotheses as they see fit.) $\mathcal{L}$ is an invertible sheaf on C.

1.1. *Negative degree line bundles have no section.* $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. Reason: $\deg \mathcal{L}$ is the number of zeros minus the number of poles (suitably counted) of any rational section. If there is a regular section (i.e. with no poles), then this is necessarily non-negative. Refining this argument gives:

1.2. *Degree 0 line bundles, and recognizing when they are trivial.* $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$, and if $h^0(C, \mathcal{L}) = 1$ then $\mathcal{L} \cong \mathcal{O}_C$. Reason: if there is a section s, it has no poles and hence no zeros. Then s gives a trivialization for the invertible sheaf. (Recall how this works: we have a natural bijection for any open set $\Gamma(U, \mathcal{L}) \leftrightarrow \Gamma(U, \mathcal{O}_U)$, where the map from left to right is $s' \mapsto s'/s$, and the map from right to left is $f \mapsto sf$.) So if there is a section, $\mathcal{L} \cong \mathcal{O}$. But we've already checked that for a geometrically integral projective variety, $h^0(\mathcal{O}) = 1$.

1.3. *Twisting $\mathcal{L}$ by a (degree 1) point changes h^0 by at most 1.* Suppose p is any closed point of degree 1 (i.e. the residue field of p is k). Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1. Reason:

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consider $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$, tensor with $\mathcal{L}$ (this is exact as $\mathcal{L}$ is locally free) to get

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0.$$

Then $h^0(C, \mathcal{L}|_p) = 1$, so as the long exact sequence of cohomology starts off

$$0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p),$$

we are done.

1.4. *Numerical criterion for $\mathcal{L}$ to be base-point-free.* Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space. Reason: given any p, our equality shows that there exists a section of $\mathcal{L}$ that does not vanish at p.

1.5. Next, suppose p and q are distinct (closed) points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 0, 1$, or 2 (by repeating the argument of Remark 1.3 twice). If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then necessarily

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}(-p)) + 1 = h^0(C, \mathcal{L}(-q)) + 1 = h^0(C, \mathcal{L}(-p-q)) + 2. \tag{1}$$

Then the linear system $\mathcal{L}$ separates points p and q, i.e. the corresponding map f to projective space satisfies $f(p) \neq f(q)$. Reason: there is a hyperplane of projective space passing through p but not passing through q, or equivalently, there is a section of $\mathcal{L}$ vanishing at p but not vanishing at q. This is because of the last equality in (1).

1.6. By the same argument as above, if p is a (closed) point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0$, 1, or 2. I claim that if this is 2, then map corresponds to $\mathcal{L}$ (which is already seen to be base-point-free from the above) separates the tangent vectors at p. To show this, I need to show that the cotangent map is *surjective*. To show surjectivity onto a one-dimensional vector space, I just need to show that the map is non-zero. So I need to give a function on the target vanishing at the image of p that pulls back to a function that vanishes at p to order 1 but not 2. In other words, I want a section of $\mathcal{L}$ vanishing at p to order 1 but not 2. But that is the content of the statement $h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-2p)) = 1$.

1.7. *Criterion for $\mathcal{L}$ to be a closed immersion.* Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a *closed immersion into projective space*, as it separates points and tangent vectors, by Theorem 1.11 from last day (Class 43).

1.8. *We know h^0 if the degree is sufficiently high.* We now bring in Serre duality. I claim that $\deg \mathcal{L} > 2g - 2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.}$$

We know $h^0(C, \mathcal{L})$ if $\deg \mathcal{L} \gg 0$. *This is important — remember this!* Reason: $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$; but $\mathcal{K} \otimes \mathcal{L}^\vee$ has negative degree (as $\mathcal{K}$ has degree $2g - 2$), and thus this invertible sheaf has no sections. Thus Riemann-Roch gives us the desired result.

1.A. EXERCISE. Suppose that k is algebraically closed, so the previous remark applies. Show that $C - p$ is affine. (Hint: Show that if $k \gg 0$, then $\mathcal{O}(kp)$ is base-point free and has at least two linearly independent sections, one of which has divisor kp. Use these two sections to map to $\mathbb{P}^1$ so that the preimage of ∞ is (set-theoretically) p. Argue that the map is finite, and that $C - p$ is the preimage of $\mathbb{A}^1$.)

1.B. FOLLOW-UP EXERCISE. Show that any non-projective integral curve over a field k (not necessarily algebraically closed) is affine.

1.C. USEFUL EXERCISE (RECOGNIZING $\mathcal{K}$ AMONG DEGREE $2g - 2$ LINE BUNDLES). Suppose $\mathcal{L}$ is a degree $2g - 2$ invertible sheaf. Show that it has $g - 1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

1.9. Conclusion. We can combine much of the above discussion to give the following useful fact. If k is algebraically closed, then $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g + 1$ implies that this is in fact a closed immersion. Remember this!

1.D. EXERCISE. Show that the statements in the previous paragraph even without the hypothesis that k is algebraically closed. (Hint: to show one of the facts about some curve C and line bundle $\mathcal{L}$, consider instead $C \otimes_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Then somehow show that if the pullback of $\mathcal{L}$ here has sections giving you one of the two desired properties, then there are sections downstairs with the same properties. You may want to use facts that we've used, such as the fact that the h^0 is preserved by extension of k, or that the property of an affine morphism being a closed immersion holds if and only if it does after an extension of k, Exercise 1.E from the previous class, Class 43.)

We're now ready to take these facts and go to the races.

2. CURVES OF GENUS 0

We are now ready to (in some form) answer the question: what are the curves of genus 0?

We have seen a genus 0 curve (over a field k) that was *not* isomorphic to $\mathbb{P}^1$: $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. We have already observed that this curve is *not* isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, because it doesn't have an $\mathbb{R}$-valued point. On the other hand, we haven't seen a genus 0 curve over an algebraically closed field with this property. This isn't a coincidence: the lack of an existence of a k-valued point is the only obstruction to a genus 0 curve being $\mathbb{P}^1$.

2.1. *Claim. — Suppose* C *is genus* 0*, and* C *has a* k*-valued (degree* 1*) point. Then* $C \cong \mathbb{P}^1_k$*.*

Thus we see that all genus 0 (integral, nonsingular) curves over an algebraically closed field are isomorphic to $\mathbb{P}^1$.

Proof. Let p be the point, and consider $\mathcal{L} = \mathcal{O}(p)$. Then $\deg \mathcal{L} = 1$, so we can apply what we know above: first, $h^0(C, \mathcal{L}) = 2$ (Remark 1.8), and second, these two sections give a closed immersion into $\mathbb{P}^1_k$ (Remark 1.9). But the only closed immersion of a curve into $\mathbb{P}^1_k$ is an isomorphism! □

As a fun bonus, we see that the weird real curve $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no *divisors* of degree 1 over $\mathbb{R}$ (*effective* divisors); otherwise, we could just apply the above argument to the corresponding line bundle.

This weird curve shows us that over a non-algebraically closed field, there can be genus 0 curves that are not isomorphic to $\mathbb{P}^1_k$. The next result lets us get our hands on them as well.

2.2. *Claim. — All genus* 0 *curves can be described as conics in* $\mathbb{P}^2_k$*.*

Proof. Any genus 0 curve has a degree -2 line bundle — the canonical bundle $\mathcal{K}$. Thus any genus 0 curve has a degree 2 line bundle: $\mathcal{L} = \mathcal{K}^\vee$. We apply Remark 1.9: $h^0(C, \mathcal{L}) = 3 \geq 2g + 1$, so this line bundle gives a closed immersion into $\mathbb{P}^2$. □

2.A. EXERCISE. Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

We will use the following result later.

2.3. *Claim. — Suppose* C *is not isomorphic to* $\mathbb{P}^1_k$ *(with no restrictions on the genus of* C*), and* $\mathcal{L}$ *is an invertible sheaf of degree* 1*. Then* $h^0(C, \mathcal{L}) < 2$*.*

Proof. Otherwise, let s_1 and s_2 be two (independent) sections. As the divisor of zeros of s_i is the degree of $\mathcal{L}$, each vanishes at a single point p_i (to order 1). But $p_1 \neq p_2$ (or else s_1/s_2 has no poles or zeros, i.e. is a constant function, i.e. s_1 and s_2 are dependent). Thus we get a map $C \to \mathbb{P}^1$ which is basepoint free. This is a finite degree 1 map of nonsingular curves, which induces a degree 1 extension of function fields, i.e. an isomorphism of function fields, which means that the curves are isomorphic. But we assumed that C is not isomorphic to $\mathbb{P}^1_k$. □

2.B. EXERCISE. Show that if k is algebraically closed, then C has genus 0 if and only if all degree 0 line bundles are trivial.

3.1. *Why not do curves of genus* 1 *now?.* It might make most sense to jump to genus 1 at this point, but the theory of elliptic curves is especially rich and beautiful, so we'll leave it for later.

In general, curves have quite different behaviors (topologically, arithmetically, geometrically) depending on whether $g = 0$, $g = 1$, or $g > 2$. This trichotomy extends to varieties of higher dimension. I gave a very brief discussion of this trichotomy for curves. For example, arithmetically, genus 0 curves can have lots and lots of points, genus 1 curves can have lots of points, and by Faltings' Theorem (Mordell's Conjecture) any curve of genus at least 2 has at most finitely many points. (Thus we knew before Wiles that $x^n + y^n = z^n$ in $\mathbb{P}^2$ has at most finitely many solutions for $n \geq 4$, as such curves have genus $\binom{n-1}{2} > 1$.) Geometrically, Riemann surfaces of genus 0 are positively curved, Riemann surfaces of genus 1 are flat, and Riemann surfaces of genus 1 are negatively curved. We will soon see that curves of genus at least 2 have finite automorphism groups, while curves of genus 1 have some automorphisms (a one-dimensional family), and the unique curve of genus 0 (over an algebraically closed field) have a three-dimensional automorphism group.

3.2. *Back to curves of genus 2.*

Over an algebraically closed field, there is only one genus 0 curve. We haven't yet seen *any* curves of genus 2 — how many are there? How can we get a hold of them.

Fix a curve C of genus $g = 2$. Then $\mathcal{K}$ is degree $2g - 2 = 2$, and has 2 sections (Exercise 1.C). I claim that $\mathcal{K}$ is base-point-free. We may assume k is algebraically closed, as base-point-freeness is independent of field extension of k. If $\mathcal{K}$ is not base-point-free, then if p is a base point, then $\mathcal{K}(-p)$ is a degree 1 invertible sheaf with 2 sections, and we just showed (Claim 2.3) that this is impossible. Thus we canonically constructed a double cover of $\mathbb{P}^1$ (unique up to automorphisms of $\mathbb{P}^1$). Conversely, any double cover $C \to \mathbb{P}^1$ arises from a degree 2 invertible sheaf with at least 2 sections, so if $g(C) = 2$, this invertible sheaf must be the canonical bundle (as the only degree 2 invertible sheaf on a genus 2 curve with at least 2 sections is $\mathcal{K}_C$, Exercise 1.C).

Hence we have a natural bijection between genus 2 curves and genus 2 double covers of $\mathbb{P}^1$.

Thus if we could classify genus 2 double covers of $\mathbb{P}^1$, we could classify all genus 2 curves. We'll do that next day, at least in the case where k is algebraically closed and characteristic 0. While we're doing that, we may as well study double covers of $\mathbb{P}^1$ instead — the theory of *hyperelliptic curves*.

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